

Non-Abelian symmetry operators from hanging branes in $\text{AdS}_5 \times S^5$

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We investigate the holographic realization of topological operators for continuous non-Abelian symmetries in quantum field theories. As a concrete case study, we focus on type IIB string theory on $\text{AdS}_5 \times S^5$ which admits an $SO(6)$ isometry, dual to the R-symmetry in 4D $\mathcal{N} = 4$ super-Yang-Mills theory. We argue that symmetry operators for continuous symmetries are generally realized by bound states of D5-branes and Kaluza-Klein (KK) monopoles hanging from the conformal boundary. Together they account for the contributions to Gauss's law constraints from the G_5 flux and the Einstein-Hilbert term respectively. We also demonstrate how the D5-KK bound state measures the representation of the end point of Wilson lines constructed by D3-branes.

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I. INTRODUCTION

The study of symmetries in quantum systems has been revolutionized by the development of generalized symmetries, where symmetry data is encoded in the topological sector of quantum field theories [1]. Holography and string theory have been a driving force, providing explicit realizations of topological operators via bulk degrees of freedom [2–25], and partially inspiring symmetry topological field theories (SymTFTs) [26–37], a powerful framework that defines the action of topological symmetries, and generalizes the action of groups by their representations. Together with fusion categories [38–45], this has yielded deep insight for finite symmetries, both invertible and noninvertible, elucidating fundamental aspects of quantum systems across condensed matter, quantum information, and high-energy physics. For continuous symmetries, however, especially non-Abelian ones, an analogous understanding is still a central unresolved problem [46–53] (see also [54]): the categorical structure is lacking, and a SymTFT construction remains an open challenge [55–61].

It is natural to expect that the holographic perspective will illuminate the topological nature and action of continuous symmetries. Indeed, a SymTFT is fundamentally a

bulk-boundary system, precisely the structure that holography is naturally built to describe. A concrete incarnation of the above challenge in holography is how to measure charges of boundary operators using bulk physics, i.e. how symmetry operators constructed from bulk degrees of freedom act on boundary operators. Answering this question is important for two reasons: (1) it would provide direct input toward a SymTFT for continuous symmetries, and (2) it is independently a fundamental open problem in AdS/CFT, probing the interplay between gauge symmetry, topology, and brane dynamics.

In this paper, we solve this problem in the prime example of holography: type IIB string theory on $\text{AdS}_5 \times S^5$, dual to 4D $\mathcal{N} = 4$ super-Yang-Mills (SYM). From the low-energy effective action [62–70], we construct the symmetry operators via Gauss's law constraints and realize them explicitly in string theory as bound states of Bogomol'nyi–Prasad–Sommerfield (BPS) D5-branes and KK monopoles anchored to the boundary in a U-shaped configuration, whose width provides the necessary regulator for continuous symmetry operators [71]. We characterize the continuous parameters labeling symmetry operators, determine their non-Abelian fusion rules, and demonstrate charge measurement via Hanany-Witten transitions.

While our construction is carried out in a specific setup, the methods extend naturally to general holographic settings [72]. In particular, the explicit identification of the bulk degrees of freedom that realize symmetry operators and their fusion provides a setup to formulate a SymTFT for continuous non-Abelian symmetries that we explore elsewhere.

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II. LOW-ENERGY DESCRIPTION

A. Setup

The holographic dual of 4D $\mathcal{N} = 4$ $SU(N)$ SYM is type IIB string theory on $\text{AdS}_5 \times S^5$ with N units of G_5 flux. Type IIB supergravity admits a consistent truncation on S^5 [63–67,73] to 5D maximal $SO(6)$ gauged supergravity [68–70]. For our purposes we do not need the full uplift formulas, and focus on the $SO(6)$ gauge fields, not considering fluctuations of the scalar fields or the external metric.

For the effective action for the $SO(6)$ gauge fields, we follow the presentation in [73]. The effective action is

$$S_{\text{eff}} = \int_{\text{AdS}_5} \left[-\frac{1}{4g^2} F_{ab} \wedge *F^{ab} - k \text{CS}_5 \right]. \quad (1)$$

The $SO(6)$ field strength is $F^{ab} = dA^{ab} + A^{ac} \wedge A_c{}^d$, where $A^{ab} = -A^{ba}$ and a, b , etc. are vector indices of $SO(6)$. The Chern-Simons 5-form satisfies $d\text{CS}_5 = \frac{1}{384\pi^2} \epsilon_{abcdef} F^{ab} \wedge F^{cd} \wedge F^{ef}$. The gauge coupling is $\frac{1}{g^2} = \frac{N^2}{8\pi^2 L}$ and the Chern-Simons level is $k = N^2 - 1$ [74]. Here $L^4 = 4\pi g_s N (\alpha')^2$. While g^{-2} receives $1/N$ corrections, the value of k is exact, as it captures the R-symmetry 't Hooft anomaly of 4D $\mathcal{N} = 4$ SYM [75].

The gauge coupling constant g^{-2} originates from two terms in the type IIB pseudoaction: the Einstein-Hilbert (EH) term, and the kinetic term for the flux G_5 . The two contributions are equal up to a factor 2 [76],

$$\frac{1}{g^2} = \frac{1}{g_{\text{EH}}^2} + \frac{1}{g_{\text{flux}}^2}, \quad \frac{1}{g_{\text{flux}}^2} = \frac{2}{g_{\text{EH}}^2}. \quad (2)$$

B. Gauss's law constraints

Building on [26,77], we analyze the effective action in a Hamiltonian framework where the radial direction of AdS_5 is identified with Euclidean time. The variation of the action with respect to the temporal components of A^{ab} gives the Gauss's law constraints $\mathbf{G}_{4ab} = 0$, where

$$\mathbf{G}_{4ab} = \frac{1}{2g^2} D * F_{ab} + \frac{1}{128\pi^2} k \epsilon_{abc_1 c_2 c_3 c_4} F^{c_1 c_2} \wedge F^{c_3 c_4}. \quad (3)$$

Here D is the $SO(6)$ covariant derivative and $*$ is the Hodge star in AdS_5 . The 4-form $\mathbf{G}_{4ab}$ is evaluated on a slice at constant AdS_5 radius [78].

Gauss's law constraints generate bulk gauge transformations. A bulk gauge transformation that is nontrivial near the conformal boundary of AdS_5 implements a global symmetry. Thus, we can read off the symmetry generators by taking Gauss's law constraints and evaluating them on a slice at constant AdS_5 radius, taking the limit in which we approach the conformal boundary. In the standard

AdS/CFT dictionary, the gauge fields have Dirichlet boundary conditions $A^{ab} = 0$. Hence, in this boundary limit, Gauss's law constraints simplify to $\mathbf{G}_{4ab} = \frac{1}{2g^2} d * F_{ab}$. We may now consider a linear combination $\theta^{ab} \mathbf{G}_{4ab}$ with constant θ^{ab} coefficients, and integrate it on a boundary 4-chain $\mathcal{B}^4$ with $\partial \mathcal{B}^4 = M^3$. Applying Stokes's theorem, we get the expected symmetry operator from the low-energy analysis

$$U(\theta; M^3) = \exp \left(\frac{i}{2g^2} \theta^{ab} \int_{M^3} *F_{ab} \right). \quad (4)$$

In (4), $*F$ and θ take values in $\mathfrak{so}(6)$ and the dual $\mathfrak{so}(6)^*$ respectively. Since the Cartan-Killing metric gives an isomorphism $\mathfrak{so}(6)^* \cong \mathfrak{so}(6)$, we will henceforth implicitly identify $\mathfrak{so}(6)^*$ and $\mathfrak{so}(6)$. $U(\theta; M^3)$ implements the $SO(6)$ global symmetry transformation associated to the group element $\exp \theta$, where $\exp : \mathfrak{so}(6) \rightarrow SO(6)$ is the exponential map, which is surjective.

C. Wilson line operators

The action (1) admits standard Wilson loop operators $\text{Tr}_{\mathbf{R}} \text{Pexp} \int (-A)$ labeled by a unitary irreducible representation (irrep) $\mathbf{R}$ of $SO(6)$. Let us recall an alternative presentation of a Wilson loop, based on a quantum mechanical model living on the support M^1 of the Wilson loop and coupled to the 5D gauge field A^{ab} . This is based on coadjoint orbits and their quantization [79–83], see also [84][Part 4, Sec. VII.7], [85,86]. The 1D action reads

$$S_{\text{WL}} = \int_{M^1} \frac{1}{2} q_{ab} (\chi^{-1} A \chi + \chi^{-1} d\chi)^{ab}, \quad (5)$$

with constants $q_{ab} = q_{[ab]}$, $\chi^a{}_b$ an $SO(6)$ -valued scalar field localized on M^1 and $(\chi^{-1} d\chi)^{ab} = (\chi^{-1})^a{}_c d\chi^c{}_d \delta^{db}$. In (5), A is implicitly pulled back from AdS_5 .

The only nonzero independent parameters q_{ab} are q_{12} , q_{34} , q_{56} and are integers. They encode the irrep labeling the Wilson line. This will be useful in discussing charge measurement later on. We refer the reader to Appendix B for justifications of the statements of this paragraph, which are standard [79–87].

Consider the end point $\mathcal{O}$ of a Wilson line in the irrep $\mathbf{R}$ ending on the conformal boundary of AdS_5 . The end point transforms as $\mathcal{O} \rightarrow D_{\mathbf{R}}(g) \mathcal{O}$ under a global $SO(6)$ transformation with parameter g . The map $D_{\mathbf{R}}$ is the homomorphism sending an abstract group element in $SO(6)$ to the matrix that represents it in the irrep $\mathbf{R}$. An $SO(6)$ transformation for $\mathcal{O}$ with $g = \exp \theta$ is implemented by the operator $U(\theta; M^3)$ of the form (4), by moving $U(\theta; M^3)$ past the Wilson line end point [88].

D. Wilson lines from D3-branes

States charged under the $SO(6)$ R-symmetry can be realized using wrapped D3-branes [89–95]. We now revisit this construction to show how the effective action for a D3-brane, with three directions compactified on S^5 , can reproduce the 1D action (5) for a Wilson line in AdS_5 .

Let x^μ and the constrained y^a denote coordinates on AdS_5 and S^5 respectively. The D3-brane world volume action contains scalar fields $X^\mu(\tau, \sigma^i)$, $Y^a(\tau, \sigma^i)$, where τ, σ^i ($i = 1, 2, 3$) are world volume coordinates. We consider a profile for the scalar fields X^μ, Y^a of the form $X^\mu(\tau, \sigma^i) = X^\mu(\tau)$, $Y^a(\tau, \sigma^i) = \chi^a_b(\tau) Y^b_{(0)}(\sigma^i)$. The functions $X^\mu(\tau)$ parametrize the D3-brane external support $M^1 \subset AdS_5$ and $Y^a_{(0)}(\sigma^i)$ describe the D3-brane internal profile. The latter wraps w times around the maximal $S^3 \subset S^5$ defined by $y^5 = 0 = y^6$. We include fluctuations of the internal profile of the brane, associated to an $SO(6)$ rotation depending on external spacetime, encoded by the τ -dependent $SO(6)$ matrix $\chi^a_b(\tau)$.

To study the topological couplings on the D3-brane, particularly the C_4 term in the Wess-Zumino action, we pull back the bulk C_4 to the D3-brane world volume using the embedding scalars $X^\mu(\tau, \sigma^i)$, $Y^a(\tau, \sigma^i)$. We report the derivation in [72]. The result is the action (5) where the only nonzero q_{ab} components are $q_{56} = Nw = -q_{65}$.

In general, for any pair of indices (a, b) with $a < b$, we define $(\Sigma^3)^{ab}$ as the $S^3 \subset S^5$ determined by $y^a = 0 = y^b$. We also set $(\Sigma^3)^{ab} = -(\Sigma^3)^{ba}$ for $a > b$ and $(\Sigma^3)^{aa} = 0$. With this notation, the D3-brane wraps $M^1 \times \Sigma^3$, where $\Sigma^3 = \frac{1}{2} n_{ab} (\Sigma^3)^{ab}$ with arbitrary integer wrapping coefficients $n_{ab} = -n_{ba}$. The result is still given by (5) with $q_{ab} = N n_{ab}$. Which components n_{ab} are nonzero translates to different directions in the Lie algebra $\mathfrak{so}(6)$. The overall factor of N is familiar from the study of maximal giant graviton states realized by D3-branes [90,91,93].

In [72] we show that the D3-brane effective action Dirac-Born-Infeld (DBI) term is subleading and does not contribute to (5).

III. SYMMETRY OPERATORS FROM HANGING BRANE BOUND STATES

As described around (2), there are two contributions to Gauss's law constraints G_{4ab} from the type IIB pseudoaction. We argue that the appropriate stringy configuration which reproduces the full symmetry operator (4) is a bound state of a D5-brane and a KK monopole corresponding, respectively, to the self-dual G_5 flux and the EH action contributions. For both we can identify the physical origin of the $\mathfrak{so}(6)$ -valued parameter θ in (4).

We take the world volume of the D5-KK bound state to be $\gamma^1 \times \Sigma^2 \times M^3$. Here γ^1 is an arc in AdS_5 whose end points lie on the conformal boundary at $z = 0$ (z denotes the radial coordinate), M^3 is a 3D subspace of the

conformal boundary, and Σ^2 is a 2D subspace of S^5 . The hanging brane configuration is motivated by [48,53] and does not depend on the choice of a metric on M^3 [72], so in this sense the operator is topological.

To define Σ^2 , we pair the six real constrained coordinates y^a into three complex coordinates, and express S^5 as a Hopf fibration over $\mathbb{CP}^2$. Then, Σ^2 can be thought of as a copy of $\mathbb{CP}^1$ inside $\mathbb{CP}^2$. The choice of complex coordinates can be parametrized by integers $m_{ab} = -m_{ba}$. Different choices correspond to different $U(1)$ subgroups of $SO(6)$. Taking products of elements from these subgroups we can construct any element in $SO(6)$; this is akin to the standard Euler angle parametrization of $SO(3)$ rotations. See Appendixes D and C for details.

A. Hanging D5-brane

Let us study the effective action for a D5-brane on $\gamma^1 \times \Sigma^2 \times M^3$, focusing on topological couplings. The 7-form anomaly polynomial arising from the Wess-Zumino action of the D5-brane contains the term $2\pi i \int_{N^7} f_2 \wedge G_5$ on an auxiliary 7-manifold N^7 whose boundary is the D5-brane world volume. $f_2 = da_1$ is the field strength of the Chan-Paton gauge field on the D5-brane, normalized to have integral periods. Since G_5 is closed on shell, this term can be rewritten as $2\pi i \int_{\gamma^1 \times \Sigma^2 \times M^3} a_1 \wedge G_5$, using Stokes's theorem (with corners). We may then consider a flat Chan-Paton connection with a nontrivial holonomy along γ^1 ,

$$\alpha_{D5} = 2\pi \int_{\gamma^1} a_1. \quad (6)$$

Here α_{D5} is an angular variable with period 2π . This can be seen from the $a_1 \wedge G_5$ coupling and the fact that G_5 has integral periods, or alternatively from the fact that α_{D5} is the parallel transport of a $U(1)$ connection. We impose Dirichlet boundary conditions on a_1 at the end points of γ^1 to ensure gauge invariance. Next we evaluate $\int_{\Sigma^2 \times M^3} G_5$ using (D1) to get

$$U_{D5}(\alpha_{D5}, m^{ab}; M^3) = \exp\left(\frac{i\alpha_{D5} m^{ab}}{2g_{\text{flux}}^2} \int_{M^3} *F_{ab}\right). \quad (7)$$

B. Hanging KK monopole

We now turn to the KK monopole, whose effective action was derived in [96] (see references therein). The world volume bosonic fields include four real scalars associated with the four transverse directions to the KK monopole, two compact scalars whose 1-form field strengths will be denoted as $\mathcal{K}_1$ and $\tilde{\mathcal{K}}_1$, and one chiral 2-form.

As in the D5-brane analysis, one can derive the 7-form anomaly polynomial via descent [72,96]. The term of interest is $2\pi i \int_{N^7} \tilde{\mathcal{K}}_1 \wedge \mathcal{K}_1 \wedge G_5$, where $d\mathcal{K}_1 = d\tilde{\mathcal{K}}_1 = 0$

by the Bianchi identities. We can regard $\tilde{\mathcal{K}}_1 \wedge \mathcal{K}_1$ as the 2-form field strength of a composite $U(1)$ gauge field v_1 , such that $dv_1 = \tilde{\mathcal{K}}_1 \wedge \mathcal{K}_1$, and then write the relevant term in the form $2\pi i \int_{\gamma^1 \times \Sigma^2 \times M^3} v_1 \wedge G_5$. Analogously to α_{D5} , we turn on a flat profile for v_1 with a nontrivial holonomy along γ^1 ,

$$\alpha_{KK} = 2\pi \int_{\gamma^1} v_1, \quad (8)$$

and impose Dirichlet boundary conditions for v_1 . The integral $\int_{\Sigma^2 \times M^3} G_5$ is done in the Appendix D. Recalling (2), the resultant operator is

$$U_{KK}(\alpha_{KK}, m^{ab}; M^3) = \exp\left(\frac{i\alpha_{KK}m^{ab}}{g_{EH}^2} \int_{M^3} *F_{ab}\right). \quad (9)$$

To reproduce the full symmetry operator (4) determined by Gauss's law constraints, the D5-KK bound state should satisfy the relation $\alpha \equiv \alpha_{D5} = 2\alpha_{KK}$. This takes into account the contributions from the G_5 flux and the EH term. All in all we get the overall operator

$$U_{D5-KK}(\alpha, m^{ab}; M^3) = \exp\left(\frac{iam^{ab}}{2g^2} \int_{M^3} *F_{ab}\right), \quad (10)$$

where g^2 is the gauge coupling given by (2). We notice that (10) matches with the operator (4) from the low-energy analysis, with the identification $\theta^{ab} = \alpha m^{ab}$. We stress again that both the D5-brane and the KK monopole are BPS objects in type IIB string theory.

IV. CHARGE MEASUREMENT FROM BRANES

The operator realized by the hanging 5-brane measures the charge of the end point of the Wilson line realized by the D3-brane. This can also be understood in terms of a Hanany-Witten transition [97]. Only the D5-brane inside the D5-KK bound state participates in the Hanany-Witten move.

For definiteness, we take a D5-brane wrapping $\Sigma^2 \subset S^5$, extending along x^0, x^1, x^2 , and along an arc γ^1 in the 2D plane spanned by x^3 and the AdS_5 radial coordinate z . The D3-brane wraps $\Sigma^3 \subset S^5$ and extends along z . Initially the D3-brane sits to the left of the hanging D5-brane in the x^3 direction. If we bring the D3-brane to the other side of the D5-brane, an F1-string is created. Its world sheet is a semidisk in the $x^3 - z$ plane, bounded by γ^1 and the conformal boundary of AdS_5 [53] (see Fig. 1). Internally, the F1-string is supported on the intersection of Σ^3, Σ^2 . We assume Σ^3 intersects transversely with Σ^2 , and denote by $\Sigma^3 \cdot \Sigma^2$ their intersection number.

The F1-string created by the Hanany-Witten transition measures charge. To see this, we focus on the topological

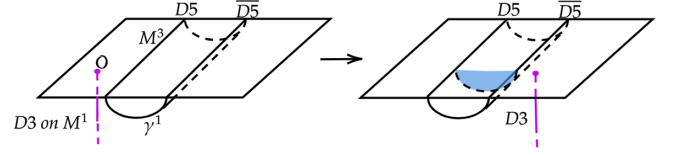


FIG. 1. A D5-brane hangs from the conformal boundary along γ^1 and wraps M^3 . A D3-brane ends on the conformal boundary on O and extends along M^1 into the AdS_5 bulk. A D5-D3 Hanany-Witten move creates an F1-string (blue surface).

couplings on the F1-string. If an F1-string with world sheet W^2 ends on a Dp -brane, it couples to the Chan-Paton gauge field a_1 on the Dp -brane, $S_{F1}^{\text{top}} \supset 2\pi \int_{\partial W^2} a_1$. The F1-string is suspended between the hanging D5-brane and the AdS_5 boundary, therefore, ∂W^2 consists of the arc γ^1 , and a segment on the boundary. The coupling of the F1-string to the D5-brane Chan-Paton gauge field happens only along the γ^1 portion of ∂W^2 . This yields the exponentiated action $\exp[2\pi i(\Sigma^3 \cdot \Sigma^2) \int_{\gamma^1} a_1] = \exp[i\alpha(\Sigma^3 \cdot \Sigma^2)]$ where we used $\alpha_{D5} = 2\pi \int_{\gamma^1} a_1$ and the effective multiplicity of the F1-string is $\Sigma^3 \cdot \Sigma^2 = \frac{1}{2} N n_{ab} m^{ab}$, in terms of the integer parameters n_{ab}, m^{ab} that specify Σ^3, Σ^2 , respectively. Identifying $q_{ab} = N n_{ab}$ and $\theta^{ab} = \alpha m^{ab}$, from earlier, we can write $\exp[i\alpha(\Sigma^3 \cdot \Sigma^2)]$ in the form $\exp(\frac{1}{2} i\theta^{ab} q_{ab})$.

We can use Hanany-Witten transitions to probe a given D3-brane with different hanging 5-branes, with various internal profiles. Each Hanany-Witten move results in a phase factor $\exp(\frac{1}{2} i\theta^{ab} q_{ab})$ with the same q_{ab} from the D3-brane, but different θ^{ab} depending on the internal profile of the 5-brane and the holonomy of its Chan-Paton gauge field. This set of phase factors allows us to reconstruct the values of the charges q_{ab} , and thereby determine the irrep $\mathbf{R}$ associated to the D3-brane.

The topological couplings of the F1-string include a coupling to the Neveu-Schwarz–Neveu-Schwarz (NSNS) 2-form B_2 that, in our setup, is trivial in the background and is also not sourced by the 5-brane insertion. Thus, this does not contribute. We also expect that the nontopological Nambu-Goto world sheet term does not affect the conclusions of this section.

We note that the D3-branes wrapping the maximal $S^3 \subset S^5$ correspond to the maximal giant gravitons of [90,91,93]. As such, they do not collapse to a point and are stabilized at finite size by the angular momentum carried on S^5 , which is encoded in the collective-coordinate rotation χ entering (5). This balances the brane tension against the coupling to the background 5-form flux. More generally, although the wrapped sphere is contractible and hence trivial in ordinary homology, the relevant configurations are nontrivial as equivariant cycles, i.e. invariant submanifolds under the relevant $U(1) \subset SO(6)$ isometry action. It is this equivariant data, rather than the (trivial) homology class, that determines the charge and topological properties of the associated

operators; we develop this viewpoint in detail in [72]. The same structure applies to the hanging 5-branes realizing the symmetry operators: their internal supports are likewise invariant submanifolds under the same isometry action, so that their topological properties are controlled by this equivariant data rather than by the (trivial) homology class.

V. FUSION OF HANGING BRANES

An operator $U(\theta; M^3)$ of the form (4), realized as a D5-KK bound state (10), implements the $SO(6)$ transformation $g = \exp \theta$. We denote it as U_g . The parallel fusion of such operators is grouplike, $U_g U_{g'} = U_{gg'}$. This can be recovered from (2), the Baker-Campbell-Hausdorff (BCH) formula, and the algebra of $\int_{M^3} *F$. To describe the latter we define $X_{ab} = \frac{i}{2g^2} \int_{M^3} *F_{ab}$ and write $[X_{ab}, X_{cd}] = \frac{1}{2} f_{ab,cd}{}^{ef} X_{ef}$, $f_{ab,cd}{}^{ef} = 8\delta_{[a}^{[e} \delta_{b]}^{f]} \delta_{[c}^e \delta_{d]}^f$. See Appendix E for more details.

Consider $g = \exp(am)$, $g' = \exp(\alpha'm')$ for the same m in $\mathfrak{so}(6)$. Fusion amounts to summing the holonomies α and α' . The two D5-KK bound states share the same internal support but have independent pairs of world volume gauge fields (a_1, v_1) and (a'_1, v'_1) respectively. We focus on the Chan-Paton gauge field a_1 . The argument for the KK monopole gauge field v_1 follows analogously.

Regard $\gamma^1 = \Gamma \sqcup \bar{\Gamma}$ as a disjoint union of two half-arcs with opposite orientations, on which the D5-brane and the $\overline{D5}$ -brane in the hanging brane solution are respectively supported. The holonomy is then the sum $\alpha_{D5} = \alpha_{\Gamma} - \alpha_{\bar{\Gamma}}$ where $\alpha_{\Gamma} = \int_{\Gamma} a_1$. As we bring two such hanging branes toward each other, each of them (momentarily) breaks up into a D5- $\overline{D5}$ pair, then the two outer components recombine to form a single hanging D5-brane on $\Gamma \sqcup \bar{\Gamma}'$ (see Fig. 2). It has the same orientation as the original arcs γ^1 and $(\gamma^1)'$, so the associated holonomy is $\tilde{\alpha} = \alpha_{\Gamma} - \alpha_{\bar{\Gamma}'}$. To ensure the inner components also recombine in a way compatible with their orientations, we see that they must move past each other and then recombine into $\Gamma' \sqcup \bar{\Gamma}$, with holonomy $\tilde{\alpha}' = \alpha_{\Gamma'} - \alpha_{\bar{\Gamma}}$.

The result is a stack of two hanging D5-branes whose effective holonomy is $\tilde{\alpha} + \tilde{\alpha}' = \alpha + \alpha'$, as desired, which is associated with the center-of-mass mode of the stack of D5-branes [8]. Due to the tracelessness of $\mathfrak{su}(2)$ -valued gauge fields, the enhanced non-Abelian component of the Chan-Paton gauge field does not contribute to the

topological coupling of interest. We can similarly keep fusing more hanging branes as long as they share the same world volume. This operation is associative. See [72] for details on the brane dynamics underlying fusion.

If instead we have $g = \exp(am)$, $g' = \exp(\alpha'm')$ with m and m' nonzero along independent factors in the Cartan subalgebra $\mathfrak{u}(1)^{\oplus 3}$, then they generate simultaneous rotations in orthogonal planes in $SO(6)$. In this sense, the two D5-KK bound states are supported on orthogonal 2D subspaces Σ^2 . What appears as a parallel fusion in the low-energy effective theory is really two noncoincident D5-KK bound states in the full 10D string theory background sharing the same external support $\gamma^1 \times M^3$. This is why we can build a symmetry operator labeled by a generic element in the maximal torus, parametrized by three angles $(\alpha, \alpha', \alpha'') \in \mathfrak{u}(1)^{\oplus 3}$. Symmetry operators “supported” along orthogonal $\mathfrak{u}(1)$ directions commute, and so the associated holonomies do not directly add.

In the most general scenario, without imposing any condition on m , m' , the internal supports of the two D5-KK bound states are neither coincident nor orthogonal. Fusion thus involves higher-order terms in the BCH formula, which determine the internal support of the final product when regarded as a single D5-KK bound state.

VI. DISCUSSION AND OUTLOOK

We have constructed the first explicit brane realization of continuous non-Abelian symmetry operators in holography, as bound states of hanging D5-branes and KK monopoles in $AdS_5 \times S^5$. The full $SO(6)$ R-symmetry operator, its continuous parameters, and non-Abelian fusion rules can be derived from the brane dynamics in the UV and matched with low-energy expectations from Gauss’s law. Charge measurement is realized via Hanany-Witten transitions. Our construction raises several questions. The symmetry operators here wrap subspaces defined by Killing vectors, in contrast to finite symmetries where branes wrap internal homological cycles; understanding this distinction should clarify the categorical structure of continuous symmetries. It would also be valuable to find a mechanism beyond Gauss’s law that fixes the factor of 2 between D5-brane and KK monopole holonomies. In [72], we generalize to backgrounds with richer topologies and study the dynamics of hanging branes. More broadly, our results provide concrete holographic data, namely, symmetry operators, their parameters, and fusion, that a SymTFT for continuous non-Abelian symmetries must encode. An emerging lesson is that the mathematics of generalized symmetries is implicitly realized by the physics of branes in string theory; our work takes an important step in this direction, and fully uncovering this correspondence is a compelling goal for the future.

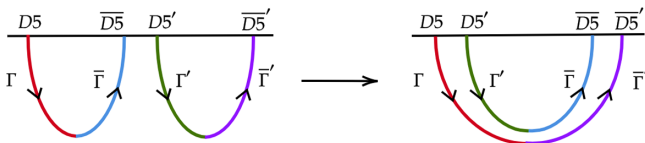


FIG. 2. Two hanging branes on $\Gamma \sqcup \bar{\Gamma}$ and $\Gamma' \sqcup \bar{\Gamma}'$ split and recombine into a hanging two-brane-stack on $\Gamma \sqcup \bar{\Gamma}'$ and $\Gamma' \sqcup \bar{\Gamma}$.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: TYPE IIB STRING THEORY SETUP

We provide additional details concerning the type IIB holographic setup that gives rise to the desired operators. Type IIB string theory on $\text{AdS}_5 \times S^5$ with N units of G_5 flux is holographically dual to 4D $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills. The 10D metric reads

$$ds_{10}^2 = ds_{\text{AdS}_5}^2 + L^2 Dy^a Dy_a, \quad Dy^a = dy^a + A^{ab} y_b. \quad (\text{A1})$$

Here L is the radius of AdS_5 , $A^{ab} = A^{[ab]}$ are the $SO(6)$ gauge fields, and $a, b = 1, \dots, 6$ are vector indices of $SO(6)$. The constrained coordinates y^a satisfy $y^a y_a = 1$ and parametrize S^5 . The 5-form flux is given as

$$G_5 = \mathcal{G}_5 + *_5 \mathcal{G}_5, \quad \mathcal{G}_5 = N \left[(V_5)^{\mathfrak{g}} + \frac{1}{2} \frac{F^{ab}}{2\pi} \wedge (\omega_{ab})^{\mathfrak{g}} \right], \quad (\text{A2})$$

where $*_5$ is the Hodge operator of the metric (A1). In our normalization, G_5 has integral periods. The 5-form V_5 and 3-forms ω_{ab} are defined as

$$V_5 = \frac{1}{\pi^3 \cdot 5!} \epsilon_{abcdef} y^a dy^b \wedge dy^c \wedge dy^d \wedge dy^e \wedge dy^f, \\ \omega_{ab} = \frac{1}{2\pi^2 \cdot 3!} \epsilon_{abcdef} y^c \wedge dy^d \wedge dy^e \wedge dy^f. \quad (\text{A3})$$

The superscript “g” in (A2) stands for the replacement $dy^a \rightarrow Dy^a$ in a differential form. It captures the coupling to the external gauge fields A^{ab} . The Hodge dual of ω_{ab} with respect to the round metric on S^5 is

$$*\omega_{ab} = \frac{1}{4\pi^2} dK_{ab}, \quad K_{ab} = 2y_{[a} dy_{b]}. \quad (\text{A4})$$

The vector fields obtained from the 1-forms K_{ab} by raising their tangent indices with the round metric are the $SO(6)$ Killing vectors on S^5 . It is important to note that

$$\int_{S^5} \omega_{ab} \wedge *\omega^{cd} = \frac{1}{3\pi} \delta_{[a}^c \delta_{b]}^d = \frac{4\pi L}{N^2 g_{\text{flux}}^2} \delta_{[a}^c \delta_{b]}^d, \quad (\text{A5})$$

where g_{flux}^2 is the coupling defined in (2).

APPENDIX B: WILSON LINES AS 1D QUANTUM MECHANICAL MODELS

The 1D Wilson line action (5) is

$$S_{\text{WL}} = \int_{M^1} \frac{1}{2} q_{ab} (\chi^{-1} A \chi + \chi^{-1} d\chi)^{ab}, \quad (\text{B1})$$

where $q_{ab} = q_{[ab]}$ are constants, χ^a_b is an $SO(6)$ -valued scalar field localized on M^1 . The action (5) is invariant under 5D $SO(6)$ gauge transformations

$$A^a_b \rightarrow (MAM^{-1} + MdM^{-1})^a_b, \quad \chi^a_b \rightarrow M^a_c \chi^c_b, \quad (\text{B2})$$

with parameter $M \in SO(6)$. Moreover, $e^{iS_{\text{WL}}}$ is invariant under the local transformation

$$\chi^a_b \rightarrow \chi^a_c h^c_b, \quad (\text{B3})$$

provided that $q_{ab} \in \mathbb{Z}$. Here h^a_b is a 0-form on M^1 valued in the subgroup S_q of $SO(6)$ that stabilizes q_{ab} , i.e. $S_q = \{g \in SO(6) | (gqg^{-1})_{ab} = q_{ab}\}$.

The parameter q_{ab} determines an element in $\mathfrak{so}(6)^* \cong \mathfrak{so}(6)$ which pairs with the $\mathfrak{so}(6)$ -valued 1-form $(\chi^{-1} A \chi + \chi^{-1} d\chi)^{ab}$. We can interpret (5) as a topological sigma model into the coadjoint orbit $SO(6)/S_q$ of q_{ab} . Integrality of q_{ab} ensures integrality of the canonical symplectic 2-form on the orbit [87].

The physics of the action (5) depends on the parameter q_{ab} only through its orbit $SO(6)/S_q$. This means that we have the freedom to “gauge-fix” q to a convenient representative of its orbit. In particular, it is always possible to choose q in the standard Cartan subalgebra of $\mathfrak{so}(6)$, consisting of antisymmetric matrices whose only nonzero independent entries are q_{12} , q_{34} , q_{56} . After this is achieved, there is a “residual gauge freedom” by the action of the Weyl group, which further allows us to select q inside the fundamental Weyl chamber in the Cartan subalgebra. More formally, it is known that each coadjoint orbit intersects the fundamental Weyl chamber in the Cartan subalgebra at one point (see e.g. [87][Chap. 5, Lemma 3]). Thus, coadjoint orbits are in 1-to-1 correspondence with points in the Weyl chamber.

Let us use the fact that q must be integral. This selects a discrete subset of points inside the Weyl chamber, namely, dominant integral elements. We recall that, by virtue of the highest-weight theorem, the following holds: (i) every finite-dimensional irrep of $\mathfrak{so}(6)$ has a unique highest-weight, which is a dominant integral; (ii) for any dominant integral element q there exists a unique finite-dimensional irrep of $\mathfrak{so}(6)$ with highest-weight q [98]. Now the crucial point is that the quantization of the classical theory with action (5) and parameter q yields a finite-dimensional Hilbert space which is precisely isomorphic to the irrep with highest-weight q , as reviewed e.g. in [85][Sec. IV.1].

All in all, knowledge of the parameter q in the action (5) contains the same information as the irrep labeling the Wilson line.

APPENDIX C: INTERNAL PROFILE OF THE 5-BRANES

The 5-brane bound state that we study is supported on the world volume $M^3 \times \gamma^1 \times \Sigma^2$, where $M^3 \times \gamma^1 \subset AdS^5$ and $\Sigma^2 \subset S^5$. We will now discuss the internal profile Σ^2 of this 5-brane bound state. To this end, we realize S^5 as a Hopf fibration over $\mathbb{CP}^2$. This can be achieved by introducing complex coordinates w_1, w_2, z via

$$zw_1 = y_1 + iy_2, \quad zw_2 = y_3 + iy_4, \quad z = y_5 + iy_6. \quad (C1)$$

We write $z = |z|e^{i\psi}$. The constraint $y^a y_a = 1$ fixes $|z|$ in terms of w_1, w_2 . The angle ψ , with period 2π , parametrizes the Hopf fiber, while w_1, w_2 are affine coordinates on the $\mathbb{CP}^2$ base. The S^5 line element reads

$$ds_{S^5}^2 = ds_{\mathbb{CP}^2}^2 + (d\psi + \mathcal{A})^2, \quad (C2)$$

where $ds_{\mathbb{CP}^2}^2$ is the Fubini-Study metric on $\mathbb{CP}^2$ and the connection 1-form $\mathcal{A}$ satisfies $d\mathcal{A} = 2J$, with J the Kähler 2-form on $\mathbb{CP}^2$. We also notice that the 1-form dual to the Killing vector ∂_ψ is

$$k_\psi = d\psi + \mathcal{A} = \frac{1}{2} m^{ab} K_{ab}. \quad (C3)$$

In the second equality we have expanded k_ψ onto the 1-forms K_{ab} defined in (A4). Without loss of generality, the expansion coefficients m^{ab} can be taken to be antisymmetric. They are computed to be

$$m^{ab} = \begin{cases} +1 & \text{if } (a, b) = (1, 2), (3, 4), (5, 6), \\ -1 & \text{if } (a, b) = (2, 1), (4, 3), (6, 5), \\ 0 & \text{otherwise.} \end{cases} \quad (C4)$$

This can be verified explicitly using the expression (A4) for K_{ab} in Hopf coordinates.

Having cast S^5 as a Hopf fibration, we can now characterize Σ^2 : it is the 2-cycle in $\mathbb{CP}^2$ which is Poincaré dual to $N \frac{d\mathcal{A}}{2\pi}$,

$$\Sigma^2 = \text{PD}_{\mathbb{CP}^2} \left[N \frac{d\mathcal{A}}{2\pi} \right]. \quad (C5)$$

The factor of N is justified below. We notice the identity

$$\frac{d\mathcal{A}}{2\pi} \wedge \frac{D\psi}{2\pi} = \frac{1}{2} m^{ab} \omega_{ab}, \quad (C6)$$

with the same coefficients m^{ab} as in (C4). One can verify (C6) from the expressions (A3) for the forms ω_{ab} evaluated in Hopf coordinates.

The identity (C6) is useful for evaluating integrals over Σ^2 ,

$$\int_{\Sigma^2} \lambda_2 = \int_{S^5} N \frac{d\mathcal{A}}{2\pi} \wedge \frac{D\psi}{2\pi} \wedge \lambda_2 = \int_{S^5} \frac{1}{2} N m^{ab} \omega_{ab} \wedge \lambda_2, \quad (C7)$$

where λ_2 is a 2-form. Based on (C7) we can formally write $\Sigma^2 = \frac{1}{2} N m^{ab} \Sigma_{ab}^2$, where integrals over Σ_{ab}^2 are defined via

$$\int_{\Sigma_{ab}^2} \lambda_2 = \int_{S^5} \omega_{ab} \wedge \lambda_2. \quad (C8)$$

While Σ^2 is a nontrivial 2-cycle in $\mathbb{CP}^2$, it is trivialized by the Hopf fibration (as it must be the case, since S^5 has no nontrivial 2-cycles) [99].

In our construction, we insert a probe KK monopole. We recall that the BPS KK monopole of type IIB string theory is a supersymmetric soliton with a 6D world volume. One of its four transverse directions plays a distinguished role and must be an isometry of the 10D metric. In the present setting, this isometry is identified with ∂_ψ . Correspondingly, the KK monopole supports Σ^2 . Thanks to the probe KK monopole, we can also wrap a D5-brane on Σ^2 .

APPENDIX D: SYMMETRY OPERATOR FROM THE 5-BRANE

The desired symmetry operator comes from the following term of the D5-brane action, $i\alpha_{D5} \int_{\Sigma^2 \times M^3} G_5$. The relevant terms are $G_5 \supset \frac{1}{2} N L^{-1} \frac{*F_{cd}}{2\pi} \wedge *\omega^{cd}$, where the L^{-1} factor comes from the L^2 in front of the S^5 metric in the 10D line element (A1) when computing $*_{10} G_5$. The integral of $*\omega^{cd}$ over Σ^2 is conveniently performed using (C7) and (A5). The result reads

$$\int_{\Sigma^2 \times M^3} G_5 = \frac{1}{2g_{\text{flux}}^2} m^{ab} \int_{M^3} *F_{ab}, \quad (D1)$$

with the same m^{ab} coefficients as in (C3), satisfying (C4). The factor of N in the Poincaré dual (C5) of Σ^2 ensures that the right-hand side has no explicit N factors. Finally, (D1) combines with $\alpha_{D5} = 2\pi \int_{\gamma^1} a_1$, to produce the operator

$$U_{D5}(\alpha_{D5}, m^{ab}; M^3) = \exp\left(\frac{i\alpha_{D5}m^{ab}}{2g_{\text{flux}}^2} \int_{M^3} *F_{ab}\right) \quad (\text{D2})$$

from the hanging D5-brane. The computation for the KK monopole follows in the same way to give

$$U_{KK}(\alpha_{KK}, m^{ab}; M^3) = \exp\left(\frac{i\alpha_{KK}m^{ab}}{g_{\text{EH}}^2} \int_{M^3} *F_{ab}\right) \quad (\text{D3})$$

where g_{EH}^2 is the coupling defined in (2). These two contributions then combine to give the full symmetry operator

$$U_{D5-KK}(\alpha, m^{ab}; M^3) = \exp\left(\frac{i\alpha m^{ab}}{2g^2} \int_{M^3} *F_{ab}\right). \quad (\text{D4})$$

1. Generalization of the symmetry operator

Our definition of Hopf coordinates in the Appendix C is subordinate to the choice (C1) of complex coordinates. The entire construction, however, can be repeated verbatim for different choices of complex coordinates. Indeed, we can consider a generalization of (C1) of the form

$$\begin{aligned} zw_1 &= y_{\sigma_1} + iy_{\sigma_2}, & zw_2 &= y_{\sigma_3} + iy_{\sigma_4}, \\ z &= y_{\sigma_5} + iy_{\sigma_6}, \end{aligned} \quad (\text{D5})$$

where $\sigma: \{1, \dots, 6\} \rightarrow \{1, \dots, 6\}$ is a permutation and $\sigma_i = \sigma(i)$. The relations (C3) and (C6), as well as the final result (D4), still hold, albeit with different m^{ab} coefficients. More precisely, (C3), (C6), and (D4) hold with $m^{ab} = m_{(\sigma)}^{ab}$ where we have introduced the notation

$$m_{(\sigma)}^{ab} = \begin{cases} +1 & \text{if } (a, b) = (\sigma_1, \sigma_2), (\sigma_3, \sigma_4), (\sigma_5, \sigma_6), \\ -1 & \text{if } (a, b) = (\sigma_2, \sigma_1), (\sigma_4, \sigma_3), (\sigma_6, \sigma_5), \\ 0 & \text{otherwise.} \end{cases} \quad (\text{D6})$$

For the trivial permutation $\sigma = (123456)$, $m_{(\sigma)}^{ab}$ equals m^{ab} in (C4).

All in all, repeating our construction with different complex coordinates, we have a 5-brane realization of any operator $U(\theta)$,

$$U(\theta; M^3) = \exp\left(\frac{i}{2g^2} \theta^{ab} \int_{M^3} *F_{ab}\right), \quad (\text{D7})$$

with parameters θ^{ab} of the form $\theta^{ab} = \alpha m_{(\sigma)}^{ab}$ for some permutation σ .

We now argue that, for any $g \in SO(6)$, the symmetry operator that realizes the transformation g can be constructed by taking the fusion product of a number of $U(\theta = \alpha m_{(\sigma)})$ operators, for suitable α, σ . We also use the result, from Sec. V and Appendix E, that the fusion of $U(\theta)$ and $U(\theta')$ yields $U(\theta'')$ with $\exp \theta \exp \theta' = \exp \theta''$. It follows that the claim is mapped to a group theory statement, namely, that any $g \in SO(6)$ can be obtained as a product of factors of the form $\exp(\alpha m_{(\sigma)})$, for suitable α, σ . To argue for this, we proceed in two steps.

First, we show how to realize an arbitrary rotation in the (a, b) -plane, for any a, b . For ease of exposition, we consider the $(1, 2)$ -plane, the general case is analogous. If we use the permutations $\sigma = (123456)$ and $\sigma = (124365)$ with parameter α , we realize the $SO(6)$ matrices

$$\begin{aligned} \exp(\alpha m_{(123456)}) &= \text{diag}(R_{12}(\alpha), R_{34}(\alpha), R_{56}(\alpha)), \\ \exp(\alpha m_{(124365)}) &= \text{diag}(R_{12}(\alpha), R_{34}(-\alpha), R_{56}(-\alpha)) \end{aligned} \quad (\text{D8})$$

where $R_{ab}(\alpha)$ is the $SO(2)$ rotation matrix on the (a, b) -plane with angle α . The desired $(1, 2)$ -element of $SO(6)$ can be obtained by multiplying the two elements above.

Finally, having realized arbitrary rotations in the (a, b) -plane for any a, b , we can realize any element in $SO(6)$ by appealing to a generalization of $SO(3)$ Euler angles for $SO(n)$ [100].

APPENDIX E: NONCOMMUTATIVE OPERATOR FUSION

The operators in described in this paper were first constructed with a specific choice of maximal torus $T \cong U(1)^3$. It is true that such a choice exists for every element in $SO(6)$. However, to exhibit the full non-Abelian structure of the $SO(6)$ symmetry, one needs to relate the D5-KK bound states constructed with respect to different choices of maximal tori. In other words, there is a change of basis involved when we try to compose the action of two symmetry operators (D4).

For concreteness, let us fix a reference maximal torus T_0 . Any given element $g \in SO(6)$ is always related to some element $t \in T_0$ by conjugation, i.e. $g = hth^{-1}$ for some $h \in SO(6)$, so a symmetry operator U_g labeled by g can be expressed similarly to (D4) as

$$U_g = \exp\left(\frac{i}{2g^2} \alpha (h m h^{-1})^{ab} \int_{M^3} *F_{ab}\right). \quad (\text{E1})$$

Geometrically, this “rotates” the internal 2D subspace,

$$\Sigma^2 \rightarrow \frac{1}{2} (h m h^{-1})^{ab} \Sigma_{ab}^2 = \frac{1}{2} m^{ab} (h^{-1} \Sigma^2 h)_{ab}. \quad (\text{E2})$$

The difference between the two expressions can be understood as active vs passive rotations. Importantly, the coefficients $\tilde{m}^{ab} = (h m h^{-1})^{ab}$ can now be nonzero for generic (a, b) unlike before, so we have effectively moved out of the maximal torus T_0 .

The parallel fusion $U_g U_{g'} = U_{gg'}$ of two D5-KK bound states labeled by generic $g, g' \in SO(6)$ is noncommutative. Let us see how this manifests at the level of the operators. Using $g = \exp(\alpha \tilde{m})$ and $g' = \exp(\alpha' \tilde{m}')$, the Baker-Campbell-Hausdorff formula gives

$$\begin{aligned} U_g U_{g'} &= \exp\left((\alpha \tilde{m} + \alpha' \tilde{m}')^{ab} X_{ab} + \frac{1}{2} \alpha \tilde{m}^{ab} \alpha' (\tilde{m}')^{cd} [X_{ab}, X_{cd}] + \dots\right) \\ &= \exp\left(\left(\alpha \tilde{m} + \alpha' \tilde{m}' + \frac{1}{2} [\alpha \tilde{m}, \alpha' \tilde{m}'] + \dots\right)^{ab} X_{ab}\right), \end{aligned} \quad (\text{E3})$$

where we have introduced the shorthand notation $X_{ab} = \frac{i}{2g^2} \int_{M^3} *F_{ab}$ for compactness. In the second step above, we noted that X_{ab} satisfies the $\mathfrak{so}(6)$ algebra $[X_{ab}, X_{cd}] = \frac{1}{2} f_{ab,cd}{}^{ef} X_{ef}$, $f_{ab,cd}{}^{ef} = 8\delta_{[a}^{[e} \delta_{b]}^{f]} \delta_{cd}$. The commutator terms are trivial if and only if g and g' commute.

Nonetheless, it is illuminating to consider a special case where $h^{-1}h'$ is in the centralizer of $(h')^{-1}g'h'$, then

$$U_g U_{g'} = \exp\left(\frac{i(h(am + \alpha'm')h^{-1})^{ab}}{2g^2} \int_{M^3} *F_{ab}\right). \quad (\text{E4})$$

Up to an overall conjugation by h , we see that the expression above recovers the discussion in Sec. V regarding the fusion of hanging branes. By the same token, we also obtain a similar fusion rule with $h \rightarrow h'$ if $h^{-1}h'$ is in the centralizer of $h^{-1}gh$.

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- [1] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, *J. High Energy Phys.* **02** (2015) 172.
 - [2] F. Apruzzi, I. Bah, F. Bonetti, and S. Schafer-Nameki, *Phys. Rev. Lett.* **130**, 121601 (2023).
 - [3] I. García Etxebarria, *Fortschr. Phys.* **70**, 2200154 (2022).
 - [4] J. J. Heckman, M. Hübner, E. Torres, and H. Y. Zhang, *Fortschr. Phys.* **71**, 2200180 (2023).
 - [5] J. J. Heckman, M. Hubner, E. Torres, X. Yu, and H. Y. Zhang, *Phys. Rev. D* **108**, 046015 (2023).
 - [6] M. Etheredge, I. Garcia Etxebarria, B. Heidenreich, and S. Rauch, *J. High Energy Phys.* **09** (2023) 005.
 - [7] M. Dierigl, J. J. Heckman, M. Montero, and E. Torres, *Phys. Rev. D* **109**, 046004 (2024).
 - [8] I. Bah, E. Leung, and T. Waddleton, *J. High Energy Phys.* **01** (2024) 117.
 - [9] F. Apruzzi, F. Bonetti, D. S. W. Gould, and S. Schafer-Nameki, *SciPost Phys.* **17**, 025 (2024).
 - [10] M. Cvetič, J. J. Heckman, M. Hübner, and E. Torres, *Phys. Rev. D* **109**, 026012 (2024).
 - [11] F. Baume, J. J. Heckman, M. Hübner, E. Torres, A. P. Turner, and X. Yu, *Phys. Rev. D* **109**, 106013 (2024).
 - [12] X. Yu, *Phys. Rev. D* **110**, 065008 (2024).
 - [13] M. Del Zotto, S. N. Meynet, and R. Moscrop, *J. High Energy Phys.* **07** (2024) 220.
 - [14] S. Hu, *J. High Energy Phys.* **12** (2024) 171.
 - [15] R. Argurio, F. Benini, M. Bertolini, G. Galati, and P. Niro, *J. High Energy Phys.* **07** (2024) 130.
 - [16] H. Y. Zhang, *J. High Energy Phys.* **05** (2026) 118.
 - [17] N. Braeger, V. Chakrabhavi, J. J. Heckman, and M. Hübner, *Phys. Rev. D* **111**, 066015 (2025).
 - [18] S. Franco and X. Yu, *J. High Energy Phys.* **11** (2024) 004.
 - [19] M. Gutperle, Y.-Y. Li, D. Rathore, and K. Roumpedakis, *J. High Energy Phys.* **09** (2024) 110.
 - [20] B. Knighton, V. Sriprachyakul, and J. Vošmera, *J. High Energy Phys.* **07** (2025) 083.
 - [21] J. J. Fernandez-Melgarejo, G. Giorgi, D. Marques, and J. A. Rosabal, *Phys. Rev. D* **111**, 066024 (2025).
 - [22] O. Bergman and F. Mignosa, *J. High Energy Phys.* **04** (2025) 047.
 - [23] F. Bonetti, M. Del Zotto, and R. Minasian, *J. High Energy Phys.* **02** (2025) 156.
 - [24] F. B. Christensen, *arXiv:2412.19887*.
 - [25] A. Caldararu, T. Pantev, E. Sharpe, B. Sung, and X. Yu, *J. Geom. Phys.* **218**, 105653 (2025).
 - [26] E. Witten, *J. High Energy Phys.* **12** (1998) 012.
 - [27] I. Bah, F. Bonetti, R. Minasian, and E. Nardoni, *J. High Energy Phys.* **06** (2019) 123.
 - [28] I. Bah, F. Bonetti, R. Minasian, and E. Nardoni, *J. High Energy Phys.* **01** (2020) 125.
 - [29] W. Ji and X.-G. Wen, *Phys. Rev. Res.* **2**, 033417 (2020).
 - [30] I. Bah, F. Bonetti, R. Minasian, and P. Weck, *J. High Energy Phys.* **02** (2021) 116.
 - [31] O. Bergman, Y. Tachikawa, and G. Zafrir, *J. High Energy Phys.* **07** (2020) 077.
 - [32] I. Bah, F. Bonetti, and R. Minasian, *J. High Energy Phys.* **03** (2021) 196.
 - [33] D. Gaiotto and J. Kulp, *J. High Energy Phys.* **02** (2021) 132.
 - [34] I. Bah, F. Bonetti, E. Leung, and P. Weck, *J. High Energy Phys.* **10** (2022) 122.

- [35] F. Apruzzi, F. Bonetti, I. García Etxebarria, S. S. Hosseini, and S. Schafer-Nameki, *Commun. Math. Phys.* **402**, 895 (2023).
- [36] O. Bergman and S. Hirano, *J. High Energy Phys.* **11** (2022) 069.
- [37] D. S. Freed, G. W. Moore, and C. Teleman, *arXiv:2209.07471*.
- [38] L. Bhardwaj and Y. Tachikawa, *J. High Energy Phys.* **03** (2018) 189.
- [39] R. Thorngren and Y. Wang, *J. High Energy Phys.* **04** (2024) 132.
- [40] R. Thorngren and Y. Wang, *J. High Energy Phys.* **07** (2024) 051.
- [41] L. Bhardwaj, L. E. Bottini, S. Schafer-Nameki, and A. Tiwari, *SciPost Phys.* **14**, 007 (2023).
- [42] L. Müller, *arXiv:2505.04761*.
- [43] L. Bhardwaj, T. Décoppet, S. Schafer-Nameki, and M. Yu, *Commun. Math. Phys.* **406**, 208 (2025).
- [44] I. Bah, E. Leung, and T. Waddleton, *arXiv:2506.04346*.
- [45] S. Kim, O. Sela, and Z. Sun, *J. High Energy Phys.* **05** (2026) 120.
- [46] E. García-Valdecasas, *J. High Energy Phys.* **04** (2023) 102.
- [47] M. Cvetič, J. J. Heckman, M. Hübner, and E. Torres, *Phys. Rev. D* **109**, 046007 (2024).
- [48] O. Bergman, E. García-Valdecasas, F. Mignosa, and D. Rodriguez-Gomez, *J. High Energy Phys.* **02** (2025) 066.
- [49] T. Waddleton, *J. High Energy Phys.* **11** (2025) 104.
- [50] M. Cvetič, J. J. Heckman, M. Hübner, and C. Murdia, *Phys. Rev. D* **112**, 106020 (2025).
- [51] H. Calvo, F. Mignosa, and D. Rodriguez-Gomez, *J. High Energy Phys.* **10** (2025) 107.
- [52] M. Najjar, *arXiv:2503.17108*.
- [53] H. Calvo, F. Mignosa, and D. Rodriguez-Gomez, *J. High Energy Phys.* **06** (2025) 196.
- [54] C. Córdova, K. Ohmori, and T. Rudelius, *J. High Energy Phys.* **11** (2022) 154.
- [55] T. D. Brennan and Z. Sun, *J. High Energy Phys.* **12** (2024) 100.
- [56] A. Antinucci and F. Benini, *Phys. Rev. B* **111**, 024110 (2025).
- [57] F. Bonetti, M. Del Zotto, and R. Minasian, *Phys. Lett. B* **871**, 140010 (2025).
- [58] A. Antinucci, F. Benini, and G. Rizi, *Fortschr. Phys.* **72**, 2400172 (2024).
- [59] A. Arbalestrier, R. Argurio, and L. Tizzano, *SciPost Phys.* **19**, 032 (2025).
- [60] F. Bonetti, M. Del Zotto, and R. Minasian, *J. High Energy Phys.* **05** (2026) 048.
- [61] Q. Jia, R. Luo, J. Tian, Y.-N. Wang, and Y. Zhang, *arXiv:2503.04546*.
- [62] M. Cvetič, H. Lu, and C. N. Pope, *Phys. Rev. D* **62**, 064028 (2000).
- [63] K. Pilch and N. P. Warner, *Nucl. Phys.* **B594**, 209 (2001).
- [64] D. Cassani, G. Dall'Agata, and A. F. Faedo, *J. High Energy Phys.* **05** (2010) 094.
- [65] J. T. Liu, P. Szepietowski, and Z. Zhao, *Phys. Rev. D* **81**, 124028 (2010).
- [66] J. P. Gauntlett and O. Varela, *J. High Energy Phys.* **06** (2010) 081.
- [67] A. Baguet, O. Hohm, and H. Samtleben, *Phys. Rev. D* **92**, 065004 (2015).
- [68] M. Gunaydin, L. J. Romans, and N. P. Warner, *Phys. Lett.* **154B**, 268 (1985).
- [69] M. Pernici, K. Pilch, and P. van Nieuwenhuizen, *Nucl. Phys.* **B259**, 460 (1985).
- [70] M. Gunaydin, L. J. Romans, and N. P. Warner, *Nucl. Phys.* **B272**, 598 (1986).
- [71] I. Bah, P. Jefferson, K. Roumpedakis, and T. Waddleton, *SciPost Phys.* **19**, 116 (2025).
- [72] I. Bah, F. Bonetti, M. Chitoto, and E. Leung, *arXiv:2602.22377*.
- [73] M. Cvetič, H. Lu, C. N. Pope, A. Sadzadeh, and T. A. Tran, *Nucl. Phys.* **B586**, 275 (2000).
- [74] D. Z. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli, *Nucl. Phys.* **B546**, 96 (1999).
- [75] E. Witten, *Adv. Theor. Math. Phys.* **2**, 253 (1998).
- [76] E. Barnes, E. Gorbatov, K. A. Intriligator, and J. Wright, *Nucl. Phys.* **B732**, 89 (2006).
- [77] D. Belov and G. W. Moore, *arXiv:hep-th/0412167*.
- [78] A purely classical analysis of Gauss's law constraints suffices for our purposes. The quantum treatment, in the presence of topological terms in the action, requires care [77].
- [79] A. P. Balachandran, S. Borchardt, and A. Stern, *Phys. Rev. D* **17**, 3247 (1978).
- [80] A. Alekseev, L. D. Faddeev, and S. L. Shatashvili, *J. Geom. Phys.* **5**, 391 (1988).
- [81] D. Diakonov and V. Y. Petrov, *Phys. Lett. B* **224**, 131 (1989).
- [82] M. Stone, *Nucl. Phys.* **B314**, 557 (1989).
- [83] O. Alvarez, I. M. Singer, and P. Windey, *Nucl. Phys.* **B337**, 467 (1990).
- [84] P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, and E. Witten, *Quantum Fields and Strings: A Course for Mathematicians: Volume 2* (American Mathematical Society, Providence, 1999), Vol. 2.
- [85] C. Beasley, *Adv. Theor. Math. Phys.* **17**, 1 (2013).
- [86] D. Tong and K. Wong, *J. High Energy Phys.* **06** (2014) 048.
- [87] A. A. Kirillov, *Lectures on the Orbit Method* (American Mathematical Soc., Providence, 2004), Vol. 64.
- [88] Indeed, recall that: (i) the operator $U(\theta; M^3)$ is constructed from Gauss's law constraints; (ii) the latter implement bulk gauge transformations; (iii) the global $SO(6)$ action originates as a special instance of a gauge transformation, whose parameter is nontrivial at the boundary.
- [89] S.-J. Rey and J.-T. Yee, *Eur. Phys. J. C* **22**, 379 (2001).
- [90] J. McGreevy, L. Susskind, and N. Toumbas, *J. High Energy Phys.* **06** (2000) 008.
- [91] M. T. Grisaru, R. C. Myers, and O. Tafjord, *J. High Energy Phys.* **08** (2000) 040.
- [92] N. Drukker and B. Fiol, *J. High Energy Phys.* **02** (2005) 010.
- [93] S. Benvenuti, L. A. Pando Zayas, and Y. Tachikawa, *Adv. Theor. Math. Phys.* **10**, 395 (2006).
- [94] J. Gomis and F. Passerini, *J. High Energy Phys.* **08** (2006) 074.
- [95] J. Gomis and F. Passerini, *J. High Energy Phys.* **01** (2007) 097.

- [96] E. Eyras, B. Janssen, and Y. Lozano, *Nucl. Phys.* **B531**, 275 (1998).
- [97] A. Hanany and E. Witten, *Nucl. Phys.* **B492**, 152 (1997).
- [98] As far as group representations are concerned, dominant integral elements are in 1-to-1 correspondence with isomorphism classes of unitary finite-dimensional irreps of the simply connected $SU(4)$. For $SO(6)$, an analogous statement holds, but the notion of integral element has to be replaced by that of analytically integral element, see e.g. [101]. While all analytically integral elements are integral, the converse is not true. Integral elements that are not analytically integral correspond to irreps of $SU(4)$ which are not irreps of $SO(6)$.
- [99] This can be rephrased as follows. The 2-form $N \frac{dA}{2\pi}$ is cohomologically nontrivial on the base $\mathbb{CP}^2$. If we pull it back to the total space of the Hopf fibration, however, it becomes cohomologically trivial, since we can write $N \frac{dA}{2\pi} = \frac{N}{2\pi} dk_\psi$, and k_ψ is a globally defined 1-form in the total space.
- [100] D. K. Hoffman, R. C. Raffenetti, and K. Ruedenberg, *J. Math. Phys. (N.Y.)* **13**, 528 (1972).
- [101] B. C. Hall, *An Elementary Introduction to Groups and Representations* (Springer, Cham, 2003), Vol. 222.