Dirac-eikonal scattering amplitude

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The fixed energy Dirac equation using scalar plus fourth component vector potentials is solved in the eikonal limit. The Dirac-eikonal amplitude can be written in the familiar integral over impact parameter form. Results of model calculations using the Dirac-eikonal amplitude are compared with those of a partial wave solution for parameters appropriate to intermediate energy proton nucleus scattering.

[NUCLEAR REACTIONS Intermediate energy. Dirac-eikonal amplitude, Dirac impulse approximation.]

I. INTRODUCTION

Recently the Dirac equation approach has been shown to provide a greatly improved starting point for understanding intermediate energy proton-nucleus scattering. There have been remarkable advances both in phenomenology and in attempts to relate input parameters to nucleon-nucleon physics.[1-5] Thus far the calculations have exclusively used a partial wave analysis of the Dirac equation. Although this is certainly adequate, it makes extraction of the underlying physics in these diffraction processes difficult. A natural starting point for such studies is the eikonal representation of the scattering amplitude. In this paper we present the eikonal form of the scattering amplitude for a spin $\frac{1}{2}$ particle governed by a Dirac equation with both scalar and fourth component of vector potentials.

Rosenfelder[6] had previously treated the eikonal limit of Coulomb (vector potential only) scattering by electrons, where the electron's mass is ignored. Friar and Wallace are also investigating eikonal approximation to the Dirac equation using a method different than ours.[7]

The convenience and insight gained by having an integral representation of the scattering amplitude needs no argument. In particular, insight into the various geometries necessary to describe spin observables is expected. The Dirac-eikonal amplitude also admits of analytic evaluation which can lead to data-to-data relations between observables.[8-10] Furthermore, extension to nuclear reactions using Dirac-eikonal distorted waves is straightforward given the Dirac-eikonal scattering solution. It is also expected that at these energies the eikonal form will be an excellent approximation. We verify this in a numerical example comparing the eikonal and partial wave Dirac results for the cross section and spin observables in a case with parameters appropriate to intermediate energy proton nucleus scattering.

In Sec. II the elastic scattering Dirac-eikonal amplitude is derived. The solution is just what one would expect from the standard eikonal solution of the (upper component only) "equivalent" Schrödinger equation; however, the Darwin term does not contribute to the elastic amplitude in the eikonal limit. In Sec. III the Dirac-eikonal amplitude is compared to Dirac partial wave results. Our conclusions follow in the last section.

II. THE DIRAC-EIKONAL ELASTIC AMPLITUDE

We begin by considering the time independent Dirac equation

$$H\psi^{(+)}_{\vec{k},s}(\vec{r})=[\vec{\alpha}\cdot\vec{p}+\beta m+\beta V_s(r)+V_v(r)]\psi^{(+)}_{\vec{k},s}(\vec{r})$$
$$=E\psi^{(+)}_{\vec{k},s}(\vec{r}) \qquad (1)$$

for a projectile with relativistic energy $E=\sqrt{k^2+m^2}$ in the presence of local Lorentz scalar and vector potentials $V_s(r)$ and $V_v(r)$.

The scattered four component spinor can be decomposed into Pauli spinors $u^{(+)}_{k,s}(\vec{r})$ and $w^{(+)}_{k,s}(\vec{r})$, usually known as the upper and lower components, respectively.

$$\psi^{(+)}_{\vec{k},s}(\vec{r})=\begin{bmatrix} u^{(+)}_{\vec{k},s}(\vec{r}) \\ w^{(+)}_{\vec{k},s}(\vec{r}) \end{bmatrix} \qquad (2)$$

which satisfy the following set of coupled differential equations:

$$\vec{\sigma}\cdot\vec{p}u^{(+)}_{\vec{k},s}-(E+m+V_s-V_v)w^{(+)}_{\vec{k},s}=0\ , \qquad (3a)$$

$$\vec{\sigma}\cdot\vec{p}w^{(+)}_{\vec{k},s}-(E-m-V_s-V_v)u^{(+)}_{\vec{k},s}=0\ . \qquad (3b)$$

Substituting for the lower component, we obtain

$$\left[\vec{\sigma}\cdot\vec{p}\frac{1}{E+m+V_s-V_v}\vec{\sigma}\cdot\vec{p}-(E-m-V_s-V_v)\right]u^{(+)}_{\vec{k},s}=0\ , \qquad (4a)$$

$$w^{(+)}_{\vec{k},s}=\frac{1}{E+m+V_s-V_v}\vec{\sigma}\cdot\vec{p}u^{(+)}_{\vec{k},s}\ . \qquad (4b)$$

Equation (4a) can be turned, after some straightfoward manipulation, into a Schrödinger-type form:

$$\left[\frac{p^2}{2m}+V_c+V_{SO}(\vec{\sigma}\cdot\vec{L}-i\vec{r}\cdot\vec{p})\right]u^{(+)}_{\vec{k},s}=\frac{k^2}{2m}u^{(+)}_{\vec{k},s}\ , \tag{5}$$

where the central and spin orbit potentials V_c and V_{SO} have been defined as follows:

$$V_c(r)=V_s(r)+\frac{E}{m}V_v(r)+\frac{V_s(r)^2-V_v(r)^2}{2m}\ , \tag{6a}$$

$$V_{SO}(r)=\frac{1}{2m[E+m+V_s(r)-V_v(r)]}\times\frac{1}{r}\frac{d}{dr}[V_v(r)-V_s(r)]\ . \tag{6b}$$

Note that from the above definitions three basic dynamical geometries, precisely those needed to properly account for the cross section and spin observable data, arise naturally from a Dirac formalism.[5] Thus far no approximations have been made and from here on we proceed in the standard way to solve the Scrödinger-type equation in the eikonal limit. We define the average momentum $\vec{K}$ and momentum transfer $\vec{q}$ in terms of the initial and final momentum $\vec{k}$ and $\vec{k}\,'$, respectively,

$$\vec{K}=\tfrac{1}{2}(\vec{k}+\vec{k}\,')\ , \tag{7a}$$

$$\vec{q}=\vec{k}-\vec{k}\,'\ . \tag{7b}$$

Within the context of a small angle approximation $(q/k\ll 1)$, following Glauber[11] we can use

$$p^2=[(\vec{p}-\vec{K})+\vec{K}]^2\sim 2\vec{K}\cdot\vec{p}-K^2 \tag{8}$$

in Eq. (5), resulting in the following linear equation in $\vec{p}$:

$$(\vec{K}\cdot\vec{p}-K^2+m\{V_c+V_{SO}[\vec{\sigma}\cdot(\vec{r}\times\vec{K})-i\vec{r}\cdot\vec{K}]\})u^{(+)}_{\vec{k},s}=0\ , \tag{9}$$

where $\vec{K}$ has been substituted into the spin orbit and Darwin terms. In the eikonal approximation the upper component of the scattering wave function,

$$u^{(+)}_{\vec{k},s}=e^{i\vec{k}\cdot\vec{r}}e^{iS(\vec{r})}\chi_s\ , \tag{10}$$

is written in terms of the initial Pauli spinor χ_s and the eikonal phase $S(\vec{r})$ which satisfies the differential equation

$$\vec{K}\cdot\vec{\nabla}S(\vec{r})=-m\{V_c(r)+V_{SO}(r)[\vec{\sigma}\cdot(\vec{r}\times\vec{K})-i\vec{r}\cdot\vec{K}]\}\ . \tag{11}$$

If we define the z axis along the direction of the average momentum $\vec{K}$, the eikonal phase, which is an operator in spin space, can be written in the usual integral form $[\vec{r}\equiv(\vec{b},z)]$:

$$iS(\vec{r})=-i\frac{m}{K}\int_{-\infty}^{z}dz'\{V_c(\vec{b},z')+V_{SO}(\vec{b},z')[\vec{\sigma}\cdot(\vec{b}\times\vec{K})-iKz']\}\ . \tag{12}$$

We can now proceed to evaluate the scattering amplitude,

$$F_{ss'}(\vec{k},\vec{k}\,',E)=-\frac{m}{2\pi}\langle\Phi_{\vec{k}\,',s}\,|\,(\beta V_s+V_v)\,|\,\psi^{(+)}_{\vec{k},s}\rangle \tag{13}$$

in terms of the scattered wave

$$\psi^{(+)}_{\vec{k},s}(\vec{r})=\left[\frac{E+m}{2m}\right]^{1/2}\begin{bmatrix}1\\ \dfrac{1}{E+m+V_s-V_v}\,\vec{\sigma}\cdot\vec{p}\end{bmatrix}e^{i\vec{k}\cdot\vec{r}}e^{iS(\vec{r})}\chi_s \tag{14}$$

and the free Dirac solution

$$\Phi_{\vec{k},s}=u(\vec{k}\,',s')e^{i\vec{k}\,'\cdot\vec{r}}\ , \tag{15}$$

where

$$u(k',s')=\left[\frac{E+m}{2m}\right]^{1/2}\begin{bmatrix}1\\ \dfrac{\vec{\sigma}\cdot\vec{k}\,'}{E+m}\end{bmatrix}\chi_{s'}\ . \tag{16}$$

Note that the normalization of the distorted wave is identical to that of the free Dirac solution. We obtain

$$F_{ss'}(\vec{k},\vec{k}\,',E)=\langle\chi_{s'}\,|\,F(\vec{k},\vec{k}\,',E)\,|\,\chi_s\rangle \tag{17a}$$

with

$$F(\vec{k},\vec{k}\,',E)=-\frac{m}{2\pi}\frac{E+m}{2m}\int d^3r\,e^{-i\vec{k}\,'\cdot\vec{r}}\left[1,\frac{\vec{\sigma}\cdot\vec{k}\,'}{E+m}\right]\begin{bmatrix}V_v+V_s & 0\\ 0 & V_v-V_s\end{bmatrix}\begin{bmatrix}1\\ \dfrac{1}{E+m+V_s-V_v}\,\vec{\sigma}\cdot\vec{p}\end{bmatrix}e^{i\vec{k}\cdot\vec{r}+iS(\vec{r})}\ . \tag{17b}$$

After performing the Dirac algebra, the scattering amplitude operator $F(\vec{k},\vec{k}\,',E)$ takes the following form:

$$F(\vec{k},\vec{k}',E)=-\frac{E+m}{2}\int\frac{d^3r}{2\pi}e^{-i\vec{k}'\cdot\vec{r}}\left[V_s+V_v-i\frac{(V_v-V_s)}{(E+m)(E+m+V_s-V_v)}\vec{\sigma}\cdot\vec{k}'\vec{\sigma}\cdot\vec{\nabla}\right]e^{i\vec{k}\cdot\vec{r}+iS(\vec{r})}\ . \tag{18}$$

If we now let the gradient in the second term act to the left, i.e., integrate by parts, and use the fact that the surface term vanishes together with Eqs. (6a) and (6b) we obtain:

$$F(\vec{k},\vec{k}',E)=-2iK\frac{(E+m)}{2}\int\frac{d^2b}{2\pi}e^{i\vec{q}\cdot\vec{b}}\int_{-\infty}^{\infty}dz\frac{1}{(E+m+V_s-V_v)}\left[-\frac{im}{K}\right]\{V_c+V_{SO}[\vec{\sigma}\cdot(\vec{b}\times\vec{K})+iKz]\}e^{iS(\vec{r})}\ . \tag{19}$$

We note that in expression (19) there are two factors which apparently hamper our reduction of (19) to the usual form. One of them is related to the sign difference of the Darwin term in the expression in curly braces with respect to the corresponding one in the eikonal phase, Eq. (12). The second one arises by noting that besides the expression in curly braces there are more z dependent terms which will not get trivially integrated even in the absence of the sign difference. As we shall see, these two factors cancel in the eikonal limit. We rewrite Eq. (19) as

$$F(\vec{k},\vec{k}',E)=-iK(E+m)\int\frac{d^2b}{2\pi}e^{i\vec{q}\cdot\vec{b}}\int_{-\infty}^{\infty}dz\frac{1}{E+m+V_s-V_v}\left[\frac{d}{dz}e^{iS(\vec{r})}+2mzV_{SO}(r)e^{iS(\vec{r})}\right]\ , \tag{20}$$

where a correction in the second term in square brackets has been included in order to account for the sign difference in the Darwin term. As was mentioned above, we can now turn the z integrand into an exact differential by noting that

$$\frac{d}{dz}\left[\frac{e^{iS(\vec{r})}}{E+m+V_s-V_v}\right]=\frac{1}{E+m+V_s-V_v}\left[\frac{d}{dz}e^{iS(\vec{r})}+2mzV_{SO}(\vec{r})e^{iS(\vec{r})}\right]\ , \tag{21}$$

where Eq. (6b), together with the relation

$$\frac{1}{z}\frac{d}{dz}=\frac{1}{r}\frac{d}{dr}\ ,$$

has been used.

The scattering amplitude operator then takes the simple form

$$F(\vec{k},\vec{k}',E)=-iK\int\frac{d^2b}{2\pi}e^{i\vec{q}\cdot\vec{b}}(e^{-\chi(b)}-1)\ , \tag{22}$$

where the profile function $\chi(b)$ is given by

$$-\chi(b)=-i\frac{m}{K}\int_{-\infty}^{\infty}dz[V_c(b,z)+V_{SO}(b,z)\vec{\sigma}\cdot(\vec{b}\times\vec{K})]\ , \tag{23}$$

and we note that the Darwin term has vanished since it is an odd function of z. If we now expressed the profile function in terms of the central and spin-orbit thickness functions, $t_c(b)$ and $t_s(b)$, respectively, through the relation

$$\chi(b)=\gamma t_c(b)+w\gamma t_s(b)\vec{\sigma}\cdot(\vec{b}\times\vec{K})\ , \tag{24}$$

where γ is the central strength and w is the spin orbit to central strength ratio, we can perform the angular integration in (22) and finally write the scattering amplitude as an integral over impact parameter, i.e.,

$$F(\vec{k},\vec{k}',E)=F_1+\vec{\sigma}\cdot\vec{n}F_2\ , \tag{25}$$

where

$$F_1=-iK\int_0^{\infty}db\,bJ_0(qb)[e^{-\gamma t_c(b)}\cosh w\gamma t_s(b)-1]\ , \tag{26a}$$

$$F_2=-K\int_0^{\infty}db\,bJ_1(qb)e^{-\gamma t_c(b)}\sinh w\gamma t_s(b)\ , \tag{26b}$$

and

$$\hat{n}=\hat{q}\times\hat{K}$$

is a unit vector perpendicular to the scattering plane.

The scattered wave solution, Eq. (14), provides a convenient representation for distorted waves to be used in distorted wave approaches to inelastic scattering based on the Dirac equation.

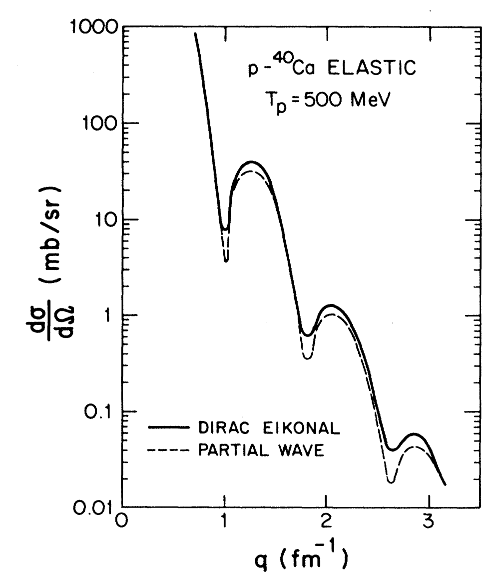

FIG. 1. The 500 MeV p-^{40}Ca elastic scattering cross section calculated using the Dirac-eikonal amplitude [Eq. (1)] (solid line) is compared to that using a Dirac partial wave code (dashed line). The optical potential parameters are given in the text.

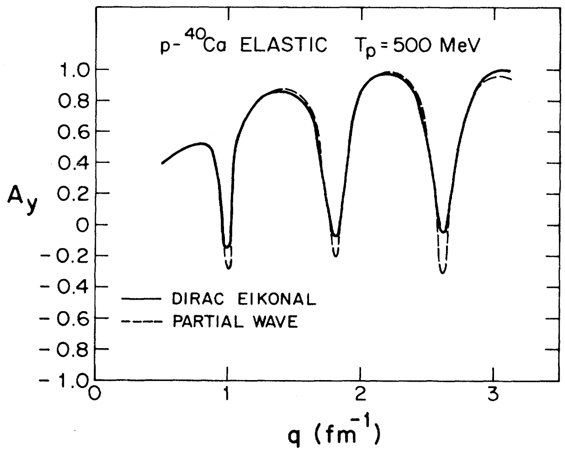

FIG. 2. Same as Fig. 1 except for analyzing power.

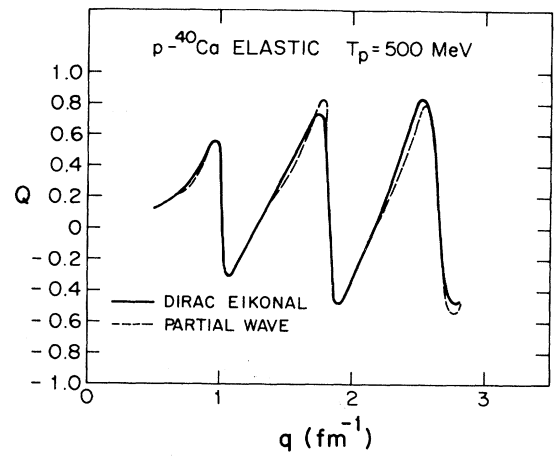

FIG. 3. Same as Fig. 1 except for spin rotation function.

III. COMPARISON WITH PARTIAL-WAVE SOLUTION

In Figs. 1—3 we compare the results obtained using our Dirac-eikonal amplitude with those from a full partial wave solution. The example we choose is p-^{40}Ca at 500 MeV where both spin observables A_y and Q have been measured and where the Dirac approach is known to give excellent agreement.[3] For simplicity we choose a semi-realistic model calculation where all geometric parameters are set equal and the short range approximation is used. The Lorentz scalar and vector potentials are given by the Woods-Saxon form:

$$V_s(r)=V_s^0[1+e^{(r-c)/\beta}]^{-1}, \quad (27a)$$

$$V_v(r)=V_v^0[1+e^{(r-c)/\beta}]^{-1}, \quad (27b)$$

where for ^{40}Ca we take $c=3.55$ fm and $\beta=0.64$ fm. Following the approach of Ref. 12, the strengths are taken from the impulse approximation values based on the Sp83 Arndt phase shift solution giving $V_s^0=(-303+i73)$ MeV and $V_v^0=(191-i86)$ MeV. From Fig. 1 we see that the Dirac-eikonal cross section successfully reproduces the partial wave result, except in the minima where the eikonal solution has too much filling. This is in contrast to the usual eikonal results which generally have too little filling in the minima. The spin observables A_y and Q shown in Figs. 2 and 3 show excellent agreement between the eikonal and partial wave results, again parting only where the cross section shows minima.

IV. CONCLUSIONS

We have derived a Dirac-eikonal elastic scattering amplitude which successfully reproduces the observables calculated from a partial wave solution. The amplitude is just what one would obtain by applying the standard nonrelativistic eikonal procedure to the two component equivalent Schrödinger equation. Note that three dynamical geometries enter in the equivalent Schrödinger potential and these give rise to the dramatic structure seen in the spin observables.[5] The Darwin term does not contribute to the elastic amplitude in the eikonal limit, but would contribute to inelastic processes where the (eikonal) distortions seen by the incoming and outgoing projectile are significantly different (see Ref. 13). It should be noted that each eikonal approximation invoked in the derivation of the Dirac-eikonal amplitude generated a noneikonal correction which can be included in a perturbative fashion.

One of the principal advantages of having an integral representation of the scattering amplitude is the geometric insights it allows. Work is currently underway to evaluate the Dirac eikonal analytically where the role the three shapes play in determining the structure in the spin observables becomes manifest, permitting the insight necessary to relate the three observables ($d\sigma/d\Omega$, P, and Q) to each other directly, i.e., data to data.[6–8]

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