Ratio of electron capture to positron emission in the elementary-particle treatment of nuclear β decay*

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Nuclear size effects and Coulomb corrections are included in the expression for the electron capture rate R_ϵ by using the "elementary-particle" approach of Armstrong and Kim. As an example, we use our result together with the corresponding expression for the positron emission rate R_{β^+} to compute R_ϵ/R_{β^+} for certain $\frac{1}{2}^+ \to \frac{1}{2}^+$ and $\frac{1}{2}^+ \to \frac{3}{2}^+$ transitions in Gd. Our ratios are in good agreement with conventional impulse-approximation calculations and for a wide range of nuclear parameters, do not exhibit the anomalies seen experimentally.

[RADIOACTIVITY Finite size and Coulomb corrections to electron capture and β-decay calculations.]

I. INTRODUCTION

Since both electron capture and positron emission are governed by the Hamiltonian for the weak interaction, the rates R_ϵ and R_{β^+} for these processes in a given parent nucleus are related. In the simplest approximation, which neglects finite nuclear size effects and Coulomb interactions between the outgoing positron and the nucleus, the ratio R_ϵ/R_{β^+} is independent of the nuclear matrix element.[1] Inclusion of size effects and Coulomb corrections modifies R_ϵ/R_{β^+} in such a way that the nuclear matrix element no longer completely cancels and it is therefore necessary to use a model to determine the residual nuclear physics effects. For the decay process, Armstrong and Kim[2] have shown that the "elementary-particle" treatment of β decay[3] provides a particularly convenient framework for including these size effects and Coulomb corrections. In the present paper, we adapt the approach of Ref. 2 to the case of electron capture by a nucleus of arbitrary spin and compute the ratio R_ϵ/R_{β^+} for the cases of $\frac{1}{2}^+ \to \frac{1}{2}^+$ and $\frac{1}{2}^+ \to \frac{3}{2}^+$ nuclear transitions. Transitions of this type are of interest in analyzing certain anomalous electron capture to positron emission ratios recently reported for gadolinium (Gd).[4]

In Sec. II, we derive the expression for R_ϵ in the elementary-particle approach. Explicit results for R_ϵ in $\frac{1}{2}^+ \to \frac{1}{2}^+$ and $\frac{1}{2}^+ \to \frac{3}{2}^+$ transitions together with the corresponding expressions for R_{β^+} are given in Sec. III and in Sec. IV we display the ratio R_ϵ/R_{β^+} for a range of the nuclear parameters encountered.

II. FORMULATION OF THE CAPTURE PROBLEM

From the elementary particle point of view, the matrix element $\mathfrak{M}_\epsilon$ for electron capture can be written

$$\mathfrak{M}_\epsilon = \frac{G}{\sqrt{2}}(2\pi)^3\delta^{(3)}(\vec{p}_f + \vec{p}_\nu - \vec{p}_i - \vec{p}_e) \times \langle f(\vec{p}_f), \nu(\vec{p}_\nu)\,|J_\alpha^{(-)}(0) l_\alpha^{(+)}(0)|\,i, e^-; \vec{P}\rangle, \qquad (1)$$

where G is the Fermi constant $10^{-5}/m_p^2$, $J_\alpha^{(-)} = V_\alpha^{(-)} + A_\alpha^{(-)}$ is the hadron weak charge lowering current, and $l_\alpha^{(+)}(0) = \bar{\psi}_\nu\gamma_\alpha(1+\gamma_5)\psi_e$ is the lepton current. The state $|i, e^-; \vec{P}\rangle$ is a bound state of the initial nucleus and electron with total momentum $\vec{P} = \vec{p}_i + \vec{p}_e$. In order to express the matrix element in a convenient form, we expand the initial state in terms of plane wave states as[5]

$$|i, e^-; \vec{P}\rangle = \sum_{\vec{k}_i, \vec{k}_e, s} |i(\vec{k}_i); e^-(\vec{k}_e), s\rangle\langle i(\vec{k}_i); e^-(\vec{k}_e), s\,|\,i, e^-; \vec{P}\rangle. \qquad (2)$$

The scalar product in Eq. (2) can, in turn, be written

$$\langle i(\vec{k}_i); e^-(\vec{k}_e), s\,|\,i, e^-; \vec{P}\rangle = \int d^3r_i d^3r_e \langle i(\vec{k}_i); e^-(\vec{k}_e), s\,|\,\vec{r}_i, \vec{r}_e\rangle\langle \vec{r}_i, \vec{r}_e\,|\,i, e^-; \vec{P}\rangle, \qquad (3)$$

where, for a nucleus of spin $\frac{1}{2}$,

$$\langle i(\vec{k}_i), s'; e^-(\vec{k}_e), s\,|\,\vec{r}_i, \vec{r}_e\rangle = \left(\frac{m_i}{E_i V}\right)^{1/2}\left(\frac{m_e}{E_e V}\right)^{1/2} u_{s'}^\dagger(\vec{k}_i) u_s^\dagger(\vec{k}_e) e^{-i\vec{k}_i\cdot\vec{r}_i} e^{-i\vec{k}_e\cdot\vec{r}_e}, \qquad (4)$$

$$\langle \vec{r}_i, \vec{r}_e | i, e^-; \vec{P}\rangle = \frac{1}{\sqrt{V}} u_i \psi(\vec{r}) e^{i\vec{P}\cdot\vec{R}}, \tag{5}$$

with

$$\vec{R} = \frac{m_i \vec{r}_i + m_e \vec{r}_e}{(m_i + m_e)}, \quad \vec{r} = \vec{r}_e - \vec{r}_i. \tag{6}$$

In Eq. (5) u_i denotes the spinor associated with the initial nucleus, $\psi(\vec{r})$ is the electron wave function, and V is a normalization volume. Using Eqs. (2) and (3), the expression for $\mathfrak{M}_\epsilon$ becomes

$$\mathfrak{M}_\epsilon = \frac{G}{\sqrt{2}} \sqrt{V} \delta^{(3)}(\vec{p}_f + \vec{p}_\nu - \vec{p}_i - \vec{p}_e) \int d^3k_e \sum_s \langle f(\vec{p}_f), \nu(\vec{p}_\nu) | J_\alpha^{(-)}(0) l_\alpha^{(+)}(0) | i(p_i + p_e - k_e); e^-(\vec{k}_e), s\rangle$$
$$\times \left(\frac{m_e}{E_e(\vec{k}_e)}\right)^{1/2} \left(\frac{E_i(\vec{p}_i + \vec{p}_e - \vec{k}_e)}{m_i}\right)^{1/2} u_s^\dagger(\vec{k}_e) \int d^3r\, e^{-i\vec{k}_e\cdot\vec{r}} u_i \psi(\vec{r}). \tag{7}$$

Since the matrix element in Eq. (7) involves only noninteracting states, it can be separated into lepton and hadron parts and the sum over s can be performed[6] to give

$$\mathfrak{M}_\epsilon = \frac{G}{\sqrt{2}} \delta^{(3)}(\vec{p}_f + \vec{p}_\nu - \vec{p}_i - \vec{p}_e) \int d^3k_e \langle f(\vec{p}_f) | J_\alpha^{(-)}(0) | i(\vec{p}_i + \vec{p}_e - \vec{k}_e)\rangle \left(\frac{m_\nu}{E_\nu V}\right)^{1/2} \bar{u}(p_\nu)\gamma_\alpha(1+\gamma_5) \int d^3r\, e^{-i\vec{k}_e\cdot\vec{r}} \psi(\vec{r}). \tag{8}$$

The remaining hadron matrix element will contain nuclear transition form factors $F_i(\vec{q}^2)$ which, as in Ref. 2, we assume to have the form

$$F_i(\vec{q}^2) = F_i(0)\mathfrak{F}(\vec{q}^2), \tag{9}$$

where $\mathfrak{F}(\vec{q}^2)$ is a common form factor with the normalization $\mathfrak{F}(0) = 1$. By expressing $\mathfrak{F}(\vec{q}^2)$ in terms of its Fourier transform

$$\mathfrak{F}(\vec{q}^2) = \int d^3x\, e^{i\vec{q}\cdot\vec{x}} \phi(\vec{x}), \tag{10}$$

Eq. (8) can be written

$$\mathfrak{M}_\epsilon = \frac{G}{\sqrt{2}} (2\pi)^3 \delta(\vec{p}_f + \vec{p}_\nu - \vec{p}_i - \vec{p}_e)$$
$$\times \langle f(\vec{p}_f) | J_\alpha^{(-)}(0) | i(\vec{p}_i)\rangle_{\vec{q}^2=0}$$
$$\times \left(\frac{m_\nu}{E_\nu V}\right)^{1/2} \bar{u}(p_\nu)\gamma_\alpha(1+\gamma_5)\langle\psi\rangle, \tag{11}$$

where

$$\langle\psi\rangle = \int d^3r\, \phi(\vec{r})\psi(\vec{r}) e^{-i\vec{p}_\nu\cdot\vec{r}}. \tag{12}$$

To calculate $\langle\psi\rangle$, we recall that for a κ electron $\psi(\vec{r})$ is[7]

$$\psi(\vec{r}) = N e^{-Zr/a_0} \left(\frac{Zr}{a_0}\right)^{\beta-1} \begin{vmatrix} \chi^s \\ \dfrac{i(1-\beta)}{Z\alpha} \vec{\sigma}\cdot\hat{r}\chi^s \end{vmatrix}, \tag{13}$$

with $\beta = [1 - (Z\alpha)^2]^{1/2}$ and

$$N = 2^{\beta-1} \left(\frac{Z}{a_0}\right)^{3/2} \left[\frac{(1+\beta)}{\pi\Gamma(1+2\beta)}\right]^{1/2}. \tag{14}$$

For convenience in subsequent calculations, we express $\psi(\vec{r})$ in terms of the free-particle electron spinors as

$$\psi(\vec{r}) = \frac{1}{2}\left(\frac{2m_e}{E_e + m_e}\right)^{1/2} N \left(\frac{Zr}{a_0}\right)^{\beta-1} e^{-Zr/a_0}$$
$$\times \left[1 + \frac{(1-\beta)}{Z\alpha} \vec{\gamma}\cdot\hat{r}\right](1+\gamma_4)u(p). \tag{15}$$

The lepton part of the matrix element Eq. (11) can now be written as

$$l'_\alpha = \bar{u}(p_\nu)\gamma_\alpha(1+\gamma_5)\langle\psi\rangle$$
$$= \bar{u}(p_\nu)\gamma_\alpha(1+\gamma_5)[A' + \vec{\gamma}\cdot\hat{p}_\nu B'](1+\gamma_4)u(p_e), \tag{16}$$

where

$$A'(\vec{p}_\nu) = \frac{1}{2}\left(\frac{2m_e}{E_e + m_e}\right)^{1/2} N \int d^3r\, \phi(\vec{r}) \left(\frac{Zr}{a_0}\right)^{\beta-1}$$
$$\times e^{-Zr/a_0} e^{-i\vec{p}_\nu\cdot\vec{r}}, \tag{17}$$

$$B'(\vec{p}_\nu) = \frac{1}{2}\left(\frac{2m_e}{E_e + m_e}\right)^{1/2} N \frac{(1-\beta)}{Z\alpha}$$
$$\times \hat{p}_\nu \cdot \int d^3r\, \phi(\vec{r})\hat{r}\left(\frac{Zr}{a_0}\right)^{\beta-1} e^{-Zr/a_0} e^{-i\vec{p}_\nu\cdot\vec{r}}. \tag{18}$$

In order to calculate the transition rate, we must compute $\mathfrak{L}'_{\alpha\beta}$, which is defined by

$$\mathfrak{L}'_{\alpha\beta} = \frac{1}{2} \sum_{s_e, s_\nu} l'_\alpha l'^*_\beta, \tag{19}$$

where the sum is over the electron spin s_e and neutrino spin s_ν. Using Eq. (16), we obtain

$$\mathcal{L}'_{\alpha\beta}=\frac{2i}{m_\nu}(-)^{\delta_{\beta 4}}\frac{(m_e+E_e)}{m_e}\{(|A'|^2+|B'|^2)[\delta_{\alpha 4}(p_\nu)_\beta+\delta_{\beta 4}(p_\nu)_\alpha-\delta_{\alpha\beta}(p_\nu)_4-\epsilon_{\alpha\beta\gamma 4}(p_\nu)_\gamma]$$
$$+2i\,\mathrm{Im}(A'^*B')[(\hat{p}_\nu)_\alpha(p_\nu)_\beta+(p_\nu)_\alpha(\hat{p}_\nu)_\beta-\delta_{\alpha\beta}\hat{p}_\nu\cdot p_\nu+\epsilon_{\alpha\beta\gamma\delta}(\hat{p}_\nu)_\gamma(p_\nu)_\delta]\}. \tag{20}$$

For the case of shell distribution, $\phi(r)=(4\pi R^2)^{-1}\delta(r-R)$ and the expressions for A' and B' take the form

$$A'(p_\nu)=a(R)\frac{\sin(E_\nu R)}{E_\nu R}, \tag{21}$$

$$B'(p_\nu)=ia(R)\left(\frac{1-\beta}{Z\alpha}\right)\left[\frac{\cos(E_\nu R)}{E_\nu R}-\frac{\sin(E_\nu R)}{(E_\nu R)^2}\right], \tag{22}$$

with

$$a(R)=\frac{1}{2}\left(\frac{2m_e}{E_e+m_e}\right)^{1/2}N\left(\frac{ZR}{a_0}\right)^{\beta-1}e^{-ZR/a_0}. \tag{23}$$

III. TRANSITION RATES FOR ELECTRON CAPTURE AND POSITRON EMISSION

To complete the calculation of R_ϵ, it is necessary to compute the nuclear counterpart of Eq. (19):

$$\frac{1}{(2s_i+1)}\sum_{s_i,s_f}h^0_\alpha h^{0*}_\beta, \tag{24}$$

where s_i and s_f denote the spins of the initial and final nuclei, and

$$h^0_\alpha=\langle f(\vec{p}_f)|J^{(-)}_\alpha(0)|i(\vec{p}_i)\rangle_{\vec{q}^2=0}. \tag{25}$$

For the $\frac{1}{2}^+\to\frac{1}{2}^+$ transition, we take[8]

$$h^0_\alpha=\frac{1}{V}\bar{u}_f[F_V\gamma_\alpha+F_A\gamma_\alpha\gamma_5]u_i, \tag{26}$$

while for the $\frac{1}{2}^+\to\frac{3}{2}^+$ transition we use

$$h^0_\alpha=\frac{1}{V}\bar{u}_{f\alpha}(F_A+F_V\gamma_5)u_i, \tag{27}$$

where $u_{f\alpha}$ is the Rarita-Schwinger wave function for spin $\frac{3}{2}$. The effect of induced terms, such as the weak magnetism contribution, will be discussed in Sec. IV.

Using Eqs. (26) and (27), the expressions for R_ϵ take the form

$$R_\epsilon(\tfrac{1}{2}^+\to\tfrac{1}{2}^+)=\frac{G^2m_e^2}{2\pi^2}\frac{(1+\beta)}{\Gamma(1+2\beta)}\left(\frac{Z}{a_0}\right)^3e^{-2ZR/a_0}\left(\frac{2ZR}{a_0}\right)^{2(\beta-1)}$$
$$\times\left[\frac{\sin^2(E_\nu R)}{(E_\nu R)^2}(3F_A^2+F_V^2)+\frac{2}{3}\left(\frac{1-\beta}{Z\alpha}\right)E_\nu R(F_V^2-F_A^2)\right](W_0+1-\epsilon_\kappa)^2, \tag{28}$$

where ϵ_κ is the binding energy for a κ electron expressed in electron masses. The corresponding expression for $\frac{1}{2}^+\to\frac{3}{2}^+$ is obtained from Eq. (28) by putting $F_V=0$ and supplying an overall factor of $\frac{2}{3}$.

To compute R_{β^+} for the decays under consideration, we need only use our expressions for h^0_α, Eqs. (26) and (27), and the formula for $\mathcal{L}_{\alpha\beta}$ given in Ref. 2, Eq. (58). In applying this formula, we keep all terms in the electron wave function which contribute to order $Z\alpha E_\nu R$. The expression for $R_{\beta^+}(\frac{1}{2}^+\to\frac{1}{2}^+)$ is

$$R_{\beta^+}=\frac{G^2m_e^5}{2\pi^3}\left[(3F_A^2+F_V^2)I_1+\frac{4}{15}Z\alpha m_eR(F_V^2-F_A^2)I_2\right], \tag{29}$$

with

$$I_1=\frac{1}{R^2m_e^2}\int_1^{W_0}d\epsilon\,\sin^2[Rm_e(W_0-\epsilon)]F(Z,\epsilon)\times(1+\tfrac{14}{15}Z\alpha m_eR\epsilon)^2\epsilon(\epsilon^2-1)^{1/2}, \tag{30}$$

$$I_2=\int_1^{W_0}d\epsilon\,F(Z,\epsilon)(W_0-\epsilon)^3\epsilon(\epsilon^2-1)^{1/2}, \tag{31}$$

where $\epsilon=E_e/m_e$, W_0 is the mass difference between the initial and final atoms expressed in electron masses, and $F(Z,\epsilon)$ is the Fermi function for positron decay. To obtain the corresponding result for $\frac{1}{2}^+\to\frac{3}{2}^+$, we again set $F_V=0$ and multiply by $\frac{2}{3}$.

IV. DISCUSSION

The electron capture to positron emission ratios can be determined with the aid of Eqs. (28) and

(29). The phase space integrals encountered in the latter equation were evaluated numerically using a convenient approximation for the Fermi function.[9] In general, our results are in good agreement with those of Gove and Martin.[10] As a particular example, Fig. 1 shows R_ϵ/R_{β^+} for the transition between the $\frac{1}{2}^+$ ground state of ^{145}Gd and the 808.5 keV $\frac{1}{2}^+$ level of ^{145}Eu. It can be seen that there is little variation in the ratio for essentially all values of $F_A{}^2/F_V{}^2$ and any reasonable nuclear radius. Thus, the anomalous ratio of 18 ± 8 measured in this transition[4] cannot be explained in terms of a simple parametrization of the nuclear form factors, which is consistent with values of R_ϵ/R_{β^+} obtained for other transitions from the same ground state.

It is, of course, possible to modify the above treatment by including the effects of induced terms such as the weak magnetism contribution. These terms can be included in the capture formalism in essentially the same way that they are included for the decay process.[2,11] However, it is known[12] that induced terms have associated Coulomb corrections of order $(\alpha Z/m_p R)$. These corrections are of the same order as the $\alpha Z E_\nu R$ corrections calculated above and hence any systematic discussion of Coulomb effects should include them. For the decay process, all such corrections can be conveniently included by using the single photon exchange approximation to the electromagnetic interaction between the β^+ and the final nucleus.[13] We are currently performing such a calculation for the anomalous transitions in ^{145}Gd to determine if some reasonable choice of form factors and nuclear parameters can account for the observed ratios.

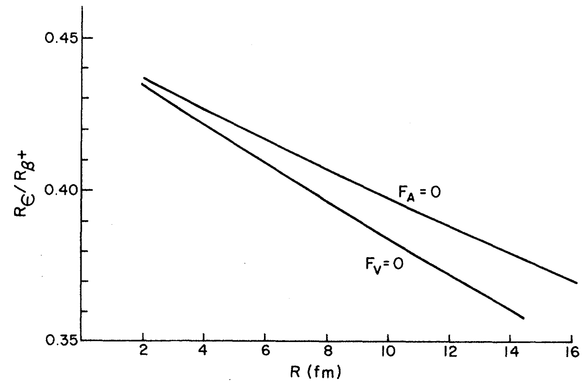

FIG. 1. The ratio R_ϵ/R_{β^+} for the transition between the ground state of ^{145}Gd and the 808.5 keV level of ^{145}Eu as a function of the radius parameter R. The tabulated value (Ref. 10) for this transition is 0.45 with $R \simeq 6$ fm. Our value is slightly smaller due to the additional Coulomb corrections included in Eqs. (28) and (29). For $0 \le (F_A/F_V)^2 < \infty$, the value of R_ϵ/R_{β^+} lies between the two lines shown.

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$$\sum_s [u_s(\vec{p})]_\alpha [u_s^\dagger(\vec{p})]_\beta = \frac{E}{M}\delta_{\alpha\beta}.$$

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