

Conventional and dual Shapiro steps in small Josephson junctions

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We propose a model describing the formation of both conventional and dual Shapiro steps in small Josephson junctions. According to our model, dual Shapiro steps occur at relatively low frequency of the microwave signal and low microwave power, whereas conventional steps occur in the opposite limit of high frequency and power. The crossover between these two regimes is controlled by a single parameter—the effective relaxation time of the environment. Our model takes into account the effect of a large inductor in the bias circuit, which has been used in recent experiments to protect the junction from the high-frequency noise of the environment. We predict the possibility of observing both types of steps in the same sample. Our model describes the I - V curves observed in the experiments with reasonable accuracy, thus opening the possibility of quantitatively fitting experimental data.

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I. INTRODUCTION

One of the main building blocks in superconducting circuits is Josephson junctions. They introduce the necessary nonlinearity to explore a plethora of fundamental quantum phenomena in combination with linear circuit elements such as inductors, capacitances, and ohmic resistors. In addition, they serve as central elements for quantum technological applications, for example, in quantum computing architectures.

The physics of Josephson junctions is governed by two conjugate degrees of freedom, namely, phase and charge. The fluctuation properties of phase and charge are not only determined by Heisenberg's uncertainty relation but also by the electromagnetic environment of the entire circuitry. “Conventional” Shapiro steps, discovered more than 60 years ago [1], occur in the parametric domain where the phase variable behaves as a classical variable. Indeed, conventional Shapiro steps appear in the current-voltage characteristics when the dynamics is dominated by microwave radiation at dc voltages

$$V_n = \frac{\hbar\omega_0}{2e}n, \text{ where } n \text{ is an integer.} \quad (1)$$

Here, ω_0 is the angular frequency of the radiation and e is the electron charge. Conventional Shapiro steps are caused by the nonlinear nature of the Josephson current-phase relation,

$$I_J(\varphi) = I_C \sin \varphi,$$

which leads to synchronization between the microwave radiation and Josephson oscillations.

Subsequently [2–4], another type of Shapiro steps has been predicted to occur in small-area Josephson junctions, sufficiently protected from noise of the environment. These are the so-called “dual” Shapiro steps, observed for dc-current biased junctions subject to an ac-current drive. Specifically, dual Shapiro steps appear at fixed current values

$$I_n = 2ef_0n, \quad (2)$$

where $f_0 = \omega_0/2\pi$. Dual steps are caused by synchronization between the microwaves and the Bloch oscillations in the lowest Bloch band of the periodic Josephson potential with the phase as a quantum degree of freedom. In their first experimental observation [5], dual steps were strongly smeared by noise and could be only seen in the differential resistance dV/dI . More recently, experiments conducted in inductively protected phase slip superconducting nanowires [6], in single Josephson junctions [7,8] and in single Cooper pair transistors [9] have demonstrated the occurrence of sharp dual Shapiro steps directly in the I - V curve. The theory of the dual Shapiro steps has been developed over many years. From the very beginning [3] it was understood that dual Shapiro

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steps can be observed only if the effective resistance of the ohmic environment exceeds the quantum of resistance of the Cooper pairs

$$R_Q = \frac{h}{4e^2} = 6.453 \text{ k}\Omega,$$

see, e.g., Refs. [10,11] for more details. Subsequently, the effect of heating and quantum fluctuations on the widths of the dual steps has been analyzed [10–12] and the noise properties associated with Bloch oscillations have been studied [11,12]. Experimentalists were the first to realize the importance of inductance in protecting the junction from high-frequency noise, thus motivating subsequent theoretical investigations into its effect [13]. Most theory models rely on the tight binding approximation for the periodic Josephson potential. This approximation leads to cosine dispersion of the lowest Bloch band, and neglects Zener tunneling to the higher Bloch bands. The effect of Zener tunneling on the I - V curve of a small Josephson junction has been considered, for example, in Refs. [14,15], but only in the dc bias regime, where Shapiro steps are absent. Recent experimental progress calls for the development of more realistic models capable of describing the experimental data in wide range of parameters. An important step towards this goal has been taken in Ref. [16]. The authors of this reference numerically investigated the model of dual steps previously introduced in Ref. [17] which includes Zener tunneling and treats quasiparticle tunneling as a shot noise process. The Monte Carlo simulations found good agreement with the experiment [5]. In the present work, we follow a similar path and develop a model of dual and conventional Shapiro steps in small Josephson junctions. In our model, we take into account Zener tunneling and, in addition, the effect of the large inductance used in the experiments to protect the junction. We neglect quasiparticle tunneling because in experiments, dissipation predominantly comes from bias resistors. We find that our model describes recent experiments [7,8] rather well, even though it overestimates the size of the dual steps in the sample with large critical current.

Our main results can be summarized as follows. We consider a junction with the Josephson energy, $E_J = \hbar I_C/2e$, smaller than the charging energy, $E_C = e^2/2C$, which typically implies *small* size junctions and may lead to Coulomb blockade. The junction is protected from the high-frequency noise by an inductor L and a resistor R as shown in Figs. 1 and 2. The properties of such electromagnetic environment can be characterized by a single parameter—the effective environment relaxation time τ_0 (21). We show that the dual Shapiro steps can be observed at bias currents well below the critical current and at sufficiently low microwave powers and frequencies. The formal condition for the existence of the dual steps, Eq. (42), can be roughly translated to

$$f_0 \tau_0^2 \sqrt{P_R^{\text{mw}}/\hbar g} \lesssim 1. \quad (3)$$

Here P_R^{mw} is the microwave power dissipated in the resistor R and $g = 4R_Q/R$ is its dimensionless conductance. In the opposite limit

$$f_0 \tau_0^2 \sqrt{P_R^{\text{mw}}/\hbar g} \gg 1, \quad (4)$$

features similar in shape to the I - V curve of the junction in the absence of microwaves should appear on the current

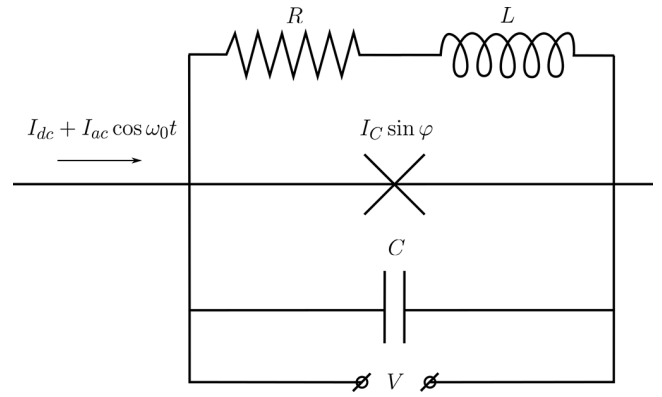


FIG. 1. Resistively and capacitively shunted Josephson junction (RCSJ model). The junction is current biased with the current $I_{dc} + I_{ac} \cos \omega_0 t$ and the voltage drop across the junction V is measured by a voltmeter.

voltage characteristics at bias voltages $V_n = (\hbar \omega_0/2e)n$. In this respect, these features resemble Shapiro steps observed in large size Josephson junctions, which look like sharp vertical current jumps at voltages V_n , thus repeating in shape the I - V curve of the junction at $P^{\text{mw}} = 0$. For this reason, we will call the features induced by microwaves at voltages V_n under the condition (4) as “conventional Shapiro steps.” Another motivation behind such terminology is that these features gradually crossover to the usual sharp Shapiro steps of a large size Josephson junction with increasing ratio E_J/E_C . We emphasize though that in small size Josephson junctions with $E_J/E_C \lesssim 1$, which we are considering here, the physical mechanism behind the formation of the conventional steps is the incoherent tunneling of Cooper pairs in presence of microwaves, the phase fluctuations are strong and there is no perfect phase-locking between microwaves and Josephson oscillations. In addition, in this limit, the conventional Shapiro steps significantly differ from sharp current jumps in shape. For example, at strong Coulomb blockade, the shape of the conventional steps turns into plateaus with high differential resistance, centered around the voltages $(\hbar \omega_0/2e)n$.

By varying the microwave power and frequency, one can switch between the two regimes as defined in (3) and (4). Therefore it should be possible to observe both dual and conventional Shapiro steps in the same sample if the

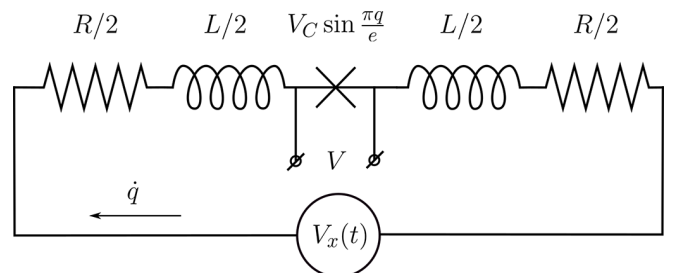


FIG. 2. Setup of the experiments [6–9], which is favorable for observation of the dual Shapiro steps. The junction is voltage biased with the voltage $V_x(t) = V_{dc} + V_{ac} \cos \omega_0 t$ via the protecting resistors and inductors. The voltage drop across the junction V is measured by a voltmeter and plotted versus the average current $\langle I \rangle$.

heating by the strong microwave signal can be kept under control. At intermediate powers and frequencies, such that $f_0 \tau_0^2 \sqrt{P_R^{\text{mw}}/\hbar g} \sim 1$, the theory becomes highly nontrivial, and we have not been able to compute the I - V curves. However, we expect that in this regime both dual and conventional steps may be visible in the same I - V curve, although both types of steps may be strongly broadened.

The paper is organized as follows: in Sec. II, we make a brief introduction to the problem and present our model; in Sec. III, we compare our results with the experiments [7,8]; and in the appendixes, we present the technical details.

II. MODEL

We consider a small Josephson junction shunted by a capacitor and by a resistor connected in series with an inductor. The bias current containing both dc and ac components is applied from the current source and the voltage drop across the junction is measured. The corresponding circuit is depicted in Fig. 1. An electrically equivalent circuit with the voltage source is shown in Fig. 2. The latter circuit has been used in the experiments [6–9]. In both circuits, the inductor plays a crucial role of cutting off the high-frequency noise. Experimental evidence indicates that sharp dual Shapiro steps cannot be observed in the absence of the inductor.

A. Duality between conventional and dual Shapiro steps

In this section, we briefly explain the origin of dual Shapiro steps in simple terms.

We choose the circuit of Fig. 1 because it is easier to model. In a lumped element description, the circuit consists of a capacitance C , a shunt resistance R , an inductance L connected in series with the resistor R , and a Josephson junction with critical current I_C . Conventional Shapiro steps are described by the equation governing the Josephson phase φ , i.e.,

$$\frac{C \hbar}{2e} \ddot{\varphi}(t) + \int_{-\infty}^t dt' Y(t-t') \frac{\hbar \dot{\varphi}(t')}{2e} + I_C \sin \varphi(t) = I(t) + \xi(t). \quad (5)$$

Here

$$Y(t) = \frac{e^{-Rt/L}}{L} \quad (6)$$

is the response function whose Fourier transform is the admittance

$$Y(\omega) = 1/Z(\omega),$$

where

$$Z(\omega) = R - i\omega L$$

is the circuit impedance. The time-dependent bias current is given by

$$I(t) = I_{\text{dc}} + I_{\text{ac}} \cos \omega_0 t, \quad (7)$$

and the noise process $\xi(t)$ is Gaussian with zero mean value and correlation function

$$\langle \xi(t) \xi(0) \rangle = k_B T \int \frac{d\omega}{\pi} e^{-i\omega t} \text{Re}\{Y(\omega)\} \quad (8)$$

with temperature T of the resistor. The noise correlation function is related to the admittance $Y(\omega)$ by the fluctuation dissipation theorem.

In the quantum description of the Josephson junction, charge and phase are operators obeying the canonical commutation relation

$$[\varphi, Q] = i2e.$$

Correspondingly, the bare current-biased Josephson junction is described by the standard Hamiltonian

$$H_J = -4E_C Q^2 + E_J(1 - \cos \varphi) - \frac{\hbar I(t)}{2e} \varphi. \quad (9)$$

The energy scales of the system are set by the charging energy

$$E_C = \frac{e^2}{2C}$$

and the Josephson energy

$$E_J = \frac{\hbar I_C}{2e}.$$

In the absence of a bias current

$$I(t) = 0,$$

Hamiltonian (9) is widely used in applications, for instance, to model superconducting qubits see, e.g., Ref. [18]. In the absence of bias, the Hamiltonian is 2π -periodic in φ : the Bloch theorem applies and predicts the existence of energy bands that are $2e$ -periodic with respect to the quasicharge q corresponding to Q . In the limit $E_J \gg E_C$ (deep energy wells of the cos potential) the lowest energy band is specified by

$$E_0(q) = \epsilon_0 - \delta_0 \cos \frac{\pi q}{e}, \quad (10)$$

where

$$\epsilon_0 = \sqrt{2E_J E_C} - \frac{E_C}{4}$$

is the energy of the lowest level at the bottom of the cosine potential well, and

$$\delta_0 = 16 \sqrt{\frac{E_J E_C}{\pi}} \left(\frac{E_J}{2E_C} \right)^{1/4} e^{-\sqrt{8E_J E_C}} \quad (11)$$

is the half-bandwidth corresponding to the tunneling between the wells. The next energy band is separated from the lowest one by an energy difference of the order of $\sqrt{8E_J E_C}$. The physical meaning of q is that of the charge transferred through the junction [3,4,19]. For a given q , the average voltage drop across the junction follows:

$$V = \frac{dE_0}{dq} = V_C \sin \frac{\pi q}{e}, \quad (12)$$

where $V_C = \pi \delta_0 / e$ is the critical voltage.

At nonzero bias current $I(t) \neq 0$, the quasicharge becomes time-dependent. In such a case, dual Shapiro steps originate from the synchronization of Bloch oscillations in the lowest energy band (10) with an external microwave signal [3,4] at the current values (2). In this situation, the relevant degree of freedom is the quasicharge. For the circuit depicted in Fig. 2,

the quasicharge obeys the equation [2–4]

$$L \ddot{q}(t) + R \dot{q}(t) + V_C \sin \frac{\pi q(t)}{e} = V_{dc} + V_{ac} \cos \omega_0 t + \xi_v(t), \quad (13)$$

where $\xi_v(t)$ is the voltage noise of the resistor. To obtain the I - V curve, one solves Eq. (13) in order to compute the time and ensemble averages (which here and below we denote by $\overline{\cdot}$ and $\langle \cdot \rangle$, respectively) of the time derivative of the quasicharge

$$\overline{\dot{q}} = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t ds \langle \dot{q}(s) \rangle = I_{dc},$$

which specifies the dc current I_{dc} through the junction. Equation (13) is valid if the system always stays in the lowest Bloch band, i.e., if Zener tunneling to the upper bands can be neglected.

Apparently, Eqs. (13) and (5) are mathematically very similar. This is the origin of the well-known duality in the dynamics of a Josephson junction. The similarity becomes even more pronounced when we consider the high-temperature (and $L \rightarrow 0$) limit of (5) while holding the ratio $k_B T/R$ fixed. In this limit, the response function tends to a Dirac distribution

$$Y(t) \propto \delta(t),$$

and the noise becomes white. Correspondingly, (5) reduces to a time-local Langevin equation with linear friction as (13). This high-temperature limit is often referred to as the resistively and capacitively shunted junction (RCSJ) model. It accurately describes the phenomenon of underdamped phase diffusion in a Josephson junction [20].

Shapiro steps emerge according to the following simple heuristic arguments. For the standard case, one uses the second Josephson relation between the total voltage drop across the junction and the phase velocity, namely,

$$V(t) = \frac{\hbar \dot{\varphi}(t)}{2e},$$

assuming that V is the sum of a dc bias and an RF component of frequency ω_0 . Then, as noticed in Ref. [21], conventional steps (1) occur when the resonance condition

$$2eV_{dc} = n\hbar\omega_0$$

is satisfied, which also implies

$$\overline{\langle \dot{\varphi} \rangle} = \omega_0 n.$$

For the dual case, Eq. (13) translates this condition into

$$\pi \overline{\langle \dot{q} \rangle} = \omega_0 n e, \quad (14)$$

from which the positions of the dual steps (2) follow. We refer to [2–4] for a first-principles derivation of these results. Dual Shapiro steps are observed at current values below the critical current I_C . More precisely, they are observed when

$$I_{dc} < I_{sw} < I_C,$$

where I_{sw} is the switching current (see Sec. III A below) and at small voltages, when the system mostly occupies the lowest Bloch band. In contrast, conventional Shapiro steps occur at higher currents and voltages, when higher bands become occupied.

B. Kinetic equation for charge distribution

In this section, we extend the above quantum mechanical model of the bare Josephson junction to include the impact of the impedance within a system + reservoir model. We assume that E_J is sufficiently smaller than E_C . This hypothesis allows us to apply a perturbative treatment. In the case of a pure dc bias, we then recover the well-known $P(E)$ theory [22], where $P(E)$ is the probability that a reservoir in thermal equilibrium exchanges energy quanta with the junction.

In the presence of an RF field, we need to obtain an effective equation of motion for the marginal probability distribution of the quasicharge. To this end, we start with the total Hamiltonian of the circuit in Fig. 1 that consists of three terms:

$$H = H_J + H_B + V, \quad (15)$$

The first term, H_J , is the Hamiltonian of the Josephson junction that appears in Eq. (9). The second term, H_B , is the Hamiltonian of the bath. It is the sum of oscillators with coordinates x_k , momenta p_k , masses m_k , and frequencies ω_k describing the shunt resistance R and the inductance L , respectively, i.e.,

$$H_B = \sum_k \left(\frac{p_k^2}{2m_k} + \frac{m_k \omega_k^2 x_k^2}{2} \right). \quad (16)$$

Finally, the third term describes the junction-bath interaction

$$V = - \sum_k \left(c_k x_k \varphi - \frac{c_k^2}{2m_k \omega_k^2} \varphi^2 \right). \quad (17)$$

The coefficients c_k define the quasicontinuous spectrum of bath oscillators via the spectral density $J(\omega)$ as

$$J(\omega) = \sum_k \frac{\pi c_k^2}{2m_k \omega_k} \delta(\omega - \omega_k) = \frac{\hbar^2 \omega}{4e^2} \frac{R}{R^2 + \omega^2 L^2}, \quad (18)$$

which is proportional to real part of the admittance $Y(\omega)$.

With the total Hamiltonian at hand, the crucial step is now to consistently eliminate the bath degrees of freedom. We accomplish this task by resorting to the path integral approach [19,23] in order to identify the quantum non-Markovian (meaning nonlocal in time) effects important at low temperature. To make the problem tractable, we make a set of approximations and proceed as discussed in detail in Appendix A. In essence, we retain the lowest order in E_J in the expansion of the nonlinear action of the reduced system (junction) and then, by time differentiation, we obtain a time-local evolution equation for the charge distribution $W(t, Q)$ from the path integral. Within our approximations, the charge Q accumulated in the capacitor C plays the role of conjugate variable to the quasiclassical Josephson phase. Finally, we reabsorb non-Markovian effects into renormalized parameters. Proceeding in this way, we arrive at

$$\begin{aligned} \frac{\partial W(t, Q)}{\partial t} = & \frac{\partial}{\partial Q} \left(\frac{Q}{RC} - I^*(t) + \frac{k_B T^*}{R} \frac{\partial}{\partial Q} \right) W(t, Q) \\ & + \Gamma(t, Q + 2e) W(t, Q + 2e) \\ & + \Gamma(t, -Q + 2e) W(t, Q - 2e) \\ & - (\Gamma(t, Q) + \Gamma(t, -Q)) W(t, Q). \end{aligned} \quad (19)$$

Before we explain in detail the basic ingredients of this equation, let us first comment on its general structure. Evidently, the terms in the first row specify an overdamped diffusion process (Smoluchowski equation) of the charge across the RC elements of the circuit in the presence of an effective bias current $I^*(t)$ and of an effective temperature T^* . The time-dependent jump rates Γ in the second and third line of the equation model Cooper pair transfer across the junction. Overall, the jump-diffusion equation (19), which generalizes the well-known $P(E)$ theory, constitutes the main result of this work.

Now we turn to the details. First, the time-dependent Cooper pair tunneling rate is given by the time integral

$$\Gamma(t, Q) = \frac{I_C^2}{8e^2} \int_0^\infty dt' e^{-(\frac{t'}{\tau_0})^2} \cos\left(\varphi_{cl}(t) - \varphi_{cl}(t - t') - \dot{\varphi}_{cl}(t)t' + \frac{2e(Q - e)}{\hbar C} t'\right), \quad (20)$$

We emphasize that this expression holds within the second-order accuracy in E_J respectively I_C . In (20), τ_0 denotes the relaxation time of the environment

$$\frac{1}{\tau_0} \approx \frac{g^{3/4}}{\sqrt{\delta + \sqrt{g}}} \sqrt{1 + \left| \ln\left(1 + \frac{2\sqrt{g}}{\delta}\right) \right|} \frac{2E_C}{\pi\hbar} + \left(\frac{\pi^2}{3} \frac{g^2}{\delta + g} \frac{k_B T}{E_C}\right)^{1/3} \frac{2E_C}{\pi\hbar}. \quad (21)$$

The relaxation time depends on g , the dimensionless conductance of the junction, given by

$$g = \frac{4R_Q}{R}, \quad (22)$$

and on the parameter δ which is proportional to the inductance

$$\delta = \frac{4L}{R^2 C}. \quad (23)$$

In (20), the argument of the cos function depends on the time-dependent phase $\varphi_{cl}(t)$. This latter quantity solves (5) when we set

$$I_C = 0$$

and

$$\xi(t) = 0.$$

In such a case, we get the explicit expression

$$\varphi_{cl}(t) = \frac{2eI_{dc}R}{\hbar} t + \varphi_{ac} \sin(\omega_0 t - \theta_0), \quad (24)$$

where

$$\varphi_{ac} = \frac{2eI_{ac}}{\hbar\omega_0} \sqrt{\frac{R^2 + \omega_0^2 L^2}{(1 - \omega_0^2 LC)^2 + \omega_0^2 R^2 C^2}} \quad (25)$$

and

$$\theta_0 = -\arctan \frac{\omega_0 RC}{1 - \omega_0^2 LC} + \arctan \frac{\omega_0 L}{R}. \quad (26)$$

Second, the time-dependent effective current $I^*(t)$ appearing in Eq. (19) differs from the input current (7). This is because

at large inductance the circuit of Fig. 1 is resonant. As a consequence, we find

$$I^*(t) = I_{dc} + I_{ac}^* \cos(\omega_0 t - \theta_0^*), \quad (27)$$

with an effective amplitude

$$I_{ac}^* = \sqrt{\frac{(1 - \omega_0^2 LC)^2 + \omega_0^2 (RC + \frac{L}{R})^2}{(1 - \omega_0^2 LC)^2 + \omega_0^2 R^2 C^2}} I_{ac} \quad (28)$$

and an effective phase shift

$$\theta_0^* = \arctan \frac{\omega_0 (RC + \frac{L}{R})}{1 - \omega_0^2 LC} - \arctan \frac{\omega_0 RC}{1 - \omega_0^2 LC}. \quad (29)$$

Roughly speaking, we can interpret this current $I^*(t)$ as the effective current seen by the junction. The effective temperature T^* , which also appears in Eq. (19), is specified by the integral

$$T^* = \frac{RC}{k_B} \int \frac{d\omega}{2\pi} \frac{\hbar\omega \coth \frac{\hbar\omega}{2k_B T}}{(1 - \omega^2 LC)^2 + \omega^2 R^2 C^2} \quad (30)$$

and is approximately

$$T^* \approx \begin{cases} T + \frac{gE_C}{\pi^2 k_B \sqrt{1-\delta}} \ln \frac{1+\sqrt{1-\delta}}{1-\sqrt{1-\delta}}, & \delta < 1, \\ T + \frac{2gE_C}{\pi^2 k_B \sqrt{\delta-1}} \arctan(\sqrt{\delta-1}), & \delta > 1. \end{cases}$$

The introduction of the effective temperature T^* shows that in the absence of Cooper pair tunneling, i.e., at

$$\Gamma(t, Q) = 0,$$

Eq. (19) reduces to the Fokker-Planck equation of an Ornstein-Uhlenbeck process (see, e.g., Ref. [[24], §3.8.4]) governing the charge distribution in a linear circuit consisting of a capacitor C , an inductor L and a resistor R . In particular, the equilibrium value for vanishing I^* of the variance of the charge is given by

$$\langle (Q - \langle Q \rangle)^2 \rangle = C k_B T^*. \quad (31)$$

We provide further details on the derivation of Eq. (16) in Appendix A.

C. Range of validity

Next, we elaborate on the range of validity of the kinetic equation (19). Equation (19) is definitely valid in the presence of a timescale separation, known as the regime of Coulomb blockade, between the sequential Cooper pair transfer and the equilibration of the reservoir. Mathematically, the regime corresponds to the condition

$$\Gamma(t, Q) \tau_0 \lesssim 1.$$

At $Q = e$, where the rate Γ reaches its maximum [see Eq. (44) below], this condition translates to

$$\sqrt{\pi} \frac{I_C^2 \tau_0^2}{16e^2} \lesssim 1,$$

and leads to a rather strict constraint on the Josephson energy of the junction, namely,

$$E_J \lesssim \frac{\hbar}{\tau_0}. \quad (32)$$

In an isolated Josephson junction at zero temperature, the Coulomb blockade disappears when the bias current reaches the threshold value

$$I_{\text{th}} = e/RC.$$

For currents exceeding I_{th} , the system experiences Zener tunneling events at the degeneracy charge value $Q = e$. Tunneling, however, depends only very weakly on the value of the dissipative charge transfer rate. Accordingly, we can replace the condition (32) with the weaker one

$$\Gamma(t, I_{\text{dc}}RC) \tau_0 \lesssim 1,$$

which further leads to

$$E_J \lesssim \frac{\hbar}{\tau_0} \exp\left(\frac{4\pi^2}{g^2} \frac{\tau_0^2 (I_{\text{dc}} - I_{\text{th}})^2}{e^2}\right). \quad (33)$$

This condition implies that Eq. (19) is valid if

$$\frac{g e}{2 \pi \tau_0} \lesssim I_{\text{th}}$$

and for sufficiently small critical currents

$$E_J \lesssim 2E_C. \quad (34)$$

With the help of (21), the first of these conditions equivalently reads

$$\frac{g^{3/4}}{2\pi\sqrt{\delta + \sqrt{g}}} + \left(\frac{1}{24\pi} \frac{g^2}{\delta + g} \frac{k_B T}{E_C}\right)^{1/3} \lesssim 1. \quad (35)$$

Condition (34) has an immediate consequence; when it is satisfied the potential barrier between neighboring wells of the Josephson cosine potential, $2E_J$, is lower than the level splitting between localized levels at the bottom of the potential well $\sqrt{8E_J E_C}$. Hence, localized levels are absent and the wave function in the ground state is strongly delocalized. As we will show below, one can stretch the limits of applicability and use Eq. (19) for a qualitative analysis of I - V curves even to the range $E_J \approx 5E_C$. For even higher Josephson energies, the half-bandwidth (11) becomes too small and the dual Shapiro steps vanish.

Another formal limitation for Eq. (19) is the smallness of the inductance, which can be expressed in the form

$$\delta \ll 1.$$

Indeed, the dynamics underlying Eq. (19) is the charge relaxation equation

$$\dot{Q}(t) + \frac{Q(t)}{RC} = I(t) + \xi(t),$$

complemented by

$$Q(t) = \frac{C \hbar \dot{\phi}(t)}{2e}.$$

This is an accurate approximation for Eq. (5) if

$$\frac{R}{L} \gg \omega_0, \frac{1}{RC}.$$

In this case, the inductance just provides the high-frequency cutoff for the environment. To be able to describe resonant systems with $\delta > 1$, in Eq. (19), we replace the bias current (7) by the effective current (27) with resonantly enhanced ac

amplitude. We provide more details on this approximation in Appendix A. With such modifications, we can apply Eq. (19) also to the analysis of strongly resonant systems with $\delta \gg 1$. An example of such a system is the experimental setup of Ref. [7], in which $\delta = 169$.

In the case of purely Ohmic dissipation, $L = 0$, which is usually considered in the literature, and at zero temperature, the condition (35) reduces to

$$g \lesssim 2\pi.$$

This estimate is consistent with the theory [25] that predicts the occurrence of a superconductor-insulator phase transition at $g = 4$ from first principles. According to this theory, for $g > 4$, the dual Shapiro steps are washed out by quantum fluctuations even in the absence of Zener tunneling [10–12]. Large inductances (large δ) extend the validity of Eq. (19) beyond that threshold. Thus dual Shapiro steps may be observed even at $g > 4$ provided $\delta \gg 1$, and the condition (35) is satisfied.

D. Numerical simulations

With the kinetic equation (19) at hand, we can now explore the junction dynamics in various regimes of operation. For this purpose, we consider the following observables.

(i) The voltage drop across the junction, which is given by an integral over the charge, i.e.,

$$V(t) = \int dQ \frac{Q}{C} W(t, Q). \quad (36)$$

(ii) Multiplying Eq. (19) by Q/C and integrating the result over Q , we obtain the identity reflecting the current conservation, namely,

$$I^*(t) = C \frac{dV}{dt}(t) + \frac{V(t)}{R} + 2e \int dQ (\Gamma(t, Q) - \Gamma(t, -Q)) W(t, Q), \quad (37)$$

from which we arrive, upon averaging over the time variable, at the dc current, i.e.,

$$I_{\text{dc}} = \frac{V_{\text{dc}}}{R} + \bar{I}_J. \quad (38)$$

Here, the first term on the right-hand side is the dc current flowing through the resistor. The second term is the time-averaged Josephson current,

$$\frac{\bar{I}_J}{2e} = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t ds \int dQ (\Gamma(s, Q) - \Gamma(s, -Q)) W(s, Q). \quad (39)$$

Now, a numerical analysis necessitates an efficient way to treat the evolution equation (19) for the time-dependent charge distribution $W(Q, t)$. A numerically efficient approach is to integrate the stochastic differential equation for the charge $Q(t)$ that generates data distributed according to the probability density $W(t, Q)$ solution of Eq. (19). This stochastic differential equation takes the form

$$dQ(t) + \left(\frac{Q(t)}{RC} - I^*(t)\right) dt + \sqrt{\frac{2k_B T^*}{R}} dw_t = 2e(dN^+(t) - dN^-(t)) \quad (40)$$

with dw_t the increment of a Wiener process, and where $dN^\pm$ denote the increments of counting processes fully characterized by the conditions

$$\begin{aligned} dN^{\pm 2}(t) &= dN^\pm(t), \\ dN^+(t)dN^-(t) &= 0, \\ E(dN^\pm(t)|Q(t)) &= \Gamma(t, \mp Q(t))dt. \end{aligned} \quad (41)$$

Here, E denotes the conditional expectation with respect to the value of the charge process at time t . We emphasize that (40) and (41) are well defined only if the tunneling rate Γ is positive. This is not guaranteed *per se* though. In fact, the rates (20) may oscillate in time and sometimes may even become negative. As we will show below, we can avoid negative values by imposing further approximations which are in line with the physics we consider in the sequel. Specifically, the negative values of the Cooper pair tunneling rate (20) play an essential role at $\varphi_{ac}(\omega_0\tau_0)^2 \sim 1$, where the dual Shapiro steps gradually cross over to the conventional ones. Even though Eq. (19) formally remains valid in this range of parameters, our numerical algorithm based on Eqs. (40) and (41) cannot be applied there. For this reason, in this paper, we focus on the regimes where either dual or conventional Shapiro steps are clearly visible. Namely, the dual steps can be observed in the adiabatic limit $\varphi_{ac}(\omega_0\tau_0)^2 \lesssim 1$, while the conventional steps are visible in the opposite limit $\varphi_{ac}(\omega_0\tau_0)^2 \gg 1$.

In this way, by solving the stochastic differential equation for many charge realizations and performing ensemble and time averages, we obtain the desired observables. More details about the numerical simulations are given in Appendix B.

III. RESULTS AND DISCUSSION

A. Adiabatic regime, dual Shapiro steps

In this section, we consider the adiabatic regime. This regime is specified by the requirements that the frequency of the microwave signal and its amplitude are sufficiently small and the condition

$$\varphi_{ac}(\omega_0\tau_0)^2 \ll 1 \quad (42)$$

is satisfied. In this regime, the classical phase changes very little during the relaxation time τ_0 of the environment. This circumstance allows us to approximate

$$\varphi_{cl}(t) - \varphi_{cl}(t - t') - \dot{\varphi}_{cl}(t)t' \approx 0 \quad (43)$$

in the expression for the Cooper pair tunneling rate (20). The approximation has the advantage to reduce the rate (20) to the time-independent expression

$$\Gamma_{ad}(Q) = \frac{\sqrt{\pi}}{16} \frac{I_C^2 \tau_0}{e^2} \exp\left(-\frac{e^2 \tau_0^2 (Q - e)^2}{\hbar^2 C^2}\right). \quad (44)$$

Equation (19) with a similar time-independent Cooper pair tunneling rate has been originally derived by Averin and Likharev in Ref. [4] to describe the formation of the dual Shapiro steps. Sharp steps appear on the I - V curve if the rate $\Gamma_{ad}(Q)$ tends to

$$\Gamma_\delta(Q) = \frac{\pi e E_J^2}{8 \hbar E_C} \delta(Q - e). \quad (45)$$

We obtain this approximation by replacing a unit normalized Gaussian distribution in (44) with a Dirac δ . The replacement requires

$$\frac{\hbar^2 C^2}{2 e^2 \tau_0^2} \ll 4 e^2.$$

We can couch this latter condition into the form

$$\frac{\hbar}{\tau_0} \lesssim 4 \sqrt{2} E_C, \quad (46)$$

which is essentially equivalent to (35).

The apparent critical current, or the switching current I_{sw} , is determined by the probability of Zener tunneling P_Z between the lowest and the first excited bands of the Hamiltonian (9) if the quasicharge grows linearly in time,

$$\dot{Q}(t) = \dot{Q}(0)t.$$

One can show [15] that at the degeneracy point $Q = e$ and in the absence of the microwave signal, the time derivative of the charge takes the value

$$\dot{Q}(0) = I_{dc} - I_{th}.$$

Hence, the probability of Zener tunneling can be estimated as

$$\begin{aligned} P_Z &= \exp\left(-\int_{-\infty}^{+\infty} dt \Gamma_\delta(Q(t))\right) \\ &\approx \exp\left(-\frac{I_Z}{I_{dc} - I_{th}}\right), \end{aligned} \quad (47)$$

where

$$I_Z = \frac{\pi E_J}{16 E_C} I_C \quad (48)$$

is the Zener current. The switching current in the absence of microwaves can be estimated in the same way as in Ref. [15], i.e., by equating the rate of Zener tunneling from the lowest Bloch band to the higher bands,

$$\Gamma_\uparrow(I_{dc}) = \frac{I_{dc}}{2e} \exp\left(-\frac{I_Z}{I_{dc} - I_{th}}\right),$$

and the rate of Cooper pair tunneling at the equilibrium value of the charge,

$$\Gamma_\downarrow(I_{dc}) \approx \Gamma_{ad}(I_{dc} R C).$$

The condition

$$\Gamma_\uparrow(I_{sw}) = \Gamma_\downarrow(I_{sw})$$

leads to

$$I_{sw} \approx \frac{e}{RC} + \left(\frac{\hbar^2 I_Z}{e^2 R^2 \tau_0^2}\right)^{1/3}. \quad (49)$$

In Fig. 3, we plot numerically simulated I - V curves for a junction with the parameters similar to those of the experiment [8]: $E_J = 60 \mu\text{eV}$, $E_C = 80 \mu\text{eV}$, $R = 275 \text{ k}\Omega$, and $L = 0.1 \mu\text{H}$. Assuming very low temperature $T = 1 \text{ mK}$, for the chosen parameters we estimate the effective temperature (30) as $T^* = 60 \text{ mK}$, the environment relaxation time (21) as $\tau_0 \approx 1.6 \times 10^{-11} \text{ s}$, and the phase oscillation amplitude (25) as $\varphi_{ac} \approx 20.3$, which results in a rather small value of the

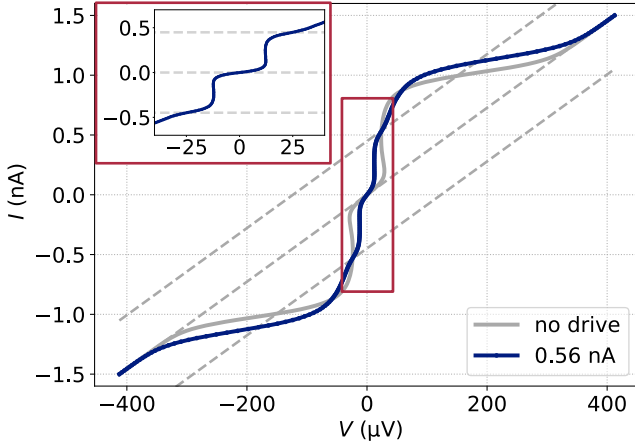


FIG. 3. Comparison of the I - V curve of a junction driven with an ac current of $I_{ac} \approx I_{ac}^* = 0.56$ nA and a frequency $f_0 = 1.4$ GHz with the undriven I - V curve of the same sample. The model parameters are similar to those of Ref. [8], $E_J = 60$ μ eV, $E_C = 80$ μ eV, $R = 275$ k Ω , $T = 1$ mK and $L = 0.1$ μ H leading to $\tau_0 \approx 1.6 \times 10^{-11}$ s and $\varphi_{ac}(\omega_0 \tau_0)^2 = 0.41$. The inset shows the driven I - V curve where the ohmic part was subtracted to increase the visibility of the steps, and the gray dashed lines indicate the expected positions of the inverse Shapiro steps. The simulation was performed in the adiabatic limit.

adiabaticity parameter $\varphi_{ac}(\omega_0 \tau_0)^2 = 0.41$. Thus the approximation of time-independent Cooper pair tunneling rate (44) is justified in this case. We also estimate the parameter

$$\frac{g^{3/4}}{2\pi\sqrt{\delta + \sqrt{g}}} + \left(\frac{1}{24\pi} \frac{g^2}{\delta + g} \frac{k_B T}{E_C} \right)^{1/3} \approx 0.06. \quad (50)$$

Hence, the condition (35) is satisfied, and our model can be used for this set of system parameters. Importantly, we use the value of the time $\tau_0 \approx 1.6 \times 10^{-11}$ s for simulating both dual (Figs. 3 and 4) and conventional Shapiro steps in the next section (Fig. 8). Due to the high value of the resistance, the

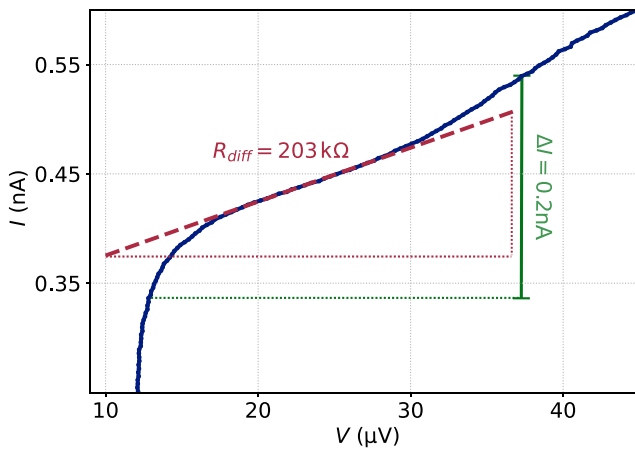


FIG. 4. First inverse step of the I - V curve of a junction described by the parameters given in the caption of Fig. 3, similar to those of Ref. [8]. The width of the step is estimated to be $\Delta I = 0.2$ nA, and the differential resistance is $R_{diff} = 203$ k Ω . The simulation was performed in the adiabatic limit.

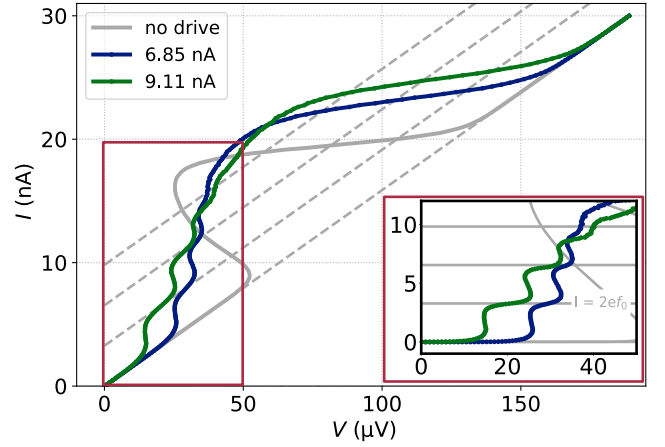


FIG. 5. I - V curves of a junction driven with different ac currents ($I_{ac}^* = 6.85$ nA, $I_{ac} = 3.82$ nA and $I_{ac}^* = 9.11$ nA, $I_{ac} = 5.08$ nA) and a frequency $f_0 = 10.215$ GHz. The remaining parameters are $E_J = 347$ μ eV, $E_C = 45$ μ eV, $R = 6.3$ k Ω , $T = 0$ mK, and $L = 3$ μ H. T^* was reduced to be $T^* = 10$ mK. These parameters are similar to those of Ref. [7]. This leads to $\tau_0 = 0.1$ ns, $\varphi_{ac}(\omega_0 \tau_0)^2 = 73$ for $I_{ac}^* = 6.85$ nA and to $\varphi_{ac}(\omega_0 \tau_0)^2 = 97$ for $I_{ac}^* = 9.11$ nA. For comparison, the undriven I - V curve is also plotted. The inset shows the I - V curve where the ohmic part was subtracted to increase the visibility of the steps, and the gray dashed lines indicate the expected positions of the inverse Shapiro steps. The simulations were performed in the adiabatic limit.

system operates in the overdamped regime with $\delta = 0.005$. The overall shape of the I - V curve resembles the experimental one. For example, the switching current, at which the I - V curve without microwave drive switches from the superconducting to the normal branch of the I - V , equals to 1 nA in our simulation and roughly equals to 2 nA in the experiment. The analytical estimate (49) gives $I_{sw} = 1$ nA in agreement with the numerical simulations. The Coulomb gap is close to 10 μ V in both cases. In the inset of Fig. 3, we plot the Josephson current $\langle I_J \rangle$ versus the voltage drop across the junction, subtracting the Ohmic contribution V_{dc}/R from the current according to Eq. (38). In this way, the horizontal dual steps at currents $\pm 2ef_0$ become clearly visible.

In Fig. 4, we plot the first dual Shapiro step in large scale. The Ohmic contribution to the current has been subtracted, i.e., here we assume a four-point measurement scheme like in the inset of Fig. 3. The width of the step is estimated to be 0.2 nA, and the differential resistance in the middle of it is $R_{diff} \approx 203$ k Ω .

In Fig. 5, we plot another set of the I - V curves with the parameters taken from the experiment [7]: $E_J = 347$ μ eV, $E_C = 45$ μ eV, $R = 6.3$ k Ω , and $L = 3$ μ H. Formally, our model cannot be applied in this case since $E_J > E_C$. Although quantitative fit of the experimental I - V curves is not possible, we include the corresponding plots anyway because they reproduce some of the features observed in the experiment. Namely, our model well reproduces the value of the switching current, the overall shape, and the positions of the dual steps. Since step positions are determined by the universal synchronization condition (14) between the frequency of the Bloch oscillations $I_{dc}/2e$ and the microwave frequency f , they

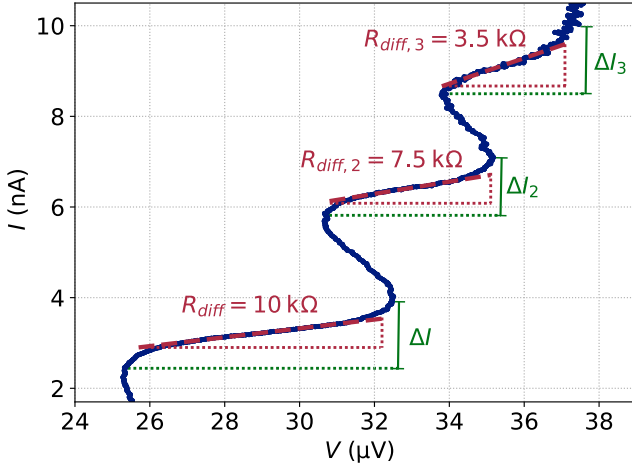


FIG. 6. Inverse Shapiro steps of the I - V curve of a junction driven with the ac currents $I_{ac}^* = 6.85$ nA, $I_{ac} = 3.82$ nA and a frequency $f_0 = 10.215$ GHz. The remaining parameters are similar to those of Ref. [7], they are given in the caption of Fig. 5. In particular, the environment relaxation times have the same value $\tau_0 = 0.1$ ns. The width of the steps are estimated to be $\Delta I = \Delta I_3 = 1.5$ nA and $\Delta I_2 = 1.3$ nA, and the differential resistances are $R_{diff} = 10$ k Ω , $R_{diff,2} = 7.5$ k Ω , and $R_{diff,3} = 3.5$ k Ω .

are not sensitive to the ratio E_J/E_C . To make the dual steps better visible, we put $T = 0$. Afterwards, the effective temperature (30) takes the value $T^* = 50$ mK and the environment relaxation time (21) becomes $\tau_0 = 0.1$ ns. We use this value of the relaxation time in Figs. 5–7, where the dual Shapiro steps are shown, as well as in Fig. 9 where we plot the conventional Shapiro steps. To increase the step sizes, we reduce the effective temperature to $T^* = 10$ mK, treating it as a fit

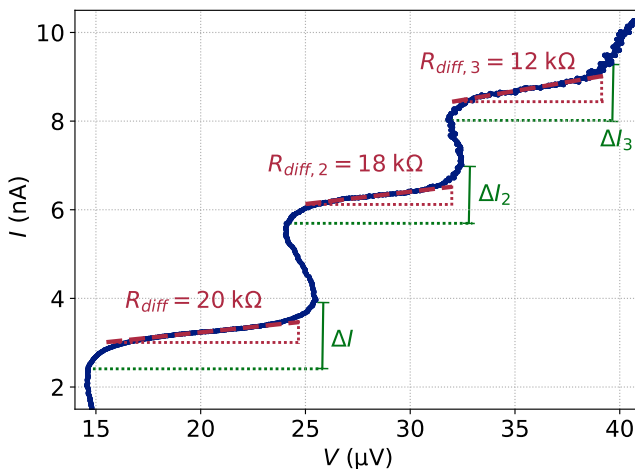


FIG. 7. Inverse Shapiro steps of the I - V curve of a junction driven with the ac currents $I_{ac}^* = 9.11$ nA, $I_{ac} = 5.08$ nA and a frequency $f_0 = 10.215$ GHz. The remaining parameters are similar to those of Ref. [7] and given in the caption of Fig. 5. The widths of the steps are estimated to be $\Delta I = 1.5$ nA and $\Delta I_2 = \Delta I_3 = 1.3$ nA, and the differential resistances are $R_{diff} = 20$ k Ω , $R_{diff,2} = 18$ k Ω , and $R_{diff,3} = 12$ k Ω .

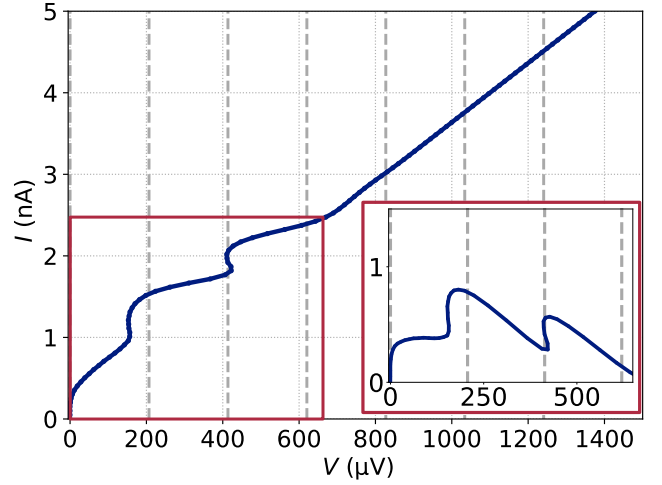


FIG. 8. IV curve of a junction driven with an ac current of the amplitude $I_{ac} = 160$ nA and a frequency $f_0 = 100$ GHz. The remaining parameters are $E_J = 60$ μ eV, $E_C = 80$ μ eV, $R = 275$ k Ω , $T = 1$ mK, and $L = 0.1$ μ H, they are the same as in Fig. 3 and similar to the parameters of Ref. [8]. This leads to $\tau_0 = 1.6 \times 10^{-11}$ s and $(\omega_0 \tau_0)^2 \varphi_{ac} = 128$. Gray dashed lines indicate the expected positions of conventional Shapiro steps. The simulation was performed in the classical limit.

parameter, and apply rather strong microwave currents; namely, we choose two values $I_{ac} = 3.82$ nA ($I_{ac}^* = 6.85$ nA) and $I_{ac} = 5.08$ nA ($I_{ac}^* = 9.11$ nA). This leads to the ac phase amplitudes $\varphi_{ac} = 1.66$ and $\varphi_{ac} = 2.2$ and the corresponding values of the adiabaticity parameter $\varphi_{ac}(\omega_0 \tau_0)^2 = 73$ and $\varphi_{ac}(\omega_0 \tau_0)^2 = 97$. Thus the condition (42) does not hold for this set of parameters. However, we still hope to get

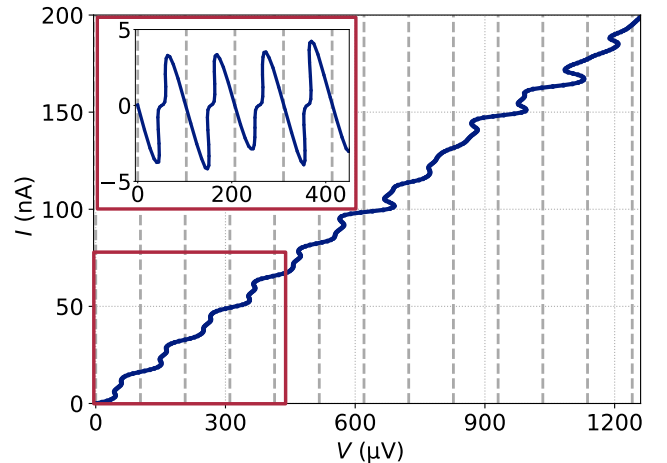


FIG. 9. I - V curve of a junction driven with an ac current of $I_{ac} = 800$ nA and a frequency $f_0 = 50$ GHz. The remaining parameters are $E_J = 347$ μ eV, $E_C = 45$ μ eV, $R = 6.3$ k Ω , $T = 0$ mK, and $L = 3$ μ H, they correspond to those of Ref. [7] and are the same as in Fig. 5. Like before, we artificially set $T^* = 10$ mK. This leads to $\tau_0 = 0.1$ ns and $(\omega_0 \tau_0)^2 \varphi_{ac} = 14620$. Gray dashed lines indicate the expected positions of conventional Shapiro steps. The simulation was performed in the classical limit.

qualitatively correct results, especially at high-bias currents where the term

$$\frac{2e(Q-e)t'}{\hbar C} \approx \frac{2e(I_{\text{dc}}RC - e)t'}{\hbar C}$$

that appears in the argument of the cosine in Eq. (20), becomes large and speeds up the convergence of the integral over time. We obtain the switching current in the absence of microwaves $I_{\text{sw}} \approx 20$ nA, in agreement with the experiment [7] and with the analytical estimate (49), which predicts $I_{\text{sw}} = 21$ nA. The inset again shows the average Josephson current, which we obtain by subtracting the Ohmic part V_{dc}/R from the main I - V curve. Three sharp steps at the current values $2ef_0$, $4ef_0$, and $6ef_0$ are clearly visible. The main disagreement between our simulations and the experiment [7] is in the size of the steps along the voltage axis. Namely, our model significantly overestimates this size. In the experiment, the step size is determined by the critical voltage in the absence of microwaves, $V_C^{\text{exp}} = 2.33$ μV . In our simulations, we find $V_C \approx 50$ μV . The reason for this discrepancy is clear: our model has been developed under the assumption $E_J \lesssim 2E_C$, while in the experiment one finds $E_J/E_C = 7.7$. Therefore the step size should be approximately rescaled by a factor $\pi\delta_0/2E_C = 0.02$, where δ_0 is given by Eq. (11). After such rescaling, we obtain good matching with the experimental values.

Next, with the parameters given above, we estimate

$$\frac{g^{3/4}}{2\pi\sqrt{\delta + \sqrt{g}}} + \left(\frac{1}{24\pi} \frac{g^2}{\delta + g} \frac{k_B T}{E_C} \right)^{1/3} \approx 0.04. \quad (51)$$

This means that the condition (35) for the validity of our model is satisfied even though the resistance R is rather small. This smallness is compensated by the large inductance, which results in the large value of $\delta = 170$.

In Figs. 6 and 7, we present the magnified dual Shapiro steps for ac currents $I_{\text{ac}}^* = 6.85$ nA and $I_{\text{ac}}^* = 9.11$ nA. The differential resistances in the middle of the first step are $R_{\text{diff}} = 10$ k Ω for $I_{\text{ac}}^* = 6.85$ nA and $R_{\text{diff}} = 20$ k Ω for $I_{\text{ac}}^* = 9.11$ nA. These values are about 10 times higher than those observed in the experiment [7]. We estimate the width of all steps as ~ 1.5 nA, which is close to the experimental value ~ 3 nA.

B. Nonadiabatic regime, conventional Shapiro steps

In this section, we demonstrate that Eq. (19) can also describe conventional Shapiro steps in Josephson junctions with a small critical current. In the nonadiabatic or strong driving limit

$$(\omega_0 \tau_0)^2 \varphi_{\text{ac}} \gg 1, \quad (52)$$

we can no longer use the approximation (43). Therefore the Cooper pair tunneling rate (20) depends on time. In fact, it quickly oscillates with the typical frequency

$$\omega_{\Gamma} \sim \omega_0 \varphi_{\text{ac}}.$$

In this regime, the system behaves almost classically. In particular, the charge accumulated in the capacitor stays close to

its classical expression at $I_C = 0$,

$$Q_{\text{cl}}(t) = \frac{\hbar C}{2e} \dot{\varphi}_{\text{cl}}(t) = I_{\text{dc}} RC + Q_{\text{ac}} \cos(\omega_0 t - \theta_0), \quad (53)$$

where the ac amplitude of charge fluctuations is

$$Q_{\text{ac}} = C \frac{\hbar \omega_0}{2e} \varphi_{\text{ac}}. \quad (54)$$

The presence of a small Josephson current causes deviations from the classical behavior (53), resulting in small corrections to the Ohmic I - V curve. These corrections are the conventional Shapiro steps.

We can further simplify the problem if we assume that (52) and

$$\varphi_{\text{ac}} \gg 1$$

simultaneously hold. In such a case, the typical frequency of the oscillations of the rate exceeds the driving frequency, $\omega_{\Gamma} \gg \omega_0$, and one can average the rate $\Gamma(t, Q)$ over time. Employing the properties of Bessel functions $J_m(x)$, we can express the exponent containing the classical phases (24) as

$$e^{i[\varphi_{\text{cl}}(t) - \varphi_{\text{cl}}(t-t')]} = e^{i \frac{2e I_{\text{dc}} R t'}{\hbar}} \sum_{m,n} J_m(\varphi_{\text{ac}}) J_n(\varphi_{\text{ac}}) e^{i m (\omega_0 t - \delta)} e^{-i n (\omega_0 (t-t') - \delta)}.$$

We observe that only the addends with $m = n$ contribute to the average over t of this sum. In addition, we can recast the term $\dot{\varphi}_{\text{cl}}(t)t'$, which also appears in the argument of the cosine function in Eq. (20), into the form

$$\dot{\varphi}_{\text{cl}}(t)t' \equiv \frac{2e Q_{\text{cl}}(t)}{C \hbar \omega_0} t'.$$

Upon combining these two approximations, we get

$$\begin{aligned} \lim_{t \nearrow \infty} \frac{1}{t} \int_0^t \Gamma(t, Q)|_{\omega_0 t = \text{const}} \\ = \sum_n \frac{I_C^2}{8e^2} J_n^2(\varphi_{\text{ac}}) \int_0^\infty dt' e^{-(\frac{t'}{\tau_0})^2} \\ \times \cos \left(\frac{2e I_{\text{dc}} R t'}{\hbar} + n \omega_0 t' + \frac{2e(Q - Q_{\text{cl}}(t) - e)}{\hbar C} t' \right). \end{aligned}$$

We evaluate the integral and conclude that the Cooper pair tunneling rate averaged over the oscillations with frequencies higher than ω_0 takes the form

$$\begin{aligned} \lim_{t \nearrow \infty} \frac{1}{t} \int_0^t \Gamma(t, Q)|_{\omega_0 t = \text{const}} \\ = \tilde{\Gamma}(\varphi_{\text{ac}}, Q - Q_{\text{ac}} \cos(\omega_0 t - \theta_0)), \end{aligned} \quad (55)$$

where the rate $\tilde{\Gamma}(\varphi_{\text{ac}}, Q)$ is expressed via the adiabatic rate (44) as

$$\tilde{\Gamma}(Q) = \sum_n J_n^2(\varphi_{\text{ac}}) \Gamma_{\text{ad}} \left(Q - C \frac{\hbar \omega_0}{2e} n \right). \quad (56)$$

The solution of the kinetic equation (19) with the rate (55) takes the form of a stationary distribution with the center following the classical trajectory,

$$W(t, Q) = \tilde{W}(Q - Q_{\text{ac}} \cos(\omega_0 t - \theta_0)). \quad (57)$$

The distribution $\tilde{W}(Q)$ can be obtained from Eq. (19) with the time-dependent current replaced by its dc component, $I^*(t) \rightarrow I_{dc}$,

$$\begin{aligned} \frac{\partial \tilde{W}}{\partial t} = & \frac{\partial}{\partial Q} \left(\frac{Q}{RC} - I_{dc} + \frac{k_B T^*}{R} \frac{\partial}{\partial Q} \right) \tilde{W} \\ & + \tilde{\Gamma}(Q + 2e) \tilde{W}(Q + 2e) + \tilde{\Gamma}(-Q + 2e) \tilde{W}(Q - 2e) \\ & - (\tilde{\Gamma}(Q) + \tilde{\Gamma}(-Q)) \tilde{W}(Q), \end{aligned} \quad (58)$$

in the limit $t \rightarrow \infty$. Here we made use of the fact that the classical charge obeys the equation

$$\dot{Q}_{cl} + \frac{Q_{cl}}{RC} = I^*(t), \quad (59)$$

which follows from Eq. (A11) of Appendix, and this equation, in turn, leads to the identity

$$\begin{aligned} & -\frac{d}{dt} (Q_{ac} \cos(\omega_0 t - \theta_0)) \\ & = \frac{Q_{ac} \cos(\omega_0 t - \theta_0)}{RC} - I_{ac}^* \cos(\omega_0 t - \theta_0^*). \end{aligned}$$

This identity can also be verified by direct substitution of the phase shifts [Eqs. (26) and (29)], of the current amplitude (28) and of the charge oscillation amplitude (54) into it. The stationary solution of Eq. (58) describes the formation of conventional Shapiro steps at the voltages (1). The positions of these steps are encoded in the rate $\tilde{\Gamma}(Q)$ (56), and their shape resembles that of the I - V curve of the junction in the absence of a microwave signal. If this I - V curve has a clear vertical superconducting branch, then conventional Shapiro will appear at voltages (1). In junctions with strong Coulomb blockade, the supercurrent branch is absent, and the conventional steps will look like replicas of the I - V curve with the Coulomb gap.

Substituting the solution (57) in the expression for the voltage (36), we obtain

$$V(t) = V_{dc} + \frac{Q_{ac}}{C} \cos(\omega_0 t - \varphi_0), \quad (60)$$

where the dc component of the voltage reads

$$V_{dc} = \int dQ \frac{Q}{C} \tilde{W}(Q). \quad (61)$$

The average Josephson current (39) in this approximation takes the form

$$I_J(V_{dc}, \varphi_{ac}) = 2e \int dQ (\tilde{\Gamma}(Q) - \tilde{\Gamma}(-Q)) \tilde{W}(Q). \quad (62)$$

To clarify the relation between Eq. (58) and the well known results on the theory of conventional Shapiro steps, we consider the limiting case of very small Josephson critical current such that $\tilde{\Gamma}(Q) \ll 1/RC$. In this limit, we can replace the charge distribution $\tilde{W}(Q)$ in the right-hand side of Eq. (62) by the solution of Eq. (58) with $\tilde{\Gamma} = 0$. This solution has the form $\tilde{W}(Q) = W_0(Q - I_{dc}RC)$, where $W_0(Q)$ is a simple Gaussian

$$W_0(Q) = \frac{\exp\left(-\frac{Q^2}{2Ck_B T^*}\right)}{\sqrt{2\pi Ck_B T^*}}. \quad (63)$$

At small $\tilde{\Gamma}$, we can replace $I_{dc} \rightarrow V_{dc}/R$ and write the charge distribution in the form $\tilde{W}(Q) = W_0(Q - CV_{dc})$. Then, with

the aid of Eq. (56), we can express the dc Josephson current (62) in the presence of microwaves via the current in the absence of microwaves $I_J^{(0)}$,

$$I_J(V_{dc}, \varphi_{ac}) = \sum_n J_n^2(\varphi_{ac}) I_J^{(0)}\left(V_{dc} - \frac{\hbar\omega_0}{2e}n, 0\right). \quad (64)$$

It is the Tien-Gordon formula [26], which well describes the I - V curves of Josephson junctions with small critical current in the classical regime [27,28].

To demonstrate further relations between our approach and the established results, we consider the expression for the Josephson current of the junction with small critical current in the absence of microwaves. In this case, the current is expressed in terms of the so-called $P(E)$ function [22,29]:

$$I_J(V_{dc}) = \frac{\pi e E_J^2}{\hbar} (P(2eV_{dc}) - P(-2eV_{dc})). \quad (65)$$

This function is defined in terms of the correlation function of the quantum phase operators,

$$P(E) = \int \frac{dt}{2\pi\hbar} e^{i\frac{Et}{\hbar}} \text{Tr}(\rho_\beta e^{i\hat{\varphi}(t)} e^{-i\hat{\varphi}(0)}), \quad (66)$$

where ρ_β is the equilibrium density matrix of the reservoir at inverse temperature β (see Ref. [22], §3.2.2 for details). Upon contrasting Eqs. (62) and (65), we observe that they become equivalent if

$$P(E) = \frac{2\hbar}{\pi E_J^2} \int dQ \Gamma_{ad}(Q) W_0\left(Q - \frac{CE}{2e}\right). \quad (67)$$

With the δ -function approximation (45) for the rate $\Gamma_{hf}(Q)$, the function (67) takes the form

$$P(E) = \frac{1}{\sqrt{16\pi E_C k_B T^*}} \exp\left(-\frac{(E - 4E_C)^2}{16E_C k_B T^*}\right), \quad (68)$$

which is the correct expression for the function (66) in the high-resistance limit $R \gg R_Q$ and at sufficiently high temperature [22,29].

To investigate the formation of the conventional Shapiro steps, we have simulated the I - V curves using Eq. (58) as a starting point. In Fig. 8, we show the I - V curve for a junction with the parameters close to those of the experiment [8], which are given in the previous section. To achieve the strongly nonadiabatic limit (52), we have applied a very strong microwave amplitude $I_{ac} = 160$ nA and chosen the frequency $f_0 = 100$ GHz. With these parameters, we obtain $\varphi_{ac} = 1.23$ and $\varphi_{ac}(\omega_0 \tau_0)^2 = 128$. Frequencies this high have been used, for example, in the experiment [28]. In accordance with the predictions of the Tien-Gordon formula (64), we observe the formation of the conventional steps, which resemble the I - V curve without microwave drive (grey line in Fig. 3) in shape. They are separated by the voltage intervals $\hbar\omega_0/2e$ and have varying magnitudes. For the chosen parameters, the conventional steps appear at rather high voltages, which are equal to and higher than the superconducting gap of aluminum electrodes. For this reason and because of the heating effects, the conventional steps may be difficult to observe in the sample used in the experiment [8].

In Fig. 9, we repeat the same simulations for a junction with the parameters of the experiment [7] and with the strong

ac amplitude $I_{ac} = 800$ nA and high frequency $f_0 = 50$ GHz. This results in $\varphi_{ac} = 13.7$ and $\varphi_{ac}(\omega_0\tau_0)^2 = 14620$. In this case, the conventional steps are also very similar in shape to the I - V curve in the absence of microwaves (grey line in Fig. 5), and they clearly follow the pattern expected from the Tien-Gordon formula (64). The possibility of observing the conventional Shapiro steps in the experiment depends on the sample design and on the Joule heating power the sample generates. In Fig. 9, we have ignored the effect of Joule heating and assumed a constant temperature of the resistors and of the aluminum electrodes of the Josephson junction at all bias voltages. However, it is known that in the absence of microwaves the aluminum electrodes should switch from the superconducting to the normal state at the critical bias voltage [30]

$$V_N = 0.829 \sqrt{\frac{R_t}{R_{lead}}} \frac{\Delta}{e}, \quad (69)$$

where $R_t \approx 1.9$ k Ω is the tunnel resistance of the Josephson junction, R_{lead} is the normal state resistance of the aluminum lead, and $\Delta \approx 100$ μ eV is the superconducting gap of aluminum. In the sample of Ref. [7], one of the aluminum leads is rather thin and long; its resistance can be roughly estimated as $R_{lead} \approx 40$ Ω , which leads to $V_N = 570$ μ V. Clearly, the Shapiro steps cannot be observed above this voltage. Moreover, they can be suppressed already at lower voltages due to additional heating by the strong microwave signal.

IV. CONCLUSION

In conclusion, we have developed a model of a Josephson junction with small critical current. Our model describes both conventional and dual Shapiro steps within the same formalism. The results of our numerical simulations agree with the recent experimental data reasonably well. We show that one can potentially observe both types of Shapiro steps in the same sample by varying the bias current and the amplitude and the frequency of the microwave signal. However, strong heating of the bias resistors by high-bias current or by high microwave power may prevent such an opportunity. Our model accounts for the effect of a large inductor in the bias circuit, which has been used in recent experiments to limit the high-frequency noise of the environment. Within our model, the effect of this inductor and of the rest of the environment is absorbed into a single parameter—the relaxation time of the environment τ_0 (21). We show that the dual Shapiro steps are formed at the I - V curve at sufficiently low bias current, frequency, and power of the microwave signal, while the conventional steps are formed in the opposite limit—sufficiently high-bias current, frequency, and microwave power. The crossover between the two regimes is determined by the time τ_0 , see the conditions (42), (52) and the expression for the switching current (49).

A point that deserves to be further explored in future work is the following. We numerically integrated our model by averaging over the realizations of the stochastic differential equation (40). This is possible as long as (20) remains positive at all times, thus ensuring that (19) is a well-posed jump-diffusion equation. In fact, (20) may oscillate and temporarily take negative values. In the present paper, we overcome this

difficulty by averaging over oscillations. It is, however, well known that quantum master equations beyond the weak coupling scaling limit may actually include nonpositive rates [31]. The circumstance may describe actual physical phenomena while at the same time restricting the range of physically admissible solutions of a master equation. Avenues to explore this issue may be provided by adapting the martingale method proposed in Ref. [32] or, alternatively, directly integrating (19), for instance, by means of the Quantum Monte Carlo method recently proposed in Ref. [33].

In conclusion, we believe that our model is an important step towards a quantitative description of the dual Shapiro steps and of related quantum phenomena in small-area Josephson junctions.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [34].

APPENDIX A: DERIVATION OF EQ. (19)

We consider the system described by the Hamiltonian (15) and assume that at the initial moment of time, $t = 0$, the density matrix of the system is factorized. In this case, the density matrix of the junction, ρ , evolves in time as [19,35,36]

$$\rho(t, \varphi_1, \varphi_2) = \int d\tilde{\varphi}_1 d\tilde{\varphi}_2 J(t, 0; \varphi_1, \varphi_2, \tilde{\varphi}_1, \tilde{\varphi}_2) \rho_0(\tilde{\varphi}_1, \tilde{\varphi}_2),$$

where ρ_0 is the initial density matrix at time $t = 0$. The time evolution kernel J is amenable to the form of the path integral

$$J = \int \mathcal{D}\varphi^+(t') \int \mathcal{D}\varphi^-(t') \exp \left(\frac{i}{\hbar} \int_0^t dt' \mathcal{L} - i \frac{\hbar}{4e^2} \int_0^t dt_1 dt_2 \varphi^-(t_1) Z^{-1}(t_1 - t_2) \dot{\varphi}^+(t_2) - \frac{\hbar}{8e^2} \int_0^t dt_1 dt_2 \varphi^-(t_1) G(t_1 - t_2) \varphi^-(t_2) \right). \quad (A1)$$

The argument of the exponential consists of

$$\mathcal{L} = \frac{\hbar^2 C}{4e^2} \dot{\varphi}^- \dot{\varphi}^+ - 2E_J \sin \varphi^+ \sin \frac{\varphi^-}{2} + \frac{\hbar I}{2e} \varphi^-$$

and

$$Z^{-1}(t_1 - t_2) = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t_1 - t_2)}}{Z(\omega)} = \frac{e^{-R(t_1 - t_2)/L}}{L},$$

$$G(t_1 - t_2) = \int \frac{d\omega}{2\pi} \text{Re} \left[\frac{\omega}{Z(\omega)} \right] \coth \frac{\hbar\omega}{2k_B T} e^{-i\omega(t_1 - t_2)}.$$

Finally,

$$Z(\omega) = R - i\omega L$$

is the impedance of the environment, and the “classical” phase $\varphi^+(t')$ and the “quantum” phase $\varphi^-(t')$, over which the path integral (A1) is taken, satisfy the boundary conditions

$$\begin{aligned}\varphi^+(t) &= \frac{\varphi_1(t) + \varphi_2(t)}{2}, & \varphi^+(0) &= \frac{\tilde{\varphi}_1 + \tilde{\varphi}_2}{2} \equiv \varphi_i^+, \\ \varphi^-(t) &= \varphi_1(t) - \varphi_2(t), & \varphi^-(0) &= \tilde{\varphi}_1 - \tilde{\varphi}_2 \equiv \varphi_i^-\end{aligned}$$

1. Limit $L \rightarrow 0$ and $k_B T \gg \hbar/RC$

In this case, we make the approximations

$$\begin{aligned}Z^{-1}(t_1 - t_2) &= \frac{\delta(t_1 - t_2 - \eta)}{R}, \\ G(t_1 - t_2) &= \frac{2k_B T}{\hbar R} \delta(t_1 - t_2),\end{aligned}\quad (\text{A2})$$

and the action in the path integral (A1) becomes local in time. Here $\eta > 0$ is an infinitesimal positive time shift which ensures that $Z^{-1}(t_1 - t_2) \equiv 0$ for $t_1 < t_2$ and the causality condition is satisfied. It is important to note that the limit of vanishing correlation must be interpreted in the post-point prescription

$$\begin{aligned}\int_0^t dt_1 dt_2 \varphi^-(t_1) Z^{-1}(t_1 - t_2) \dot{\varphi}^+(t_2) \\ \rightarrow \int_0^t dt_1 \varphi^-(t_1^+) \dot{\varphi}^+(t_1).\end{aligned}$$

This is because according to the general properties of response functions,

$$Z^{-1}(t_1 - t_2) \begin{cases} > 0 & t_1 > t_2 \\ = 0 & t_1 \leq t_2 \end{cases}$$

to obey causality [[36], §2.1] In this limit, the density matrix satisfies a first order in time, “Markovian,” partial differential equation. Namely, by taking the time derivative of Eq. (A1), we obtain

$$\begin{aligned}i\hbar \frac{\partial \rho}{\partial t} &= -8E_C \frac{\partial^2 \rho}{\partial \varphi^+ \partial \varphi^-} - i \frac{\hbar}{RC} \varphi^- \frac{\partial \rho}{\partial \varphi^-} - \frac{\hbar I(t)}{2e} \varphi^- \rho \\ &\quad - i \frac{\hbar k_B T}{4e^2 R} (\varphi^-)^2 \rho + \frac{\hbar I_C}{e} \sin \varphi^+ \sin \frac{\varphi^-}{2} \rho.\end{aligned}\quad (\text{A3})$$

We solve this equation by means of perturbation theory using as small parameter the critical current I_C . Hence, at order zero in the expansion, we set $I_C = 0$, and we are able to find the fundamental solution, which we denote by J_T , of the resulting “thermal” high-temperature Markovian equation. Such a solution satisfies the following boundary condition at equal times:

$$J_T(t', t'; \varphi^+, \varphi^-, \varphi_i^+, \varphi_i^-) = \delta(\varphi^+ - \varphi_i^+) \delta(\varphi^- - \varphi_i^-).$$

We can still write J_T in the form of the path integral (A1) by setting $E_J = 0$ and using the time local kernels (A2). As a result, the path integral becomes Gaussian, and we can

explicitly evaluate it:

$$\begin{aligned}J_T &= \frac{\hbar C}{8\pi e^2 K(t - t')} e^{i(\frac{Q_{cl}(t)}{2e} \varphi^- - \frac{Q_{cl}(t')}{2e} \varphi_i^-)} \\ &\quad \times e^{i \frac{\hbar}{4e^2 R} (\varphi^+ - \varphi_{cl}(t) - \varphi_i^+ + \varphi_{cl}(t')) \varphi^- (t - t')} \\ &\quad \times e^{-\frac{k_B T}{4e^2 R} (t - t') (\varphi^- (t - t'))^2} \\ &\quad \times e^{\frac{k_B T C}{4e^2} (\varphi^- - \varphi_i^-) \varphi^- (t - t') - \frac{k_B T C}{8e^2} ((\varphi^-)^2 - (\varphi_i^-)^2)}.\end{aligned}\quad (\text{A4})$$

Here we have introduced the response function of the environment

$$K(t) = RC(1 - e^{-t/RC}),\quad (\text{A5})$$

and the time-dependent combination of the initial and the final quantum phases φ^-, φ_i^-

$$\phi^-(t - t') = RC \frac{\dot{K}(t - t') \varphi^- - \varphi_i^-}{K(t - t')}.\quad (\text{A6})$$

The classical time-dependent phase $\varphi_{cl}(t)$ and the charge $Q_{cl}(t)$ are defined in Eqs. (24) and (53). We also note that the function J_T depends on the difference $\varphi^+ - \varphi_i^+$, which reflects the phase translational invariance of the system at $I_C = 0$. The validity of the expression (A4) can be checked by inserting it back into Eq. (A3).

We need the thermal kernel (A4) as an intermediate result, which allows us to verify the validity of a more general non-Markovian kernel derived below.

To obtain the high-temperature version of Eq. (19), we perform several additional steps. First, we express Eq. (A3) in the integral form

$$\begin{aligned}\rho(t, \varphi^+, \varphi^-) &= \frac{\hbar I_C}{e} \int_{-\infty}^t dt' \int d\varphi_i^+ d\varphi_i^- \\ &\quad \times J_T(t, t'; \varphi^+, \varphi_i; \varphi^-, \varphi_i^-) \sin \varphi_i^+ \sin \frac{\varphi_i^-}{2} \rho(t', \varphi_i^+, \varphi_i^-),\end{aligned}$$

and substitute this expression back into the last term of Eq. (A3). Afterwards, Eq. (A3) acquires the form

$$\begin{aligned}i\hbar \frac{\partial \rho}{\partial t} &= -8E_C \frac{\partial^2 \rho}{\partial \varphi^+ \partial \varphi^-} - i \frac{\hbar}{RC} \varphi^- \frac{\partial \rho}{\partial \varphi^-} - \frac{\hbar I(t)}{2e} \varphi^- \rho \\ &\quad - i \frac{\hbar k_B T}{4e^2 R} (\varphi^-)^2 \rho + \frac{\hbar^2 I_C^2}{e^2} \sin \varphi^+ \sin \frac{\varphi^-}{2} \int_{-\infty}^t dt' \\ &\quad \times \int d\varphi_i^\pm J_T(t, t'; \varphi^+ - \varphi_i^+; \varphi^-, \varphi_i^-) \\ &\quad \times \sin \varphi_i^+ \sin \frac{\varphi_i^-}{2} \rho(t', \varphi_i^\pm).\end{aligned}$$

Now we perform the Markovian approximation: we evaluate the time integral over t' in the last term under the assumption that the density matrix ρ evolves in time as if I_C were equal to zero during the convergence time of the integral. Upon rewriting, the resulting equation in terms of the charge distribution

$$W(t, Q) = \int d\varphi^+ d\varphi^- \frac{e^{-i \frac{Q\varphi^-}{2e}}}{4\pi e} \rho\left(t, \varphi^+ + \frac{\varphi^-}{2}, \varphi^+ - \frac{\varphi^-}{2}\right),\quad (\text{A7})$$

we arrive at an equation similar to Eq. (19). This set of transformations corresponds to the standard Bloch-Redfield approximation known in the theory of open quantum systems.

2. Low-temperature and high-inductance limit

Next, we consider the low-temperature and high-inductance limit, where the system dynamics is non-Markovian. In this case, only a phenomenological treatment is possible. A reasonable approximation is to keep using the Markovian kinetic equation (19) because it is convenient for numerical simulations of Shapiro steps. To capture as much of non-Markovian effects as possible, we renormalize the parameters of this equation. To derive Eq. (19) with modified parameters, we start from the approximate Markovian equation for the density matrix similar to the high-temperature Eq. (A3),

$$i\hbar \frac{\partial \rho}{\partial t} = -8E_C \frac{\partial^2 \rho}{\partial \varphi^+ \partial \varphi^-} - i \frac{\hbar}{RC} \varphi^- \frac{\partial \rho}{\partial \varphi^-} - \frac{\hbar I^*(t)}{2e} \varphi^- \rho - i \frac{\hbar k_B T^*}{4e^2 R} (\varphi^-)^2 \rho + \frac{\hbar I_C}{e} \sin \varphi^+ \sin \frac{\varphi^-}{2} \rho. \quad (\text{A8})$$

Here the modified bias current has the form

$$I^*(t) = I_{dc} + I_{ac}^* \cos(\omega_0 t - \varphi_0^*), \quad (\text{A9})$$

where the amplitude I_{ac}^* and the phase φ_0^* are defined in Eq. (28). These parameters are chosen in such a way that the solutions of the equation

$$C \frac{\hbar \ddot{\varphi}_{cl}}{2e} + \int_{-\infty}^t dt' \frac{e^{-\frac{R(t-t')}{L}}}{L} \frac{\hbar \dot{\varphi}_{cl}(t')}{2e} = I(t) \quad (\text{A10})$$

and of the equation

$$C \frac{\hbar \ddot{\varphi}_{cl}}{2e} + \frac{1}{R} \frac{\hbar \dot{\varphi}_{cl}}{2e} = I^*(t) \quad (\text{A11})$$

coincide and are equal to the classical phase (24). Thus, although Eq. (A8) formally describes the overdamped classical dynamics of Eq. (A11), it approximately captures the resonant behavior of Eq. (A10) at large inductance L via the enhancement of the ac current amplitude (28). At low frequencies $\omega \ll R/L$, the two classical equations (A10) and (A11) become equivalent. Therefore Eq. (A8), based on the overdamped classical equation (A11), should correctly capture the low-frequency quantum dynamics, which is important for obtaining the correct dc I - V curve of the junction.

Next, we treat the nonlinear term $\propto I_C$ in Eq. (A8) by means of perturbation theory in the same way as we did above for the Markovian equation (A3). We find the evolution kernel J in the zeroth order by solving the Gaussian path integral (A1) with $I_C = 0$ and with the boundary conditions

$$\begin{aligned} \varphi^+(t) &= \frac{\varphi_1 + \varphi_2}{2}, & \varphi^+(t') &= \frac{\varphi_{1i} + \varphi_{2i}}{2}, \\ \varphi^-(t) &= \varphi_1 - \varphi_2, & \varphi^-(t') &= \varphi_{1i} - \varphi_{2i}. \end{aligned} \quad (\text{A12})$$

It is a straightforward, but rather lengthy, procedure, here we follow the recipes of the review [35]. To express the result, we define the response function of the environment at finite

inductance,

$$K(t) = \frac{\kappa_+(1 - e^{-\kappa_- t})}{\kappa_-(\kappa_+ - \kappa_-)} - \frac{\kappa_-(1 - e^{-\kappa_+ t})}{\kappa_+(\kappa_+ - \kappa_-)}. \quad (\text{A13})$$

Here the two relaxation rates are defined as

$$\kappa_{\pm} = \frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}. \quad (\text{A14})$$

For large $L > R^2 C/2$ ($\delta > 1$), the rates $\kappa_{\pm}$ become complex, and the response function oscillates in time. In contrast, for small inductance $L \rightarrow 0$, one finds $\kappa_+ \rightarrow \infty$, $\kappa_- = 1/RC$ and the function $K(t)$ reduces to the form (A5). Evaluating the path integral, we obtain the following expression for the function J :

$$\begin{aligned} J &= \frac{\hbar C}{8\pi e^2 K(t-t')} e^{i(\frac{Q_{cl}(t)}{2e} \varphi^- - \frac{Q_{cl}(t')}{2e} \varphi_i^-)} \\ &\times e^{i \frac{\hbar}{4e^2 R} (\varphi^+ - \varphi_{cl}(t) - \varphi_i^+ + \varphi_{cl}(t')) \varphi^- (t-t')} \\ &\times e^{-f_1(t-t')(\varphi^- (t-t'))^2} \\ &\times e^{-f_2(t-t')(\varphi^-)^2 - f_3(t-t')\varphi^- \varphi^- (t-t')}. \end{aligned} \quad (\text{A15})$$

Here the function $\varphi^-(t-t')$ is still defined by Eq. (A6), but with the modified response function (A13), and the functions f_1 , f_2 , and f_3 are expressed as

$$\begin{aligned} f_1(t) &= f(t) - K(t)\dot{f}(t) + \frac{K^2(t)}{2}\ddot{f}(0), \\ f_2(t) &= R^2 C^2 \left(-\dot{K}(t)\dot{f}(t) + \frac{1+K^2(t)}{2}\ddot{f}(0) \right), \\ f_3(t) &= RC(1 - \dot{K}(t))\dot{f}(t) \\ &\quad - RC K(t)(\ddot{f}(t) - \dot{K}(t)\ddot{f}(0)). \end{aligned} \quad (\text{A16})$$

Finally, the function $f(t)$, appearing in the expressions above, is given by the integral

$$f(t) = \frac{\hbar}{4e^2 R} \int \frac{d\omega}{2\pi} \frac{\coth \frac{\hbar\omega}{2k_B T} (1 - \cos \omega t)}{\omega((1 - \omega^2 LC)^2 + \omega^2 R^2 C^2)}. \quad (\text{A17})$$

At high temperatures $k_B T \gg \hbar/RC$ and for $L = 0$, one finds

$$f(t) = \frac{k_B T}{4e^2 R} (t - RC(1 - e^{-\frac{t}{RC}})), \quad (\text{A18})$$

and the fundamental solution (A15) reduces to the thermal kernel (A4). As we noted above, this fact provides an additional verification of the validity of Eq. (A15) and helps to resolve the uncertainties in the choice of the saddle point solutions while evaluating the path integral.

Next, we repeat the steps taken previously and formally express Eq. (A8) in the integral form

$$\begin{aligned} \rho(t, \varphi^+, \varphi^-) &= \frac{\hbar I_C}{e} \int_{-\infty}^t dt' \int d\varphi_i^+ d\varphi_i^- \\ &\times J(t, t'; \varphi^+, \varphi_i; \varphi^-, \varphi_i^-) \\ &\times \sin \varphi_i^+ \sin \frac{\varphi_i^-}{2} \rho(t', \varphi_i^+, \varphi_i^-). \end{aligned}$$

Substituting this integral in the last term of Eq. (A8), we obtain

$$\begin{aligned} i\hbar \frac{\partial \rho}{\partial t} = & -8E_C \frac{\partial^2 \rho}{\partial \varphi^+ \partial \varphi^-} - i \frac{\hbar}{RC} \varphi^- \frac{\partial \rho}{\partial \varphi^-} - \frac{\hbar I^*(t)}{2e} \varphi^- \rho \\ & - i \frac{\hbar k_B T^*}{4e^2 R} (\varphi^-)^2 \rho + \frac{\hbar^2 I_C^2}{e^2} \sin \varphi^+ \sin \frac{\varphi^-}{2} \int_{-\infty}^t dt' \\ & \times \int d\varphi_i^\pm J(t, t'; \varphi^+ - \varphi_i^+; \varphi^-, \varphi_i^-) \sin \varphi_i^+ \\ & \times \sin \frac{\varphi_i^-}{2} \rho(t', \varphi_i^\pm). \end{aligned} \quad (\text{A19})$$

Next, we explain in some detail how Eq. (19) is derived from Eq. (A19). First, we note that in the limit of small I_C , which we are considering here, we can neglect the dependence of $\rho(t', \varphi_i^+, \varphi_i^-)$ in the last term of Eq. (A19) on the phase φ_i^+ because at $I_C = 0$ the translational invariance holds. Next, writing the product of two sines in the form

$$\sin \varphi^+ \sin \varphi_i^+ = \frac{-\cos(\varphi^+ + \varphi_i^+) + \cos(\varphi^+ - \varphi_i^+)}{2},$$

we observe that the term containing $\cos(\varphi^+ + \varphi_i^+)$ vanishes after the integration over φ_i^+ because J depends on the difference $\varphi^+ - \varphi_i^+$ and $\rho(t', \varphi_i^+, \varphi_i^-)$ does not depend on φ_i^+ . Thus Eq. (A19) can be written in the form

$$\begin{aligned} i\hbar \frac{\partial \rho}{\partial t} = & -8E_C \frac{\partial^2 \rho}{\partial \varphi^+ \partial \varphi^-} - i \frac{\hbar}{RC} \varphi^- \frac{\partial \rho}{\partial \varphi^-} - \frac{\hbar I^*(t)}{2e} \varphi^- \rho - i \frac{\hbar k_B T^*}{4e^2 R} (\varphi^-)^2 \rho + \frac{\hbar^2 I_C^2}{2e^2} \sin \frac{\varphi^-}{2} \int_{-\infty}^t dt' \int d\varphi_i^+ d\varphi_i^- \\ & \times J(t, t'; \varphi^+ - \varphi_i^+; \varphi^-, \varphi_i^-) \cos(\varphi^+ - \varphi_i^+) \sin \frac{\varphi_i^-}{2} \rho(t', \varphi^+, \varphi_i^-). \end{aligned} \quad (\text{A20})$$

Now we rewrite Eq. (A20) in terms of the charge distribution (A7), which gives

$$\begin{aligned} \frac{\partial W}{\partial t}(t, Q) = & \frac{\partial}{\partial Q} \left(\frac{Q}{RC} - I^*(t) + \frac{k_B T^*}{R} \frac{\partial}{\partial Q} \right) W(t, Q) + \frac{I_C^2}{16e^2} \frac{1}{4\pi e} \sum_{\nu=\pm 1} \sum_{\mu_1, \mu_2=\pm 1} (-1)^{\frac{\mu_1+\mu_2-2}{2}} \int_{-\infty}^t dt' \int dQ' \\ & \times \sqrt{\frac{\pi}{f_2(t-t')}} e^{-\frac{16e^4 R^2}{\hbar^2} (f_1(t-t') - \frac{f_3^2(t-t')}{4f_2(t-t')})} e^{i\nu(\varphi_{cl}(t) - \varphi_{cl}(t') - \dot{\varphi}_{cl}(t')K(t-t'))} \\ & \times e^{-\frac{(Q - \mu_1 e - Q_{cl}(t) - (Q' - Q_{cl}(t') + \mu_2 e)K(t-t'))^2}{16e^2 f_2(t-t')}} e^{i\nu \frac{eR}{\hbar} \frac{f_3(t-t')}{f_2(t-t')}} (Q - Q_{cl}(t) - \mu_1 e - (Q' - Q_{cl}(t') + \mu_2 e)K(t-t')) \\ & \times e^{i\nu \mu_2 \frac{2e^2}{\hbar C} K(t-t')} e^{i\nu \frac{2e}{\hbar C} K(t-t')Q'} W(t', Q'). \end{aligned} \quad (\text{A21})$$

Here indices ν, μ_1, μ_2 come from expressing cosines and sines in the form

$$\begin{aligned} \cos(\varphi^+ - \varphi_i^+) = & \sum_{\nu=\pm 1} \frac{e^{i\nu(\varphi^+ - \varphi_i^+)}}{2}, \quad \sin \frac{\varphi^-}{2} = \sum_{\mu_1=\pm 1} (-1)^{(\mu_1-1)/2} \frac{e^{i\mu_1 \varphi^-/2}}{2i}, \\ \sin \frac{\varphi_i^-}{2} = & \sum_{\mu_2=\pm 1} (-1)^{(\mu_2-1)/2} \frac{e^{i\mu_2 \varphi_i^-/2}}{2i}. \end{aligned} \quad (\text{A22})$$

The next step is to shift the integration parameter in Eq. (A21) as

$$Q' \rightarrow Q' - \mu_2 e + Q_{cl}(t') + \frac{Q - \mu_1 e - Q_{cl}(t)}{\dot{K}(t-t')}$$

and write it in the form

$$\begin{aligned} \frac{\partial W}{\partial t}(t, Q) = & \frac{\partial}{\partial Q} \left(\frac{Q}{RC} - I^*(t) + \frac{k_B T^*}{R} \frac{\partial}{\partial Q} \right) W(t, Q) + \frac{I_C^2}{16e^2} \frac{1}{4\pi e} \sum_{\nu=\pm 1} \sum_{\mu_1, \mu_2=\pm 1} (-1)^{\frac{\mu_1+\mu_2-2}{2}} \int_{-\infty}^t dt' \int dQ' \\ & \times \sqrt{\frac{\pi}{f_2(t-t')}} e^{-\frac{16e^4 R^2}{\hbar^2} (f_1(t-t') - \frac{f_3^2(t-t')}{4f_2(t-t')})} e^{i\nu(\varphi_{cl}(t) - \varphi_{cl}(t') - \dot{\varphi}_{cl}(t) \frac{K(t-t')}{\dot{K}(t-t')})} e^{i\nu \frac{2e}{\hbar C} \frac{K(t-t')}{\dot{K}(t-t')}} (Q - \mu_1 e) e^{-\frac{\dot{K}^2(t-t')}{16e^2 f_2(t-t')}} Q'^2 \\ & \times e^{i\nu \frac{2eR}{\hbar} (\frac{K(t-t')}{RC} - \frac{f_3(t-t')K(t-t')}{2f_2(t-t')})} Q' \\ & \times W\left(t', Q' - \mu_2 e + Q_{cl}(t') + \frac{Q - \mu_1 e - Q_{cl}(t)}{\dot{K}(t-t')}\right). \end{aligned} \quad (\text{A23})$$

To proceed further, we assume that the function $f(t)$ quickly grows with time so that one can make a Markovian approximation and get rid of the time integral in the last term of Eq. (A23). We also assume that the charge distribution weakly changes during the timescale τ_0 , at which the time integral converges. Then, at times $t - t' \lesssim \tau_0$, we can approximate

$$\sqrt{\frac{\pi}{f_2(t)}} e^{-\frac{\dot{K}^2(t)}{16e^2 f_2(t)}} Q'^2 e^{i\nu \frac{2eR}{\hbar} (\frac{K(t)}{RC} - \frac{f_3(t)K(t)}{2f_2(t)})} Q' \approx \frac{4\pi e}{|\dot{K}(t)|} e^{-\frac{16e^4 R^2}{\hbar^2} f_2(t) (\frac{K(t)}{RC \dot{K}(t)} - \frac{f_3(t)}{2f_2(t)})^2} \delta(Q'). \quad (\text{A24})$$

This approximation is possible if $8f_2(\tau_0) \lesssim 1$, in which case the spread of the charge Q' during the integration time τ_0 is much less than e , i.e., $\sqrt{\delta Q'^2} \tau_0 \ll e$. The condition $8f_2(\tau_0) \lesssim 1$ in combination with the expression for the time τ_0 (21) again leads to the condition (35), which ensures the validity of our approximations at all stages. The approximation (A24) allows one to carry out the integration over Q' in Eq. (A23), resulting in

$$\begin{aligned} \frac{\partial W}{\partial t}(t, Q) &= \frac{\partial}{\partial Q} \left(\frac{Q}{RC} - I^*(t) + \frac{k_B T^*}{R} \frac{\partial}{\partial Q} \right) W(t, Q) \\ &+ \frac{I_C^2}{16e^2} \sum_{v=\pm 1} \sum_{\mu_1, \mu_2=\pm 1} (-1)^{\frac{\mu_1+\mu_2-2}{2}} \\ &\times \int_{-\infty}^t dt' \frac{1}{|\dot{K}(t-t')|} e^{-F(t-t')} e^{iv \frac{2e}{\hbar C} \frac{K(t-t')}{K(t-t')} (Q-\mu_1 e)} \\ &\times e^{iv(\varphi_{cl}(t)-\varphi_{cl}(t')-\dot{\varphi}_{cl}(t) \frac{K(t-t')}{K(t-t')})} \\ &\times W\left(t', \frac{Q-\mu_1 e - Q_{cl}(t)}{\dot{K}(t-t')} - \mu_2 e + Q_{cl}(t')\right). \end{aligned} \quad (\text{A25})$$

Here the function $F(t)$ is defined as

$$\begin{aligned} F(t) &= \frac{64\pi^2}{g^2} \left(f_1(t) + \frac{K^2(t)f_2(t)}{R^2 C^2 \dot{K}(t)} - \frac{K(t)f_3(t)}{RC \dot{K}(t)} \right) \\ &= \frac{64\pi^2}{g^2} \left(f(t) - \frac{K(t)}{\dot{K}(t)} \dot{f}(t) + \frac{K^2(t)}{2\dot{K}^2(t)} \ddot{f}(0) \right). \end{aligned} \quad (\text{A26})$$

We assume that the time integral in Eq. (A25) quickly converges. To determine the corresponding timescale τ_0 , we expand the function $F(t)$ at small times. In the limit of small inductance, such that $\delta \lesssim 2\sqrt{g}$ and $\kappa_+ \gg \kappa_-$, the relevant times belong to the interval $\kappa_+^{-1} < t < \kappa_-^{-1}$, in which one can approximate

$$F(t) \approx \frac{4gE_C^2}{\pi^2 \hbar^2} t^2 \ln(\kappa_+ t) + \frac{8}{3\pi} g \frac{k_B T E_C^2}{\hbar^3} t^3. \quad (\text{A27})$$

For higher inductances, $2\sqrt{g} \lesssim \delta \leq 1$, the relevant timescales are $t \lesssim \kappa_+^{-1}$, where one finds

$$\begin{aligned} F(t) &\approx \frac{4}{\pi^2} \frac{g^2 k_B T E_C^3}{\delta \hbar^4} t^4 + \frac{16g^3 E_C^4}{\pi^4 \hbar^4 \delta^2} \\ &\times \left(\ln \frac{\sqrt{LC}}{t} + \frac{7}{4} - \gamma - \frac{2-\delta}{4\sqrt{1-\delta}} \ln \frac{1+\sqrt{1-\delta}}{1-\sqrt{1-\delta}} \right) t^4. \end{aligned} \quad (\text{A28})$$

Combining the two expressions and making some additional approximations, we conclude that at short times the function $F(t)$ can be reasonably well approximated as

$$F(t) = \begin{cases} \frac{t^2}{\tau_0^2}, & \delta < 2\sqrt{g}, \\ \frac{t^4}{\tau_0^4}, & \delta > 2\sqrt{g}, \end{cases} \quad (\text{A29})$$

where the relaxation time τ_0 is defined by Eq. (21) in the main text. Equation (21) has been derived by approximate interpolation between the expansions (A27) and (A28).

Next, we observe that since the condition $\kappa_- \tau_0 \lesssim 1$ holds, one can expand the function $K(t-t')$ at small time

differences $t-t'$, which results in

$$\dot{K}(t-t') \approx 1, \quad \frac{K(t-t')}{\dot{K}(t-t')} \approx t-t'. \quad (\text{A30})$$

After that, Eq. (A25) acquires the form

$$\begin{aligned} \frac{\partial W}{\partial t}(t, Q) &= \frac{\partial}{\partial Q} \left(\frac{Q}{RC} - I^*(t) + \frac{k_B T^*}{R} \frac{\partial}{\partial Q} \right) W(t, Q) \\ &+ \frac{I_C^2}{16e^2} \sum_{v=\pm 1} \sum_{\mu_1, \mu_2=\pm 1} (-1)^{\frac{\mu_1+\mu_2-2}{2}} \\ &\times \int_{-\infty}^t dt' e^{-F(t-t')} e^{iv \frac{2e}{\hbar C} (Q-\mu_1 e)(t-t')} \\ &\times e^{iv(\varphi_{cl}(t)-\varphi_{cl}(t')-\dot{\varphi}_{cl}(t)(t-t'))} \\ &\times W(t', Q - Q_{cl}(t) + Q_{cl}(t') - (\mu_1 + \mu_2)e). \end{aligned} \quad (\text{A31})$$

Now we make the following observation: if we neglect the last term $\propto I_C^2$ in Eq. (A31), then its solution takes a simple form $W(t, Q) = W_0(Q - Q_{cl}(t))$, where $W_0(Q)$ is the stationary solution defined in Eq. (63), which just follows the classical solution without changing its shape. Since we are considering the perturbation theory in I_C , we can assume that during the short time interval between t and t' the distribution W , appearing in the last term under the time integral, evolves in time in the same way as the solution at $I_C = 0$, i.e., the function $W_0(Q - Q_{cl}(t))$, does. Under this assumption, we find

$$\begin{aligned} W(t', Q - (\mu_1 + \mu_2)e - Q_{cl}(t) + Q_{cl}(t')) \\ = W(t, Q - (\mu_1 + \mu_2)e). \end{aligned} \quad (\text{A32})$$

Performing this replacement together with the summation over the indexes v , μ_1 , and μ_2 in Eq. (A31), we transform it to the form (19) with the Cooper pair tunneling rate taking the form

$$\begin{aligned} \Gamma(t, Q) &= \frac{I_C^2}{8e^2} \int_0^\infty dt' e^{-F(t')} \cos \left(\varphi_{cl}(t) - \varphi_{cl}(t-t') \right. \\ &\quad \left. - \dot{\varphi}_{cl}(t)t' + \frac{2e}{\hbar C} (Q-e)t' \right). \end{aligned} \quad (\text{A33})$$

The integration over time in this formula cannot be carried out exactly because of the complicated time dependence of the function $F(t)$, which grows exponentially at long times $\kappa_- t \gtrsim 1$. Moreover, such integration would make little sense anyway because we have already made several approximations based on the assumption that the time t' is small. Therefore the short timescale τ_0 , at which the time integral in Eq. (A33) converges, is the only parameter which can be reliably extracted from the model. For this reason, we simply approximate $F(t) \approx t^2/\tau_0^2$ for the whole range of parameters, which leads to the expression for the rate (20) given in the main text.

APPENDIX B: DETAILS OF THE NUMERICAL IMPLEMENTATION

In order to determine the I - V curves, the stochastic differential equation given in Eq. (40) has to be solved numerically for different realizations of the charge evolution. To do so, at each time step t the probability P of making a jump is determined by

$$P(t) = 1 - \exp(-(\Gamma(t, Q) + \Gamma(t, -Q))dt), \quad (B1)$$

where the time step is chosen sufficiently small to ensure that the probability to jump twice within dt is negligible. If a jump occurs, we update the value of the charge Q by increasing or decreasing it by the charge of a Cooper pair

$$Q(t + dt) = Q(t) \pm 2e, \quad (B2)$$

with probability determined by the ratio $\Gamma(Q)/(\Gamma(Q) + \Gamma(-Q))$. If no jump takes place, we compute the next value

of Q according to the Euler scheme

$$Q(t + dt) = -\left(\frac{Q(t)}{RC} - I^*(t)\right)dt + \sqrt{\frac{2k_B T^*}{R}}\sqrt{dt}X, \quad (B3)$$

where X is randomly drawn from a normal distribution (mean equal 0 and unit variance). This process is repeated for 1000 charge ensembles to determine the ensemble average of the evolution of the charge. Furthermore the time evolution is determined for several oscillations of ω_0 to ensure the average of the charge is converged. The voltage is then given by $V = \langle Q \rangle / C$. For the simulations, in this paper, a typical time step of $dt = 0.05/\max(\Gamma(Q))$ is used, and the evolution of the charge is simulated for between 10^5 – 10^6 time steps to ensure convergence. For more information on the numerical implementation, see Ref. [34].

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