

Topological pseudospin Hall effect and multifrequency corner modes in kagome-based lattices

Shenglong Guo^{1,*}, Qinhui Jiang^{1,*}, Yuma Kawaguchi², Bo Li^{1,3} and Mengyao Li^{1,†}

¹*Tsinghua Shenzhen International Graduate School, Tsinghua University, Shenzhen 518055, China*

²*Department of Electrical Engineering, The City College of New York, New York, New York 10031, USA*

³*Suzhou Laboratory, Suzhou 215000, China*



(Received 27 June 2025; revised 9 April 2026; accepted 29 June 2026; published 22 July 2026)

Topological phases and modes, including the pseudospin Hall effect with robust edge transport and higher-order corner states, provide versatile control of wave propagation and modal confinement in classical wave platforms. Motivated by the need to systematically engineer multigap interface and corner localization with enhanced design freedom in kagome-derived lattices, we theoretically investigate two types of extended kagome lattices within a tight-binding framework. These lattices support a series of localized modes, including pseudospin Hall interface states and corner states in multiple band gaps and at different frequencies. Meanwhile, the extended kagome designs introduce additional tunable structural parameters, offering enhanced flexibility for tailoring topological and localized states. By constructing two types of extended kagome lattices with distinct topological properties, we identify multiple groups of corner states, pseudospin Hall effect achieved by band folding or accidental degeneracy, and type-II corner states without long-range interactions. These results highlight extended kagome lattices as a flexible platform for topological waveguiding and multifrequency corner-state localization.

DOI: [10.1103/rt47-39ft](https://doi.org/10.1103/rt47-39ft)

I. INTRODUCTION

Topological phases give rise to a variety of novel phenomena and applications not only in condensed matter systems, but also in a broad range of classical platforms, including photonics [1–14], acoustics [15–18], mechanics [19,20], and electronics [21–25]. In classical systems, by leveraging time-reversal symmetry and avoiding extreme requirements such as strong magnetic bias, nonmagnetic topological phases, including pseudospin Hall phases [21,26–29], valley Hall phases [26,30], and higher-order topological insulators (HOTI) [31–40] have attracted broad interest. These phases enable direction-selective and robust waveguiding through pseudospin-orbit coupling, or by valley contrast when the K-point momentum is preserved [41]. In pseudospin Hall systems, the analogy between two-dimensional Dirac physics in quantum materials and classical platforms guides the control of Dirac bands and edge states, enabling robust waveguiding [42], pseudospin engineering [5,43–47], helical coupling, and lasing [48,49].

While pseudospin-Hall and valley-Hall responses mainly concern robust edge and interface transport, another important question is whether topological systems can also support richer corner-state localization. In particular, achieving

multiple corner modes within a single system, especially across multiple band gaps, has recently been shown to be possible [50], but remains challenging. In addition, type-II corner localization [51–53], whose modal distributions extend along the two edges adjacent to a corner, is often associated with next-nearest-neighbor or other long-range couplings. It is therefore of interest to ask whether such localized modes can be achieved in a simpler and more flexible setting. In this context, kagome lattices have attracted considerable attention as promising platforms with rich band-structure physics [54–60], although a pseudospin-Hall description is not naturally available in the conventional kagome setting. Motivated by these considerations, there is strong interest in robust topological platforms that support multiple localized modes and offer a broader design space for richer phenomena, yet realizing such systems remains challenging.

To address these challenges, we theoretically propose two new geometries of creating pseudospin-selection topological waveguiding based on kagome lattice, and adopt a tight-binding description to capture the essential topology of the proposed kagome-derived lattices. One of the extended kagome lattices forms twofold degeneracy of Dirac cones at the Γ point after band folding, broadening the accessible topological phenomena in kagome systems, such as pseudospin-selective guided modes and pseudospin-Hall-based adiabatic waveguiding metasurfaces [61]. The other system takes advantage of parametric tuning and achieves accidental degeneracy of double Dirac cones, enabling topological phases in a system that originally did not support them. Moreover, the platform of extended kagome systems supports rich localized-mode behavior, such as multiple groups of corner modes in multiple band gaps, in which type-II corner states are possible to emerge even without the existence

*These authors contributed equally.

†Contact author: mengyaoli@sz.tsinghua.edu.cn

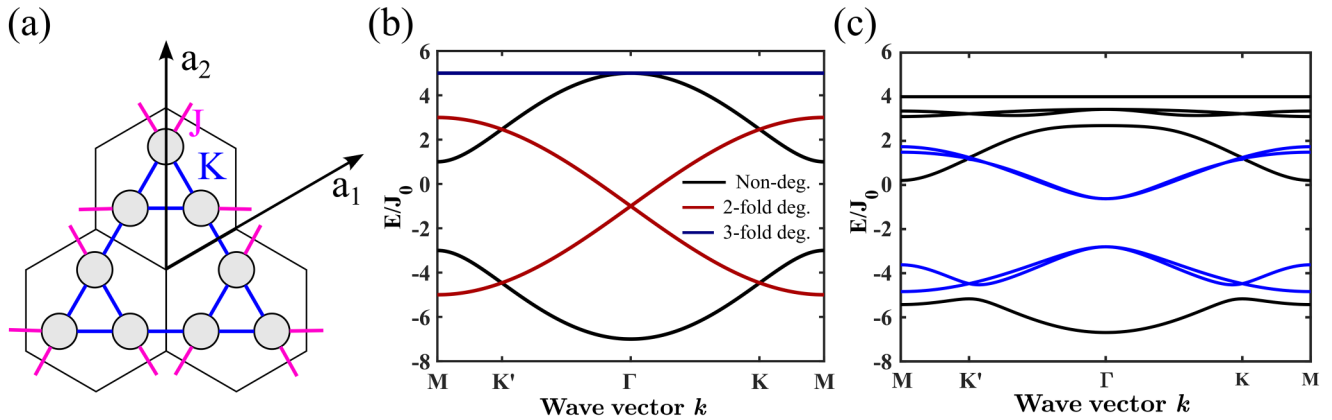


FIG. 1. Unit cells with different intracell coupling forms of ext-kagome-I. (a) The geometry of ext-kagome-I. Blue lines represent intracell couplings, and magenta lines represent intercell couplings. (b) Manually grouping three kagome unit cells and BZ folding of an unperturbed kagome lattice with nine sites per unit cell, where the configuration is as (a) but for the $J = K$ case. The colors indicate different degrees of degeneracy. (c) Dirac cone degeneracy breaking by introducing unbalanced intracell and intercell couplings. The intracell couplings have the same strength within each unit cell. Blue bands are highlighted as a pair of Dirac pseudospin bands that are degenerate at the Γ point ($K = -2.0$, $J = -1.0$).

of long-range couplings. These platforms open possibilities for robust on-chip waveguiding and trapping with enhanced modal integration, and may also support multifunctional topological lasers [62,63] by exploiting different edge or corner modes as distinct channels.

II. DESIGN AND ANALYTICS

The kagome lattice contains three sites per primitive unit cell and hosts a characteristic band structure consisting of one flat band and two Dirac cones at the K and K' valleys. To explore the folded Dirac physics at the Brillouin-zone center of kagome lattices, we group three primitive kagome cells into an enlarged nine-site cell. This enlargement reduces the reciprocal-lattice vectors, shrinks the Brillouin zone, and folds the original K and K' valleys onto the Γ point. Following this idea, we construct two kagome-derived lattice geometries, which we refer to as extended kagome lattices and denote as ext-kagome-I and ext-kagome-II, respectively.

The unit-cell configuration of the first type of extended kagome lattice (ext-kagome-I) is shown in Fig. 1(a), where three adjacent kagome unit cells, adopting a one upper and two lower configuration, are grouped into an enlarged cell. A reference band structure of the manually folded unperturbed kagome lattice, induced by the enlarged unit cell, is shown in Fig. 1(b). We emphasize that this Brillouin-zone folding is introduced only for comparison with the perturbed band structures, and the resulting degeneracies and band multiplicities should not be interpreted as the physical band properties of the unperturbed kagome lattice. Next, we introduce a lattice perturbation to break the translational symmetry and lift the Dirac-point degeneracy. Specifically, all intracell couplings are increased or decreased uniformly, which opens a band gap while preserving the twofold degeneracy of the Dirac cones at the Γ point. The corresponding band structure of ext-kagome-I is obtained from the tight-binding model (TBM). When the intracell/intercell couplings (K , J) are set to $K = -2.0$ and $J = -1.0$, the double-Dirac-cone degeneracy is

clearly observed in Fig. 1(c). In energy spectral representations throughout the work, we introduce a reference value $J_0 = 1.0$, set by the absolute value of the intercell coupling in this case. The doubly degenerate Dirac cones at the Γ point provide the essential condition for constructing an effective pseudospin description and the associated pseudospin Hall response, which serves as a classical wave analog of the Kramers-like degeneracy in time-reversal-invariant fermionic systems.

We next examine whether the other extended kagome geometry configuration can support similar pseudospin-related topological physics. The second type of extended kagome unit cell (ext-kagome-II) comprises two upper and one lower conventional kagome unit cells, as shown in Fig. 2(a). A natural symmetry-breaking perturbation is to apply the same strategy used in ext-kagome-I by uniformly expanding or shrinking the unit cell, so that all intracell couplings remain equal, as illustrated in Fig. 2(a). Similarly, we use the TBM to characterize the ext-kagome-II and calculate the band structure of the system. However, as shown in Fig. 2(b), the pseudospin degeneracy at the Γ point is lifted under this perturbation, so this modulation does not provide the condition required to support pseudospin Hall effect.

To construct pseudospin Hall band structures in the ext-kagome-II configuration, another perturbation strategy is applied to tune selected intracell couplings independently, while imposing constraints that preserve the C_{3v} symmetry, as shown in Fig. 2(c). Under this perturbation, an accidental double-Dirac-cone degeneracy can be realized at the Γ point, which provides the condition required for the pseudospin Hall effect. As an example, under parameter configuration $K_1 = -1.1$, $K_2 = K_3 = -0.6$, $K_4 = -1.8$, $J = -1.9$, the tight-binding calculation yields a double-Dirac-cone degeneracy at the Γ point, as shown in Fig. 2(d). We further find that this condition is not unique, and that multiple parameter combinations can give rise to an accidental double-Dirac-cone degeneracy at the Γ point while still producing distinct band structures. This flexibility allows different topological and

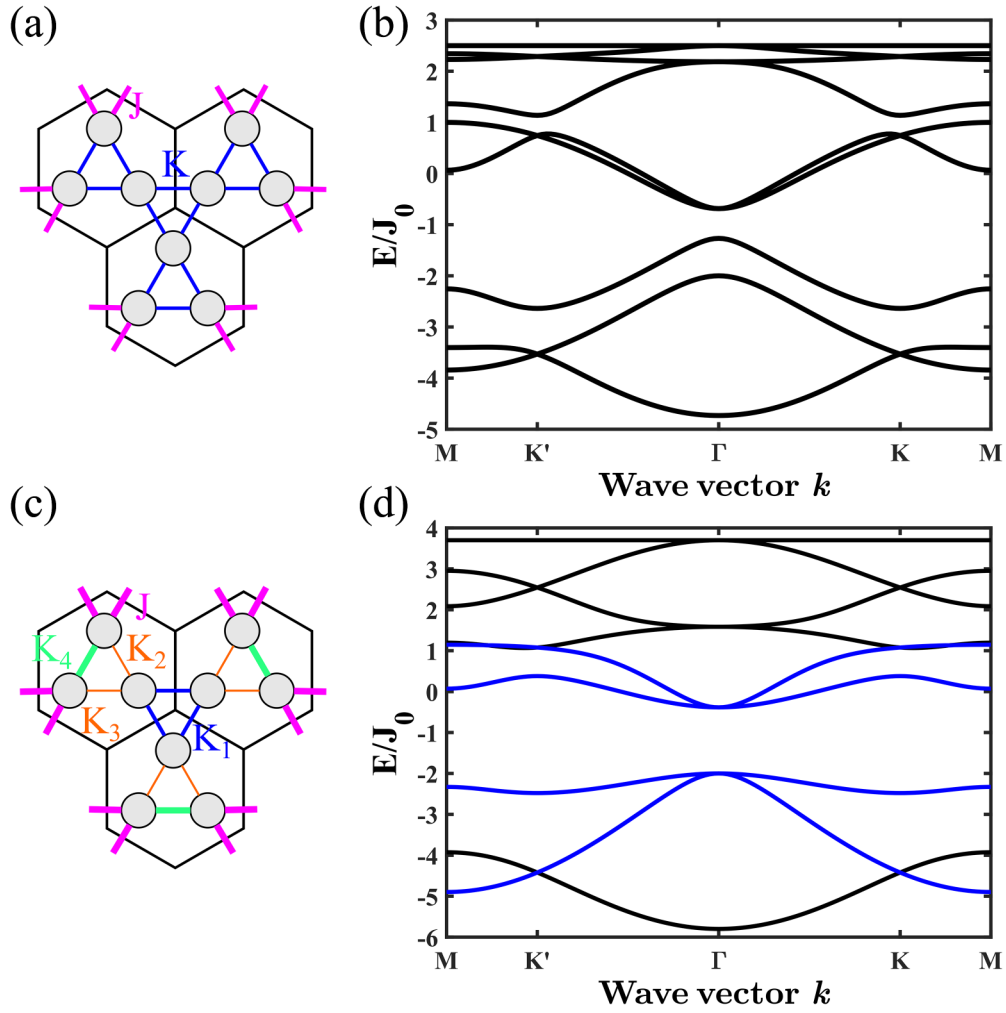


FIG. 2. Unit cells with different intracell coupling forms of ext-kagome-II. (a) The coupling of the ext-kagome-II lattice, whose intracell coupling strengths are the same. (b) The energy band structure of the geometry in (a), in which the doubly degenerated Dirac cone disappears. (c) The coupling of the ext-kagome-II lattice, whose intracell coupling strength is tunable under the C_{3v} unit-cell symmetry constraints, requires $K_2 = K_3$ in the ext-kagome-II lattices. (d) The energy band structure of the geometry in (c). The blue bands highlight the band pair forming a Dirac-type degeneracy at the Γ point. Different line colors between nearest-neighbor sites of (a) and (c) represent different kinds of coupling, and line thickness represents the strength of the coupling.

trivial domain pairs to be constructed within the same design framework. (see Sec. II of the Supplemental Material [64]).

In order to theoretically characterize extended kagome lattices and their topological properties, we start from the nearest-neighbor tight-binding model, derive the effective 4×4 Hamiltonian, and evaluate the corresponding Z_2 -type topological invariant, following an approach analogous to the analytical formalism of the conventional honeycomb-derived pseudospin Hall photonic crystal (Wu-Hu structure) [65]. The base vectors are $\vec{a}_1 = a(0, 1)$ and $\vec{a}_2 = a(\sqrt{3}/2, 1/2)$

(see Sec. I of Supplemental Material [64] and Fig. S1), where a is the lattice constant of ext-kagome-I unit cell. In the TBM, the system can approximately be described by Schrödinger equation as follows:

$$\hat{H}|\psi\rangle = E|\psi\rangle, \quad (1)$$

where the wave function under the Wannier basis is expressed as

$$|\psi\rangle = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9)^T \quad (2)$$

and the Hamiltonian for the system under TBM is equal to

$$H_1 = \begin{pmatrix} 0 & K & K & 0 & 0 & J e^{-i a k_y} & 0 & J e^{-i a k_y} & 0 \\ K & 0 & K & K & 0 & 0 & 0 & 0 & J e^{i(\frac{1}{2}\sqrt{3} a k_x - \frac{a k_y}{2})} \\ K & K & 0 & 0 & J e^{i(-\frac{1}{2}\sqrt{3} a k_x - \frac{a k_y}{2})} & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & K & K & 0 & 0 & J e^{i(\frac{1}{2}\sqrt{3} a k_x - \frac{a k_y}{2})} \\ 0 & 0 & J e^{i(\frac{1}{2}\sqrt{3} a k_x + \frac{a k_y}{2})} & K & 0 & K & J e^{i(\frac{1}{2}\sqrt{3} a k_x + \frac{a k_y}{2})} & 0 & 0 \\ J e^{i a k_y} & 0 & 0 & K & K & 0 & 0 & K & 0 \\ 0 & 0 & K & 0 & J e^{i(-\frac{1}{2}\sqrt{3} a k_x - \frac{a k_y}{2})} & 0 & 0 & K & K \\ J e^{i a k_y} & 0 & 0 & 0 & 0 & K & K & 0 & K \\ 0 & J e^{i(-\frac{1}{2}\sqrt{3} a k_x + \frac{a k_y}{2})} & 0 & J e^{i(-\frac{1}{2}\sqrt{3} a k_x + \frac{a k_y}{2})} & 0 & 0 & K & K & 0 \end{pmatrix}. \quad (3)$$

Here, the tight-binding Hamiltonian is formulated in a single-mode basis, where one effective localized mode is retained for each of the nine sublattice sites in the unit cell. To evaluate the topological properties of the system, we are aiming to obtain the effective Hamiltonian of the four bands of doubly degenerated Dirac cones near the Γ point. Other states far from the vicinity of the Dirac point should be ignored, but their interaction with the bands under study needs to be considered in the effective Hamiltonian.

Utilizing the representation transformation of a unitary matrix U_1 transforms the TBM Hamiltonian into an energy representation of H_{01} .

$$H^u = U_1^\dagger H U_1 = U_1^\dagger (H_{01} + H_{p1}) U_1 = H_{01}^u + H_{p1}^u \quad (4)$$

$$H_{01}^u = \begin{pmatrix} 4J & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2J & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2J & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2J & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2J & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & J & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & J & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J \end{pmatrix} \quad (5)$$

Define the subspace projection operator P_{sub} of four degenerating energy bands near the Dirac point and the projection operator $P_{\text{sub}}^\perp$ of its orthogonal complement. Decomposing H_{p1}^u into the component of orthogonal complement of the subspace H_{bd}^u and the component of interaction with the subspace and its orthogonal complement H_{bo}^u .

$$H_{p1}^u = H_{bd}^u + H_{bo}^u \quad (6)$$

$$H_{bd}^u = P_{\text{sub}} H_{p1}^u P_{\text{sub}} + P_{\text{sub}}^\perp H_{p1}^u P_{\text{sub}}^\perp \quad (7)$$

$$H_{bo}^u = P_{\text{sub}} H_{p1}^u P_{\text{sub}}^\perp + P_{\text{sub}}^\perp H_{p1}^u P_{\text{sub}} \quad (8)$$

We utilized the Schrieffer-Wolff transformation [66] to calculate H_{e1} :

$$H_{e1} = H_{01}^u + H_{bd}^u + \frac{1}{2} [S, H_{bo}^u] \quad (9)$$

where the anti-Hermitian operator S satisfies:

$$H_{bo}^u + [S, H_{01}^u] = 0. \quad (10)$$

Expanding H_{e1} as quadratic term about k_x, k_y around Γ point, and keep the linear term about $(K-J)$ assuming small perturbations when $K = J + o(K-J)$:

$$H_{e1} = \begin{pmatrix} \theta_1(k) + \varepsilon(k) & v_1(-k_x - i k_y) + \alpha(k_x - i k_y)^2 & \gamma(k_x - i k_y)^2 & \eta(k_x^2 + k_y^2) + \tau \\ v_1(-k_x + i k_y) + \alpha(k_x + i k_y)^2 & -\theta_1(k) + \varepsilon(k) & \eta(k_x^2 + k_y^2) + \tau & \kappa(k_x + i k_y)^2 \\ \gamma(k_x + i k_y)^2 & \eta(k_x^2 + k_y^2) + \tau & \theta_1(k) + \varepsilon(k) & v_1(k_x - i k_y) + \alpha(k_x + i k_y)^2 \\ \eta(k_x^2 + k_y^2) + \tau & \gamma(k_x - i k_y)^2 & v_1(k_x + i k_y) + \alpha(k_x - i k_y)^2 & -\theta_1(k) + \varepsilon(k) \end{pmatrix}, \quad (11)$$

where $\varepsilon(k) = (K-J)k^2/54 + (2K+J)/3$, $\theta_1(k) = \mu_1 + \beta_1 k^2$, $\mu_1 = K-J$, $\beta_1 = K/12$, $v_1 = (2K+7J)/18$, $\alpha = (4K+5J)/216$, $\gamma = 7(J-K)/108$, $\eta = -(4K+5J)/108$, $\kappa = 5(K-J)/108$, $\tau = (J-K)/3$.

In the process of calculating the topological invariant, we ignore the identity matrix H_I located on the diagonal, matrix located on the off-block-diagonal term H_{bo} , and keep terms of H_α to the first order of k . Thus, the effective Hamiltonian is decomposed into two block-diagonal parts, and the effective Hamiltonian has a similar form to the Bernevig-Hughes-Zhang model [46].

$$H_1 = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = H_{BHZ} \quad (12)$$

$$H_\pm^1 = \begin{pmatrix} \mu_1 + \beta_1 k^2 & v_1(-k_x \mp ik_y) \\ v_1(-k_x \pm ik_y) & -\mu_1 - \beta_1 k^2 \end{pmatrix} \quad (13)$$

The form of the $H_\pm$ is the same as the Dirac Hamiltonian, so there is an analytical solution for the spin Chern number of the system [47]. Lastly, we obtain the Z_2 invariant [67] of this system as

$$C_1 = \frac{1}{2}(\text{sgn}(K-J) + 1). \quad (14)$$

Accordingly, the expanded phase ($J < K < 0$) is topologically nontrivial ($C_1=1$), whereas the shrunk phase ($K < J < 0$) is topologically trivial ($C_1=0$).

In addition to the Z_2 invariant, the band inversion associated with the double-Dirac-cone degeneracy also supports the topological interpretation. Near the Γ point, the dipole modes $p_\pm$ and quadrupole modes $d_\pm$ lie on different doubly degenerate bands under trivial and topological cases (see Sec. III of Supplemental Material [64] and Fig. S5), which is a typical characterization of pseudospin Hall systems.

For ext-kagome-II structure, we adopt similar simplification methods to calculate the Z_2 invariant. Thus, the effective Hamiltonian is decomposed into two block-diagonal parts, and the effective Hamiltonian has a similar form to the Bernevig-Hughes-Zhang model. The form of the $H_\pm^2$ is the same as the Dirac Hamiltonian.

$$H_\pm^2 = \begin{pmatrix} \mu_2 + \beta_2 k^2 & v_2(-k_x \mp ik_y) \\ v_2(-k_x \pm ik_y) & -\mu_2 - \beta_2 k^2 \end{pmatrix} \quad (15)$$

The topological invariant of this system is as follows:

$$C_2 = \frac{1}{2}(\text{sgn}(\mu_2) - \text{sgn}(\beta_2)) \quad (16)$$

$$\begin{aligned} \mu_2 = [2J^2 - 6JK_3 - 6JK_4 - K_1(4J + 3K_2 + 3K_3 + 3K_4) + K_2(-6J + 5K_3 + 5K_4) \\ + 2K_1^2 + 4K_2^2 + 4K_3^2 + 4K_4^2 + 5K_3K_4]/[6(K_2 + K_3 + K_4)] \end{aligned} \quad (17)$$

$$\beta_2 = J[-6J/(2J + K_1) - 2J/(K_2 + K_3 + K_4) + 2K_1/(K_2 + K_3 + K_4) + 3]/12 \quad (18)$$

The H_{e2} is expanded to the quadratic term and linear term at the Γ point and $K_i = J$ ($i = 1, 2, 3, 4$), but there is greater freedom in the choice of multiple intracell coupling parameters K_i instead of the uniformly applied intracell coupling K in the ext-kagome-I. Note that only when the condition for all intracell coupling coefficients $K_i = J + o(K_i - J)$ meets, the approximation is valid.

We note that, due to the underlying band-folding construction, valley-derived components may be present in the proposed lattices [26]. However, in this work, the topological properties are discussed in a pseudospin framework, consistent with the calculated Z_2 invariant and the topological features at the Γ point. For completeness, a valley-resolved analysis is provided in Sec. I of Supplemental Material [64].

III. RESULTS

A. Topological pseudospin Hall Interface Modes in Extended Kagome lattices

To visualize the band structures of topological interface states and their mode distributions, we investigate a super-

cell in Fig. 3(a) consisting of three ext-kagome-I lattice domains with different intracell-to-intercell coupling ratios, arranged in a sandwich structure corresponding to a trivial-topological-trivial configuration. The middle layer of the supercell consists of topologically nontrivial ext-kagome-I structures ($K = -1.0$, $J = -1.5$), whereas the upper and lower layers are topologically trivial ext-kagome-I structures ($K = -1.5$, $J = -1.0$). Each layer consists of 12 unit cells with the same parameters. The corresponding band structure is shown in Fig. 3(b). The blue bands are interface states localized at the internal boundaries between the topological and trivial domains. These interface bands are twofold degenerate and originate from the sandwichlike configuration, which contains two equivalent interfaces. Similar to the Wu-Hu structure, the interface-state bands exhibit a small gap [29,68], because the abrupt geometrical change at the interface between the topological and trivial domains slightly reduces the lattice symmetry and induces weak mixing between the interface states. The corresponding mode profiles, i.e., the spatial distributions of the Bloch-eigenvector amplitudes in the tight-binding model, are shown in Fig. 3(c), where the

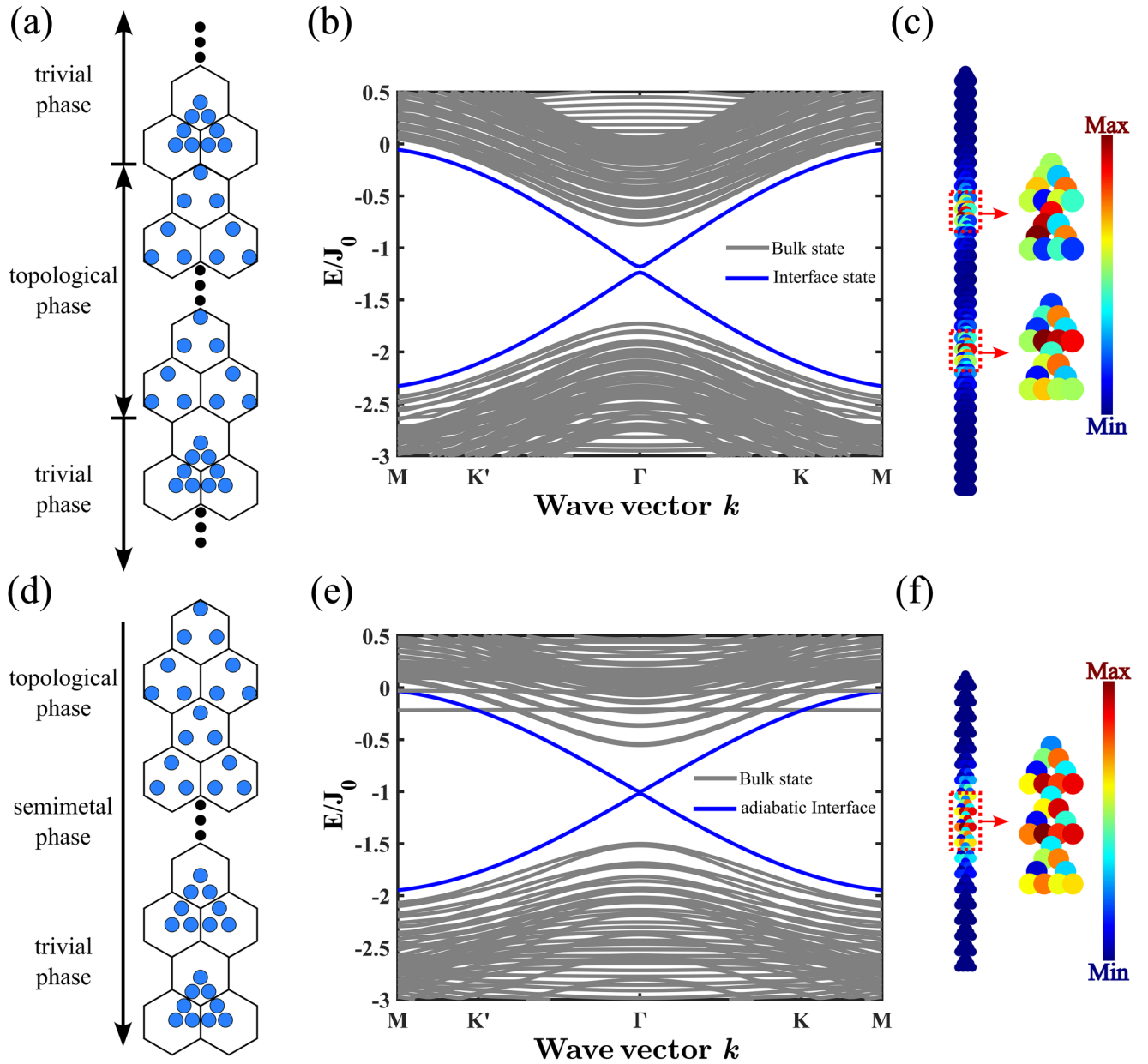


FIG. 3. Supercell of ext-kagome-I. (a) Geometry demonstration of a supercell composed of two kinds of ext-kagome-I that consists of topological extended kagome lattices ($K = -1.0$, $J = -1.5$) as the middle layer, while the upper and lower layers are trivial shrunk structures ($K = -1.5$, $J = -1.0$). (b) Band structure of the supercell in (a). Blue lines highlight the interface bands, and gray lines represent the bulk bands. (c) Mode distribution of the interface states in (b), the inset image on the right is an enlarged view of the interfaces between the topological and trivial domains. The amplitude of the mode profiles is color-coded in the figure. (d) Geometry demonstration of the supercell with adiabatically variable site positions in the unit cells. Unit cells from top to bottom are linearly varied from the topological to the trivial phase. (e) Band structure of the supercell in (d). Blue lines represent the adiabatic interface bands, and gray lines represent the bulk bands. (f) Mode profile for the adiabatic interface state in (e), the image on the right is an enlarged view of the interface between the topological phase, Dirac semimetal, and the trivial phase. The color bar shows the amplitude of the mode profiles.

states are localized at the two internal boundaries between the topological and trivial domains. The insets of Fig. 3(c) show the mode profiles of the two unit cells adjacent to the interfaces between extended kagome lattices with different topological phases.

Motivated by the controllable lattice modulation in ext-kagome-I, we further investigate whether this structure can serve as an adiabatic waveguide platform, analogous to the Wu-Hu structure. The adiabatic linear transition of the

topological interfaces between unit cells in a supercell is an emerging method that separates bulk modes to reduce the gap of the modes that emerge at the interface, thereby achieving an almost perfect pseudospin Hall effect [61,69]. We construct another supercell in Fig. 3(d) that consists of unit cells with adiabatically varying intercell and intracell couplings. From top to bottom, these couplings are varied linearly, producing a transition from the expanded configuration through the balanced configuration to the shrunk configuration. The band

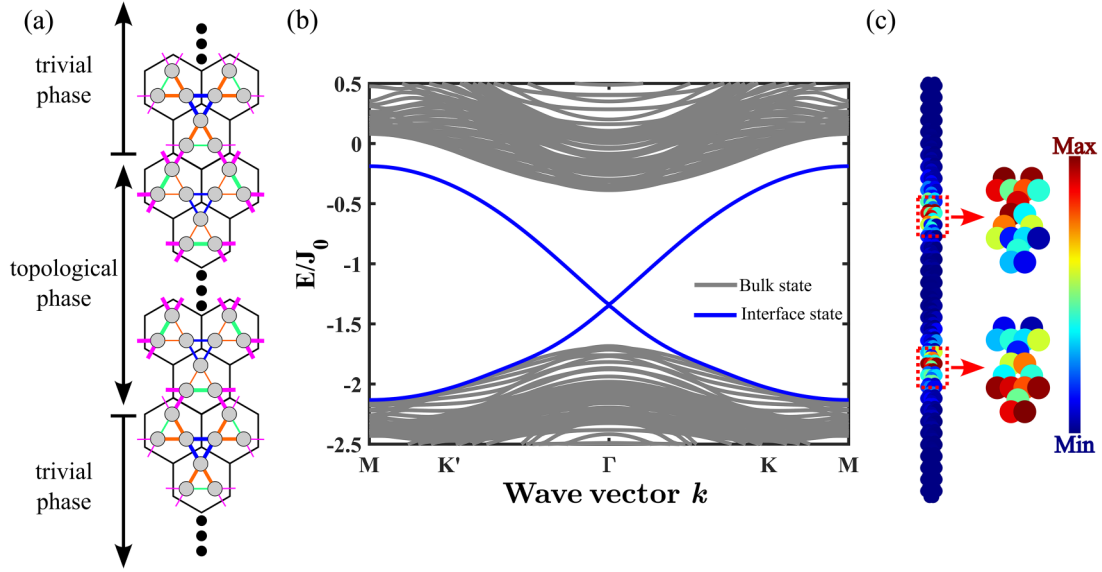


FIG. 4. Supercell and interface states of the ext-kagome-II with domains. (a) Geometry demonstration of a supercell composed of the ext-kagome-II lattices with different intracell coupling strengths. (b) The band structure of the supercell in (a). Gray lines represent bulk bands, and blue lines denote interface bands. (c) The mode profile of the interface state. The inset illustrations are enlarged images of the interfaces.

structure of the supercell in Fig. 3(e) shows a significantly smaller gap between the adiabatic waveguide modes at the interface, whose size is approximately 3/10 of the gap of the interface modes shown in Fig. 3(b). Since one interface is formed during adiabatic variation, the blue bands in Fig. 3(e) no longer have identical copies. Meanwhile, the profile of the waveguide mode with adiabatically changing localized at the interfaces is demonstrated in Fig. 3(f). The insets in Fig. 3(f) show the interface of the ext-kagome-I with expanded, balanced, and shrunk unit cells. Compared to the abrupt interface transformation of the step domain mode profile, the adiabatic linear interface transformation increases the width of the localized domain, which is also tunable by the gradient and profile of adiabatic parameters. The adiabatic metasurface method provides possibilities to design new pseudospin waveguide devices with better interface mode control for the extended kagome platform.

For the ext-kagome-II case, we employ a similar sandwich supercell structure as in Fig. 3(a) above, which consists of a middle topological layer surrounded by upper and lower trivial lattices. However, unlike in the ext-kagome-I, there is no simple coupling reversal method to achieve band inversion and ensure topological transitions. As a result, we need to identify two different parameter configurations of the ext-kagome-II that belong to different topological phases and have nearly coincident bulk band gaps. We identify the different nontrivial or trivial topological phases by calculating Z_2 invariants under different parameters, then we scan through the parameter space to select a good match for the bulk band gaps. The pairs of parameters we used are $K_1 = -1.1$, $K_2 = -0.6$, $K_3 = -0.6$, $K_4 = -1.8$, $J = -1.9$ for nontrivial phase and $K_1 = -1.8$, $K_2 = -1.6$, $K_3 = -1.6$, $K_4 = -0.8$, $J = -0.6$ for trivial phase, and constructed the supercell as shown in Fig. 4(a). Each layer consists of 12 unit cells. The interface states (blue line) associated with the pseudospin Hall effect are presented in Fig. 4(b), and the corresponding mode profile

of the interface states is exhibited in Fig. 4(c). The insets in Fig. 4(c) present the enlarged views of the interface of the ext-kagome-II supercell.

To verify the helical nature and pseudospin-selective excitation of the interface states in the extended kagome structures, we construct waveguides and analytically apply pseudospin-selective sources to observe the interface mode propagation. We introduce a helical excitation source at the middle of the domain wall within the TBM. The excitation is implemented by applying complex-valued on-site sources through the coupled-mode theory. The pseudospin-selective source is obtained by identifying the eigenmodes associated with the left- and right-propagating branches, where each source carries a phase term of $2n\pi/3$ (with $n = 0, 1, 2$). By tuning the phase sequence, left- or right-handed helicity can be realized, which enables selective coupling to opposite pseudospin interface states. As shown in Fig. 5, all presented waveguides consist of a topological upper nontrivial domain and a lower trivial domain. Every domain is composed of 11 rows and 21 columns of unit cells. The excitation source is placed in the middle position of the waveguides. The source of the ext-kagome-I waveguide is applied to the lattice sites of the topological domain on the bottom side of the interface. The circular arrows in the figures represent the excitation sources in the waveguides [5,70,71]. Note that the plotted Bloch-eigenvector amplitudes represent a reduced-order modal distribution, rather than the full continuum field profile.

When the source frequency lies within the interface-state frequency range, unidirectional propagation can be excited. As shown in Figs. 5(a) and 5(b), the mode propagates along the left and right directions, respectively, and the amplitude gradually attenuates, which is introduced by the intrinsic loss term in the Hamiltonian to avoid the mode circling back to the source. The excitation with opposite handedness propagates in the opposite direction, which demonstrates the ext-kagome-I

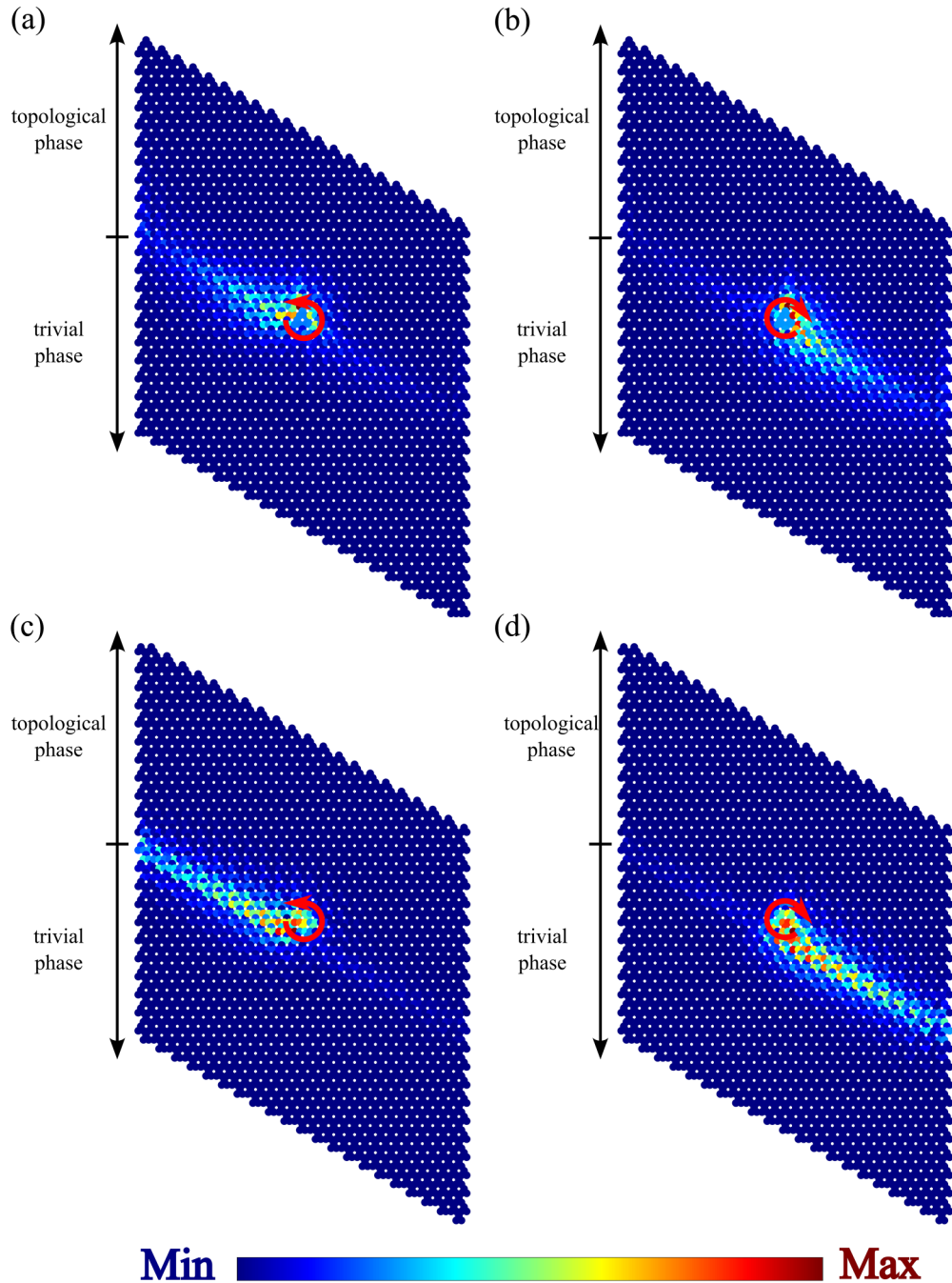


FIG. 5. The pseudospin excitation of the extended kagome. (a) The pseudospin excitation of ext-kagome-I. (b) The excitation with the opposite pseudospin to that in (a) of ext-kagome-I. (c) The pseudospin excitation of ext-kagome-II. (d) The excitation with opposite pseudospin to that in (c) of ext-kagome-II. The color bar in (a)–(d) shows the amplitude of the excitation mode. The circular arrows denote the chiral excitation sources, with the arrows indicating the handedness (left-handed or right-handed) of the circular polarizations.

possessing the pseudospin Hall effect, similar to the Wu-Hu structure [12]. As shown in Figs. 5(c) and 5(d), the waveguiding behavior of the ext-kagome-II is similar to that of ext-kagome-I. When we apply the source with opposite handedness, the propagation direction changes. Similarly, the propagation changing with the source handedness reversal demonstrates that ext-kagome-II also hosts the pseudospin Hall effect and helical interface states. A slight back propagation is observed at the probe located five unit cells away from the source, with backscatter ratio (measured by flux) of

approximately 8% (ext-kagome-I) and 1.6% (ext-kagome-II) of backscattering, which might be the result of imperfect interface modes due to the band gaps between interface modes, leading to scattering between pseudospin channels.

B. Corner States in Extended Kagome Lattices

Conventional breathing kagome lattices are known to hold Wannier-type higher-order topology, where corner-localized

modes, including type-I corner states and type-II corner states, are discovered. To explore the existence of corner-localized modes in extended kagome lattices, we investigate finite triangular structures and their mode distributions. First, we analytically construct a finite-sized triangular structure of the ext-kagome-I lattice. Each side of the finite triangular structure is composed of 12 topological unit cells with parameter ($J/K = 5$), and without any long-range couplings.

The energy spectrum of the finite triangular structure, depicted in Fig. 6(a), reveals the existence of the corner states. Figures 6(c) and 6(d) are the enlarged eigenfrequency images of the red dashed box and green dashed box in Fig. 6(a), respectively. Interestingly, not only multiple groups of type-I corner states (red dots) are observed, but this system also exhibits type-II corner states (green dots), which are not localized at the outermost corner, but near the two adjacent unit cells next to the outermost ones. Previous research [51,52] has shown that conventional kagome possesses one group of type-I corner states and two groups of type-II corner states under long-range interactions. Here, our system simultaneously hosts both type-I corner states and type-II corner states in the absence of long-range interactions. From the energy spectrum, the structure of ext-kagome-I supports four sets of type-I corner states with different frequencies inside different band gaps, and two groups of type-II corner states. All groups of corner modes demonstrated here clearly lie in the band gap, and type-II corner states have a similar profile of emerging from the interface spectrum, like in conventional kagome systems. For comparison, we construct the same finite triangular structure using the trivial phase of the reverse parameters ($K/J = 5$) in Fig. 6(b) that does not support any corner states or interface states. The interface states of different frequencies in Figs. 6(c) and 6(d) (see Sec. V of Supplemental Material [64]) are similar to the conventional kagome structure. Note that these finite structures do not have inversion symmetries due to the periodic directions and geometry of unit cells, which induce different corner mode distributions compared to conventional breathing kagome lattices.

Next, we demonstrate the mode profiles of type-I corner states in Figs. 6(e)–6(h) and type-II corner states in Figs. 6(i)–6(j) at different frequencies. From the mode profiles, we observe that all these modes are well localized at the corners. The nearly threefold degeneracy of the corner states originates from the underlying C_3 symmetry of the finite structure, as well as the spatial separation of the corners, which suppresses their mutual interactions. This symmetry makes the three corners locally equivalent, so that the corresponding corner states appear with the same eigenfrequency. Interestingly, some type-I corner states are highly localized to the top corners of unit cells at the corner, and the type-I corner states in Fig. 6(f) are localized to the dimer beside the corner. Figures 6(i)–6(j) demonstrates one type of the type-II corner states, which occurs beside the unit cell at the corner. All the inset illustrations are the enlarged mode profiles of corner states for clarity.

To demonstrate the existence of the corner states in the ext-kagome-II system, we also construct a finite triangular-shaped ext-kagome-II structure with each edge composed of

12 unit cells with a set of parameters that places the lattice in the topological regime ($K_1 = -2.0$, $K_2 = -1.2$, $K_3 = -1.2$, $K_4 = -0.8$, $J = -1.5$). The energy spectrum of the finite triangular structure in Fig. 7(a) demonstrates the existence of corner states (red dots). As shown in Fig. 7(b), the corner states at lower frequency emerge between two gapped edge states. As shown in Fig. 7(c), the corner states at higher frequency emerge between bulk states and edge states. Then we demonstrate the mode profiles of corner states corresponding to different frequencies in Figs. 7(d) and 7(e). The lower-frequency corner states are mainly concentrated on the two outermost sites. By contrast, the higher-frequency corner mode profiles have their maximum on one of the inner sites of the corner unit cell. Moreover, we find that the mode profiles of the corner states are slightly less confined, due to their location in smaller band gaps. We have also examined multiple parameter sets that are identified as topological by the emergence of edge states in the band structure, and found that they support similar corner states.

On the relation between the emergence of corner states and higher-order topology, we need to clarify that the higher-order topology properties in extended kagome lattices are unknown, despite showing very similar properties with HOTI corner modes, including band structures, mode distributions, and robustness to disorders (see Sec. VI of Supplemental Material [64]). While we conducted Wannier center and bulk polarization calculations (see Sec. IV of Supplemental Material [64]), and interestingly, a large bulk polarization indicates bulk-edge-corner correspondence as far as we investigated, a quantized bulk polarization needed for HOTI classification cannot be steadily found and shows parameter dependence in both cases. The lack of robust quantization of the bulk polarization may be related to the presence of multiple groups of corner states in the extended kagome systems, some of which coexist within a single bulk band gap. To our knowledge, existing theoretical frameworks do not yet provide a means to characterize the individual groups separately in terms of their bulk signatures and topological origins. We would also like to stress that, based on the theoretical tools currently available to us, we do not yet have a complete understanding of the physical origin of these nonquantized Wannier-center configurations. While the two-parameter regimes exhibit clearly distinguishable Wannier-geometry patterns, these differences do not correspond to any established quantized topological invariant or symmetry-based classification known in the literature. Developing a theoretical framework that can systematically interpret such nonquantized Wannier structures and multifrequency corner states requires further investigation, which we regard as an interesting direction for future work.

IV. CONCLUSIONS

In this work, we proposed and constructed a new platform for pseudospin Dirac physics and rich corner modes in two types of kagome-based structural systems. By varying the unit-cell selection and coupling tuning, the two systems exhibit distinct topological properties, together with multiple groups of corner modes distributed in different band gaps. We

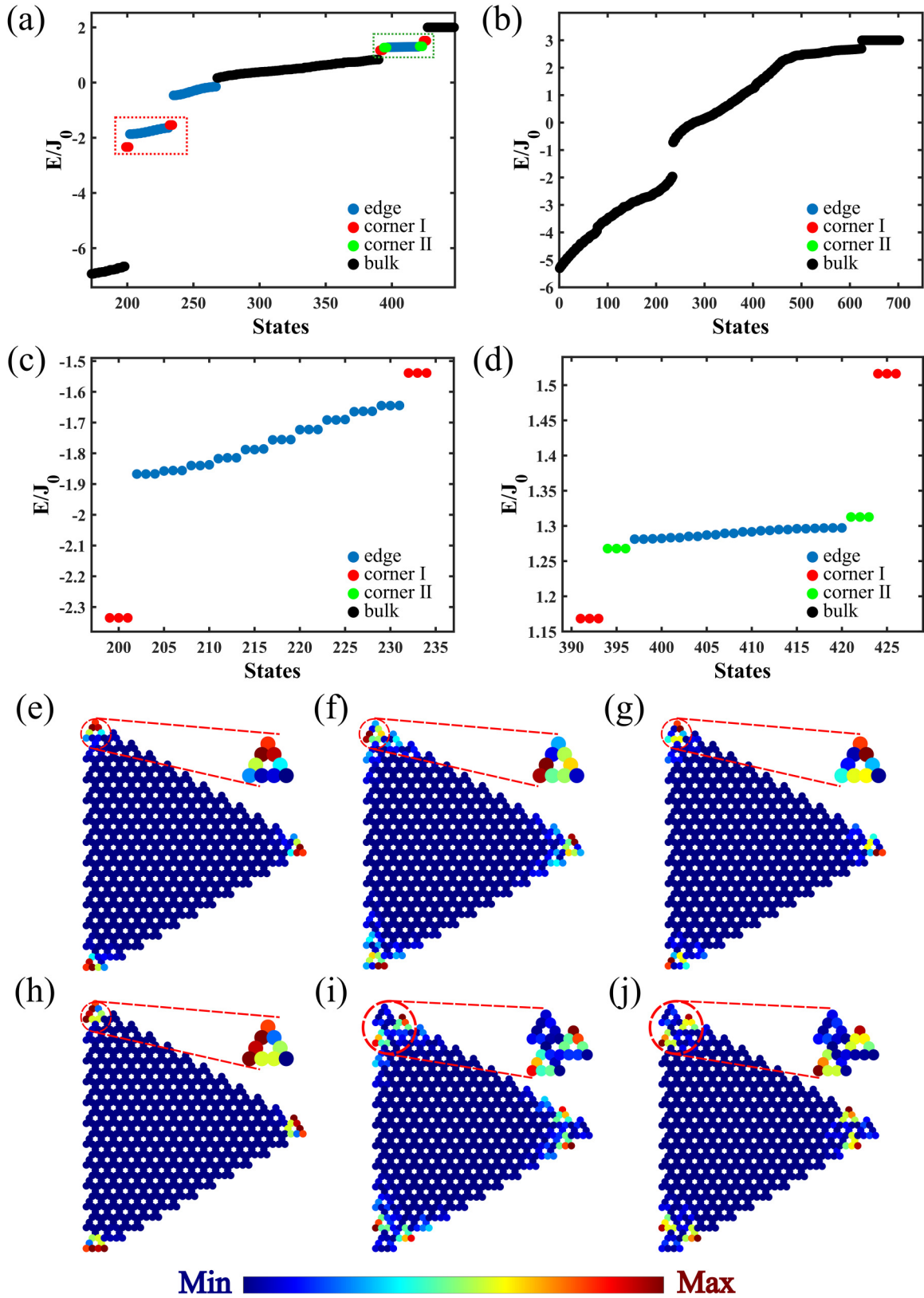


FIG. 6. The finite structure of ext-kagome-I. (a) The energy spectrum of the ext-kagome-I triangular array for $J/K = 5$. Black, blue, red, and green dots represent the bulk states, edge states, type-I corner states, and type-II corner states, respectively. (b) The energy spectrum of a triangular-shaped array composed of the trivial phase for $K/J = 5$. (c) Enlarged image of the red dashed box in (a). (d) The enlarged image of the green dashed box in (a). (e)–(h) Mode profiles of the type-I corner states with frequencies from low to high corresponding to those in (a). (i)–(j) Mode profiles of the type-II corner states with frequencies from low to high. The color bar shows the amplitude of the mode profile.

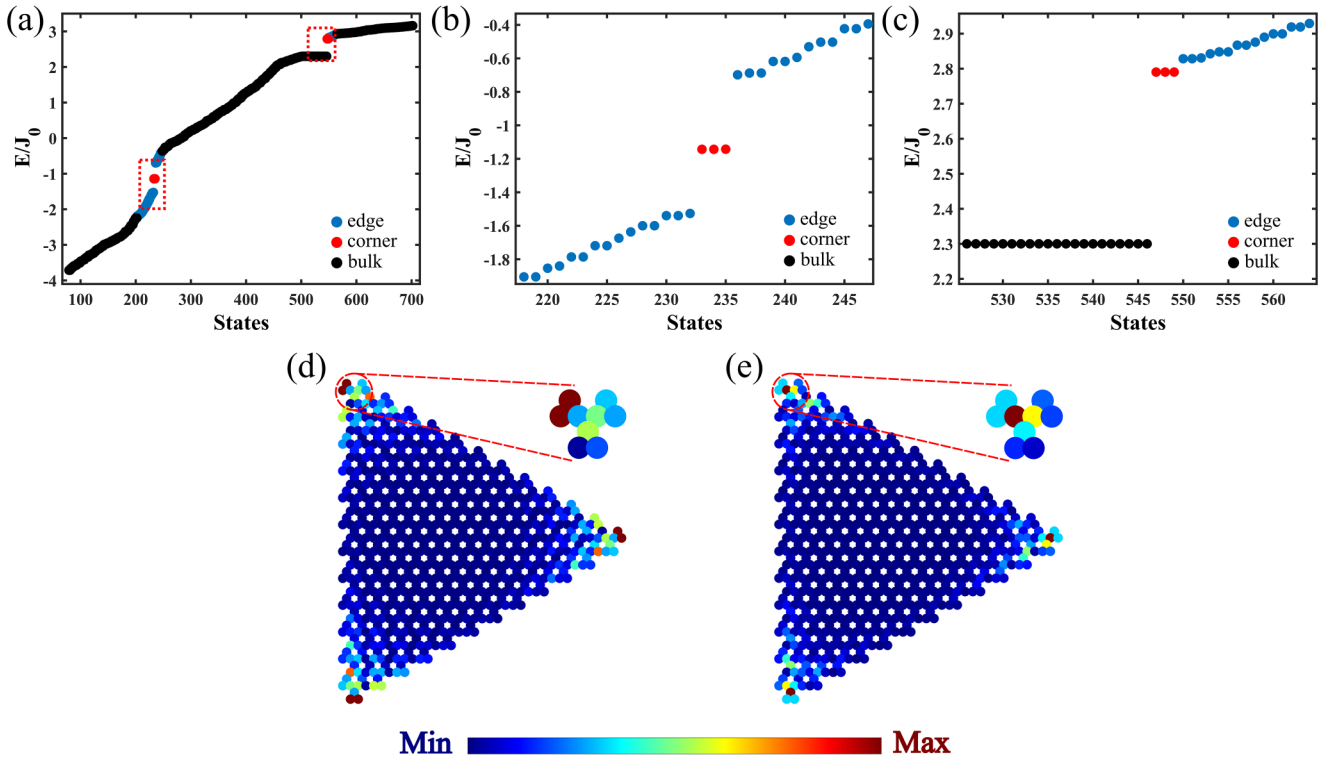


FIG. 7. The finite structure of ext-kagome-II. (a) The energy spectrum of the ext-kagome-II triangular-shaped array for $K_1 = -2.0$, $K_2 = -1.2$, $K_3 = -1.2$, $K_4 = -0.8$, and $J = -1.5$. Black, blue, and red dots represent the bulk states, edge states, and corner states, respectively. (b) The enlarged image of the lower red dashed box in (a). (c) The enlarged image of the upper red dashed box in (a). (d) Mode profiles of the corner states corresponding to the lower frequency in (b). (e) Mode profiles of the corner state corresponding to the higher frequency in (c). The color bar shows the amplitude of the mode profile in (d)–(e).

further show that type-II corner states can be achieved in these systems without long-range couplings or local corner perturbations, providing additional routes to realize corner-localized modes with different mode profiles. Since the proposed systems can be tuned toward kagome and breathing-kagome lattices, they also offer broader possibilities for topological waveguiding and trapping, as preliminarily illustrated by the adiabatic waveguide example. Our results also suggest that pseudospin-Hall-like interface states can be realized in reduced-symmetry systems such as those with C_{3v} symmetry instead of C_{6v} . Because the designs are based on generic lattices and coupling engineering, the proposed platform is compatible with fabrication approaches similar to those used in existing topological photonic and other classical-wave structures. It may therefore be realizable in photonic, acoustic, mechanical, and electronic systems, and may provide a useful route toward multifunctional topological waveguides and cavities with greater design freedom.

ACKNOWLEDGMENTS

M.L. acknowledges valuable discussions with Alexander Khanikaev. This project acknowledges financial support from the National Key R&D Program of China (Grant No. 2023YFB3811400), Shenzhen Science and Technology Innovation Commission (Grant No. RCBS20231211090547075), National Natural Science Foundation of China (Grant No. 12404435), the Guangdong Science and Technology Commission (Grant No. 2025A1515010774), and Tsinghua Shenzhen International Graduate School Scientific Research Startup Funding (Grant No. QD2023004C).

The authors declare no conflict of interest.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

- [1] S. Raghu and F. D. M. Haldane, Analogs of quantum-Hall-effect edge states in photonic crystals, *Phys. Rev. A* **78**, 033834 (2008).
- [2] F. D. M. Haldane and S. Raghu, Possible realization of directional optical waveguides in photonic crystals with broken time-reversal symmetry, *Phys. Rev. Lett.* **100**, 013904 (2008).

- [3] Z. Wang, Y. Chong, J. D. Joannopoulos, and M. Soljačić, Observation of unidirectional backscattering-immune topological electromagnetic states, *Nature (London)* **461**, 772 (2009).
- [4] M. Hafezi, E. A. Demler, M. D. Lukin, and J. M. Taylor, Robust optical delay lines with topological protection, *Nat. Phys.* **7**, 907 (2011).

- [5] A. B. Khanikaev, S. Hossein Mousavi, W.-K. Tse, M. Kargarian, A. H. MacDonald, and G. Shvets, Photonic topological insulators, *Nat. Mater.* **12**, 233 (2012).
- [6] K. Fang, Z. Yu, and S. Fan, Realizing effective magnetic field for photons by controlling the phase of dynamic modulation, *Nat. Photonics* **6**, 782 (2012).
- [7] K. Fang and S. Fan, Controlling the flow of light using the inhomogeneous effective gauge field that emerges from dynamic modulation, *Phys. Rev. Lett.* **111**, 203901 (2013).
- [8] Y. Plotnik, *et al.*, Observation of unconventional edge states in 'photonic graphene', *Nat. Mater.* **13**, 57 (2013).
- [9] M. C. Rechtsman, J. M. Zeuner, Y. Plotnik, Y. Lumer, D. Podolsky, F. Dreisow, S. Nolte, M. Segev, and A. Szameit, Photonic Floquet topological insulators, *Nature (London)* **496**, 196 (2013).
- [10] X. Cheng, C. Jouvaud, X. Ni, S. H. Mousavi, A. Z. Genack, and A. B. Khanikaev, Robust reconfigurable electromagnetic pathways within a photonic topological insulator, *Nat. Mater.* **15**, 542 (2016).
- [11] T. Ozawa *et al.*, Topological photonics, *Rev. Mod. Phys.* **91**, 015006 (2019).
- [12] L. H. Wu and X. Hu, Scheme for achieving a topological photonic crystal by using dielectric material, *Phys. Rev. Lett.* **114**, 223901 (2015).
- [13] L. Lu, J. D. Joannopoulos, and M. Soljačić, Topological photonics, *Nat. Photonics* **8**, 821 (2014).
- [14] L. Lu, J. D. Joannopoulos, and M. Soljačić, Topological states in photonic systems, *Nat. Phys.* **12**, 626 (2016).
- [15] Z. Yang, F. Gao, X. Shi, X. Lin, Z. Gao, Y. Chong, and B. Zhang, Topological acoustics, *Phys. Rev. Lett.* **114**, 114301 (2015).
- [16] M. Xiao, G. Ma, Z. Yang, P. Sheng, Z. Q. Zhang, and C. T. Chan, Geometric phase and band inversion in periodic acoustic systems, *Nat. Phys.* **11**, 240 (2015).
- [17] S. D. Huber, Topological mechanics, *Nat. Phys.* **12**, 621 (2016).
- [18] X. Ni, M. Weiner, A. Alu, and A. B. Khanikaev, Observation of higher-order topological acoustic states protected by generalized chiral symmetry, *Nat. Mater.* **18**, 113 (2019).
- [19] F. Ma, Z. Tang, X. Shi, Y. Wu, J. Yang, D. Zhou, Y. Yao, and F. Li, Nonlinear topological mechanics in elliptically geared isostatic metamaterials, *Phys. Rev. Lett.* **131**, 046101 (2023).
- [20] T. Shah, C. Brendel, V. Peano, and F. Marquardt, Colloquium: Topologically protected transport in engineered mechanical systems, *Rev. Mod. Phys.* **96**, 021002 (2024).
- [21] C. L. Kane and E. J. Mele, Quantum spin Hall effect in graphene, *Phys. Rev. Lett.* **95**, 226801 (2005).
- [22] C. L. Kane and E. J. Mele, Z_2 topological order and the quantum spin Hall effect, *Phys. Rev. Lett.* **95**, 146802 (2005).
- [23] D. Xiao, W. Yao, and Q. Niu, Valley-contrasting physics in graphene: Magnetic moment and topological transport, *Phys. Rev. Lett.* **99**, 236809 (2007).
- [24] M. Z. Hasan and C. L. Kane, Colloquium: Topological insulators, *Rev. Mod. Phys.* **82**, 3045 (2010).
- [25] X.-L. Qi and S.-C. Zhang, Topological insulators and superconductors, *Rev. Mod. Phys.* **83**, 1057 (2011).
- [26] M. Saba, S. Wong, M. Elman, S. S. Oh, and O. Hess, Nature of topological protection in photonic spin and valley Hall insulators, *Phys. Rev. B* **101**, 054307 (2020).
- [27] B. A. Bernevig and S.-C. Zhang, Quantum spin Hall effect, *Phys. Rev. Lett.* **96**, 106802 (2006).
- [28] D. Hsieh, D. Qian, L. Wray, Y. Xia, Y. S. Hor, R. J. Cava, and M. Z. Hasan, A topological Dirac insulator in a quantum spin Hall phase, *Nature (London)* **452**, 970 (2008).
- [29] L. H. Wu and X. Hu, Topological properties of electrons in honeycomb lattice with detuned hopping energy, *Sci. Rep.* **6**, 24347 (2016).
- [30] K. Qian, D. J. Apigo, C. Prodan, Y. Barlas, and E. Prodan, Topology of the valley-Chern effect, *Phys. Rev. B* **98**, 155138 (2018).
- [31] Q. Wei, X. Zhang, W. Deng, J. Lu, X. Huang, M. Yan, G. Chen, Z. Liu, and S. Jia, 3D hinge transport in acoustic higher-order topological insulators, *Phys. Rev. Lett.* **127**, 255501 (2021).
- [32] C.-H. Xia, H.-S. Lai, X.-C. Sun, C. He, and Y.-F. Chen, Experimental demonstration of bulk-hinge correspondence in a three-dimensional topological Dirac acoustic crystal, *Phys. Rev. Lett.* **128**, 115701 (2022).
- [33] B. Xie, H.-X. Wang, X. Zhang, P. Zhan, J.-H. Jiang, M. Lu, and Y. Chen, Higher-order band topology, *Nat. Rev. Phys.* **3**, 520 (2021).
- [34] A. Vakulenko, S. Kiriushchikina, M. Wang, M. Li, D. Zhirihin, X. Ni, S. Guddala, D. Korobkin, A. Alu, and A. B. Khanikaev, Near-field characterization of higher-order topological photonic states at optical frequencies, *Adv. Mater.* **33**, e2004376 (2021).
- [35] F. Schindler, A. M. Cook, M. G. Vergniory, Z. Wang, S. S. P. Parkin, B. A. Bernevig, and T. Neupert, Higher-order topological insulators, *Sci. Adv.* **4**, eaat0346 (2018).
- [36] H. Xue, Y. Yang, F. Gao, Y. Chong, and B. Zhang, Acoustic higher-order topological insulator on a kagome lattice, *Nat. Mater.* **18**, 108 (2019).
- [37] B. Xie, G. Su, H. F. Wang, F. Liu, L. Hu, S. Y. Yu, P. Zhan, M. H. Lu, Z. Wang, and Y. F. Chen, Higher-order quantum spin Hall effect in a photonic crystal, *Nat. Commun.* **11**, 3768 (2020).
- [38] A. El Hassan, F. K. Kunst, A. Moritz, G. Andler, E. J. Bergholtz, and M. Bourennane, Corner states of light in photonic waveguides, *Nat. Photonics* **13**, 697 (2019).
- [39] K. Prabith, G. Theocharis, and R. Chaunsali, Nonlinear corner states in a topologically nontrivial kagome lattice, *Phys. Rev. B* **110**, 104307 (2024).
- [40] J. Jiang, Z. Ji, F. Mei, J. Ma, L. Xiao, and S. Jia, Ultracold-molecule higher-order topological phases induced by long-range dipolar interactions in optical-tweezer lattices, *Phys. Rev. A* **110**, 013309 (2024).
- [41] N. P. Armitage, E. J. Mele, and A. Vishwanath, Weyl and Dirac semimetals in three-dimensional solids, *Rev. Mod. Phys.* **90**, 015001 (2018).
- [42] Q. Chen, L. Zhang, S. Xu, Z. Wang, E. Li, Y. Yang, and H. Chen, Robust waveguiding in substrate-integrated topological photonic crystals, *Appl. Phys. Lett.* **116**, 231106 (2020).
- [43] X.-D. Chen, Z.-L. Deng, W.-J. Chen, J.-R. Wang, and J.-W. Dong, Manipulating pseudospin-polarized state of light in dispersion-immune photonic topological metacrystals, *Phys. Rev. B* **92**, 014210 (2015).
- [44] Z.-G. Chen, J. Mei, X.-C. Sun, X. Zhang, J. Zhao, and Y. Wu, Multiple topological phase transitions in a gyromagnetic photonic crystal, *Phys. Rev. A* **95**, 043827 (2017).
- [45] Q. Yan, E. Cao, Q. Sun, Y. Ao, X. Hu, X. Shi, Q. Gong, and H. Misawa, Near-field imaging and time-domain dynamics of photonic topological edge states in plasmonic nanochains, *Nano Lett.* **21**, 9270 (2021).

- [46] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Quantum spin Hall effect and topological phase transition in HgTe quantum wells, *Science* **314**, 1757 (2006).
- [47] S. Barik, A. Karasahin, C. Flower, T. Cai, H. Miyake, W. DeGottardi, M. Hafezi, and E. Waks, A topological quantum optics interface, *Science* **359**, 666 (2018).
- [48] M. A. Bandres, S. Wittek, G. Harari, M. Parto, J. Ren, M. Segev, D. N. Christodoulides, and M. Khajavikhan, Topological insulator laser: Experiments, *Science* **359**, eaar4005 (2018).
- [49] Z. K. Shao, H. Z. Chen, S. Wang, X. R. Mao, Z. Q. Yang, S. L. Wang, X. X. Wang, X. Hu, and R. M. Ma, A high-performance topological bulk laser based on band-inversion-induced reflection, *Nat. Nanotechnol.* **15**, 67 (2020).
- [50] Z. Xiong, Y. Liu, Z.-K. Lin, Z.-L. Kong, and J.-H. Jiang, Triband higher-order topological insulators, *Phys. Rev. Appl.* **24**, 024049 (2025).
- [51] M. Li, D. Zhirihin, M. Gorlach, X. Ni, D. Filonov, A. Slobozhanyuk, A. Alù, and A. B. Khanikaev, Higher-order topological states in photonic kagome crystals with long-range interactions, *Nat. Photonics* **14**, 89 (2019).
- [52] Y.-Z. Li, Z.-F. Liu, X.-W. Xu, Q.-P. Wu, X.-B. Xiao, M.-R. Liu, L.-L. Chang, and R.-L. Zhang, Topological bounded corner states induced by long-range interactions in Kagome lattice, *Europhys. Lett.* **132**, 20005 (2020).
- [53] L. Tao, Y. Liu, X. Zhou, L. Du, M. Li, R. Ji, K. Song, and X. Zhao, Multi-type topological states in higher-order photonic insulators based on kagome metal lattices, *Adv. Opt. Mater.* **11**, 2300986 (2023).
- [54] L.-W. Wang, Z.-K. Lin, and J.-H. Jiang, Non-Hermitian topological phases and skin effects in kagome lattices, *Phys. Rev. B* **108**, 195126 (2023).
- [55] M. Li, X. Ni, M. Weiner, A. Alù, and A. B. Khanikaev, Topological phases and nonreciprocal edge states in non-Hermitian Floquet insulators, *Phys. Rev. B* **100**, 045423 (2019).
- [56] Z. Li, *et al.*, Realization of flat band with possible nontrivial topology in electronic Kagome lattice, *Sci. Adv.* **4**, eaau4511 (2018).
- [57] J.-X. Yin, B. Lian, and M. Z. Hasan, Topological kagome magnets and superconductors, *Nature (London)* **612**, 647 (2022).
- [58] X. Li, D. Wang, H. Hu, and Y. Pan, Designer artificial chiral kagome lattice with tunable flat bands and topological boundary states, *Nanotechnology* **35**, 145601 (2024).
- [59] S. Wong, M. Saba, O. Hess, and S. S. Oh, Gapless unidirectional photonic transport using all-dielectric kagome lattices, *Phys. Rev. Res.* **2**, 012011(R) (2020).
- [60] N. Lera, D. Torrent, P. San-Jose, J. Christensen, and J. V. Alvarez, Valley Hall phases in kagome lattices, *Phys. Rev. B* **99**, 134102 (2019).
- [61] S. Kiriushchikina, A. Vakulenko, D. Smirnova, S. Guddala, Y. Kawaguchi, F. Komissarenko, M. Allen, J. Allen, and A. B. Khanikaev, Spin-dependent properties of optical modes guided by adiabatic trapping potentials in photonic Dirac metasurfaces, *Nat. Nanotechnol.* **18**, 875 (2023).
- [62] P. St-Jean, V. Goblot, E. Galopin, A. Lemaître, T. Ozawa, L. Le Gratiet, I. Sagnes, J. Bloch, and A. Amo, Lasing in topological edge states of a one-dimensional lattice, *Nat. Photonics* **11**, 651 (2017).
- [63] Y. Gong, S. Wong, A. J. Bennett, D. L. Huffaker, and S. S. Oh, Topological insulator laser using valley-Hall photonic crystals, *ACS Photonics* **7**, 2089 (2020).
- [64] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/rt47-39ft> for further details; Sec. I: The effective Hamiltonian and topological property; Sec. II: Parameter space of the ext-kagome II; Sec. III: Field profiles evolution with band inversion and acoustic simulation; Sec. IV: Wannier bands of the extended kagome; Sec. V: Field profiles of edge state; Sec. VI: Perturbation of the edge and corner states, which includes Refs. [65,67,68].
- [65] M. A. Gorlach, X. Ni, D. A. Smirnova, D. Korobkin, D. Zhirihin, A. P. Slobozhanyuk, P. A. Belov, A. Alu, and A. B. Khanikaev, Far-field probing of leaky topological states in all-dielectric metasurfaces, *Nat. Commun.* **9**, 909 (2018).
- [66] J. R. Schrieffer and P. A. Wolff, Relation between the Anderson and Kondo Hamiltonians, *Phys. Rev.* **149**, 491 (1966).
- [67] R. Yu, X. L. Qi, A. Bernevig, Z. Fang, and X. Dai, Equivalent expression of Z_2 topological invariant for band insulators using the non-Abelian Berry connection, *Phys. Rev. B* **84**, 075119 (2011).
- [68] N. Parappurath, F. Alpeggiani, L. Kuipers, and E. Verhagen, Direct observation of topological edge states in silicon photonic crystals: Spin, dispersion, and chiral routing, *Sci. Adv.* **6**, eaaw4137 (2020).
- [69] A. Vakulenko, S. Kiriushchikina, D. Smirnova, S. Guddala, F. Komissarenko, A. Alu, M. Allen, J. Allen, and A. B. Khanikaev, Adiabatic topological photonic interfaces, *Nat. Commun.* **14**, 4629 (2023).
- [70] X.-D. Chen, F.-L. Zhao, M. Chen, and J.-W. Dong, Valley-contrasting physics in all-dielectric photonic crystals: Orbital angular momentum and topological propagation, *Phys. Rev. B* **96**, 020202(R) (2017).
- [71] J.-K. Yang, Y. Hwang, and S. S. Oh, Evolution of topological edge modes from honeycomb photonic crystals to triangular-lattice photonic crystals, *Phys. Rev. Res.* **3**, L022025 (2021).