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Energy Width of Ferromagnetic-Exchange Magnons in a Magnetic Field

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The energy width Γ_k due to exchange scattering of magnons in a magnetic field is of the form $\Gamma_k \sim k^2 \epsilon_k F(k^2, \mu H/Dk^2)$ for the energy $\epsilon_k = \mu H + Dk^2 \ll k_B T$ in a ferromagnet. The factor $F(k^2, \mu H/Dk^2)$ is related to the availability of phase space for scattering and is shown to be regular at $k=0$ for any nonzero magnetic field, in contrast to the well-known logarithmic singular behavior for $H=0$. The dependence of Γ_k on H is calculated and discussed.

INTRODUCTION

Spin-wave damping has been studied in ferromagnetic systems by means of microscopic, phenomenological, and hydrodynamic approaches. The hydrodynamic theory[1] predicts the energy width Γ_k of a magnon to be proportional to k^4, where k is the wave vector. The hydrodynamic prediction is expected to be valid in the long-wavelength regime and applies when isotropic exchange scattering is responsible for the damping. However, microscopic calculations[2-4] give this energy width as $k^4(a \ln^2 k + b \ln k + c)$, where a, b, and c are constants independent of k. The phenomenological Hamiltonian gives the same result.[5,6] The logarithmic factors do not affect the property that magnons are good elementary excitations at long wavelengths, but the logarithmic factors do suggest that the expansion used in the hydrodynamic theory of the deviation from equilibrium in power of gradients may not be reliable in ferromagnetic systems in the absence of magnetic fields.

In this paper we will show that the logarithmic singular factor is an artifact and arises only because the system with only the pure isotropic exchange interaction does not have a preferred direction in space. An external magnetic field eliminates this directional uncertainty and removes the logarithmic singularity, even though the field does not affect the scattering matrix element. This is our central result. The quantitative calculation of the dependence of Γ_k on the magnetic field H is given. It should be pointed out that our calculation is concerned only with exchange scattering. In most of the real ferromagnets we know that dipolar scattering in three-magnon processes[7] is dominant at very small values of the wave vector. We consider it important, however, to establish the magnitude of the exchange contribution.

ENERGY WIDTH IN LONG-WAVELENGTH LIMIT

Expanding the isotropic exchange Hamiltonian in terms of magnon operators, the four-magnon scattering processes give the energy width[5,6]

$$\Gamma_{k_1} = 8\pi (C\mu^2)^2 \sum_{2,3,4} (\vec{k}_1 \cdot \vec{k}_2)(\vec{k}_3 \cdot \vec{k}_4) \times [n_2(1+n_3+n_4) - n_3 n_4] \times \Delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4)\, \delta(\epsilon_{k_1} + \epsilon_{k_2} - \epsilon_{k_3} - \epsilon_{k_4}) \tag{1}$$

for a magnon of wave vector k_1, where C is the phenomonological exchange constant, $\mu = g\mu_B \cong 2\mu_B$ is the magnetic moment per unit spin, and $n_i = n(\epsilon_{k_i}) = (e^{\beta \epsilon_{k_i}} - 1)^{-1}$ is the Bose-Einstein factor. This expression for the energy width agrees with the one derived microscopically[2-4] and can be interpreted as the sum of the Dyson cross section[8] for exchange scattering, which is essentially the factor $(\vec{k}_1 \cdot \vec{k}_2) \times (\vec{k}_3 \cdot \vec{k}_4)$, over the phase space for scattering. The dependence of the cross section alone agrees with the k^4 dependence of the energy width in the hydrodynamic theory. A magnetic field shifts only the dispersion relation $\epsilon_k = \mu H + Dk^2$, with $D = 2C\mu M$ and M the magnetization, and the Bose-Einstein factor $n(\epsilon_k)$ will not diverge at $k=0$ for $H \neq 0$. Thus we expect the summation over the phase space for scattering in Eq. (1) to give rise to a regular factor for $H \neq 0$ instead of the logarithmic singular factor in the $H=0$ case.

With the identity

$$n_2(1+n_3+n_4) - n_3 n_4 = (1+n_1)^{-1} n_2 (1+n_{1+2})(1+n_3+n_4)$$

and the symmetry between indices 3 and 4, we sum and integrate over k_3 and k_4 to obtain

$$\Gamma_{k_1} = \tfrac{1}{8}\pi^{-3} C\mu^3 M^{-1} [1 + n(\epsilon_{k_1})]^{-1} k_1^2 I, \tag{2}$$

with

$$I = (\beta D)^{-1} \int_0^\infty dk_2\, k_2^{\,4} n(\epsilon_{k_2}) [1 + n(\epsilon_{k_1} + \epsilon_{k_2})] \int_{-1}^{1} d\mu_2\, \mu_2^2 \times (k_1^2 + k_2^2 + 2k_1 k_2 \mu_2)^{-1/2} \ln\left(\frac{\sinh[\frac{1}{2}\beta(\mu H + Dk_+^2)]}{\sinh[\frac{1}{2}\beta(\mu H + Dk_-^2)]}\right), \tag{3}$$

where

$$k_\pm^2 = \tfrac{1}{2}\{(k_1^2 + k_2^2) \pm [(k_1^2 + k_2^2)^2 - 4k_1^2 k_2^2 \mu_2^2]^{1/2}\},$$

$$\mu_2 = \cos(\vec{k}_1, \vec{k}_2).$$

To simplify the remaining integrals, we follow the procedure of Harris[2] to separate the k_2 integration range $(0, \infty)$ in I into two parts: $I = I_a + I_b$ with I_a from 0 to ξk_1 and I_b from ξk_1 to ∞. If the number ξ is chosen such that $\xi \gg 1$ but $\beta\epsilon_{\xi k_1} \ll 1$, then we may approximate $n(\epsilon_k)$ by $(\beta\epsilon_k)^{-1}$ in I_a and neglect Dk_1^2 and μH in comparison with Dk_2^2 in I_b. With the long-wavelength and weak-field condition

$$\beta\epsilon_{k_1} \ll 1, \tag{4}$$

which describes the regime which interests us, we can set

$$\xi = (\beta\epsilon_{k_1})^{-1/4}\left(\frac{1-\beta\mu H/(\beta\epsilon_{k_1})^{1/2}}{1-\mu H/\epsilon_{k_1}}\right)^{1/2} \gg 1;$$

then

$$\beta\epsilon_{\xi k_1} = \beta(\mu H + D\xi^2 k_1^2) = (\beta\epsilon_{k_1})^{1/2} \ll 1\,.$$

The final answer should be independent of the number ξ.

With the approximation $n(\epsilon_k) = (\beta\epsilon_k)^{-1}$ in $0 < k_2 < \xi k_1$, we have

$$\begin{aligned}
I_a &= (\beta D)^{-1}\int_0^{\xi k_1} dk_2\, k_2^4 \\
&\quad \times (\beta\mu H + \beta D k_2^2)^{-1}(2\beta\mu H + \beta D k_1^2 + \beta D k_2^2)^{-1} \\
&\quad \times \int_{-1}^{1} d\mu_2\, \mu_2^2 (k_1^2 + k_2^2 + 2k_1k_2\mu_2)^{-1/2}\ln\left(\frac{\mu H + Dk_+^2}{\mu H + Dk_-^2}\right) \\
&= \tfrac{1}{8}(\beta D)^{-3}\int_0^{\xi} dq\, q(1+q^2)^{5/2}(q^2+R)^{-1}(q^2+1+2R)^{-1} \\
&\quad \times \int_{-p}^{p} dy\, y^2(1+y)^{-1/2} \\
&\quad \times \ln\left(\frac{2R(1+q^2)^{-1}+1+(1-y^2)^{1/2}}{2R(1+q^2)^{-1}+1-(1-y^2)^{1/2}}\right),
\end{aligned} \tag{5}$$

with

$$k_2 = qk_1, \qquad \mu_2 = y(1+q^2)/2q,$$

$$p = 2q/(1+q^2), \qquad R = \mu H / Dk_1^2\,.$$

Here R is the ratio of the magnetic field energy to the exchange energy. It is noted that I_a depends on the ratio R but not on μH and Dk_1^2 separately. Further calculation of I_a is given in the Appendix. The result is

$$\begin{aligned}
I_a = \tfrac{2}{3}(\beta D)^{-3}\{&\ln^2\xi + 2[\tfrac{1}{3} - R + R^{3/2}\tan^{-1}(R^{-1/2}) \\
&- \tfrac{1}{2}\ln(1+R)]\ln\xi + \tfrac{3}{2}G(R)\}\,.
\end{aligned} \tag{6}$$

The function $G(R)$ is given by Eq. (A10). As $R \to \infty$; i.e., $k_1 \to 0$ with nonzero H field, we have in the limit $R \gg 1$,

$$G(R) \cong \tfrac{1}{6}\ln^2 R - \tfrac{1}{18}\pi^2 + \tfrac{1}{5}R^{-1}\ln R - 0.173R^{-1}\,. \tag{7}$$

We plot $G(R)$ in Fig. 1.

Now we calculate the remaining part I_b. With the approximation of neglecting μH and Dk_1^2 in comparison to Dk_2^2 for $\xi k_1 < k_2 < \infty$, we have

$$\begin{aligned}
I_b = (\beta D)^{-1}\int_{\xi k_1}^{\infty} dk_2\, k_2^3 e^{\beta D k_2^2}[e^{\beta D k_2^2} - 1]^{-2}\int_{-1}^{1} d\mu_2\,\mu_2^2 \\
\times \ln\left(\frac{2\sinh(\frac{1}{2}\beta D k_2^2)}{\beta D k_1^2\mu_2^2 + \beta\mu H}\right).
\end{aligned} \tag{8}$$

With

$$\begin{aligned}
\int_0^1 d\mu_2\,\mu_2^2\ln(\mu_2^2+R) \\
&= \tfrac{1}{3}\ln(1+R) - \tfrac{2}{3}[\tfrac{1}{3} - R + R^{3/2}\tan^{-1}(R^{-1/2})],
\end{aligned}$$

$$\begin{aligned}
\int_{\beta D\xi^2 k_1^2}^{\infty} dx\, x e^x(e^x-1)^{-2} \\
&= [-x(e^x-1)^{-1} + \ln(1-e^{-x})]\Big|_{\beta D\xi^2k_1^2}^{\infty} \\
&\cong 1 - \ln(\beta D\xi^2 k_1^2),
\end{aligned}$$

and

$$\begin{aligned}
\int_{\beta D\xi^2 k_1^2}^{\infty} dx\, x e^x(e^x-1)^{-2}\ln(e^{x/2}-e^{-x/2}) \\
&= \{\tfrac{1}{2}\ln^2(1-e^{-x}) + [1 - x(e^x-1)^{-1}]\ln(1-e^{-x}) \\
&\qquad - \tfrac{1}{2}(x^2+2x)(e^x-1)^{-1}\}\Big|_{\beta D\xi^2k_1^2}^{\infty} \\
&\cong 1 - \tfrac{1}{2}\ln^2(\beta D\xi^2 k_1^2),
\end{aligned}$$

where $x = \beta D k_2^2$, we obtain

$$\begin{aligned}
I_b = \tfrac{2}{3}(\beta D)^{-3}\{&-\ln^2\xi - 2[\tfrac{1}{3} - R + R^{3/2}\tan^{-1}(R^{-1/2}) \\
&- \tfrac{1}{2}\ln(1+R)]\ln\xi + \tfrac{1}{4}\ln^2(\beta D k_1^2) \\
&- [\tfrac{5}{6} - R + R^{3/2}\tan^{-1}(R^{-1/2}) \\
&- \tfrac{1}{2}\ln(1+R)][\ln(\beta D k_1^2) - 1]\}.
\end{aligned} \tag{9}$$

In I_a and I_b terms that depend on the constant ξ cancel each other, and we have the result for the energy width from Eqs. (2), (6), and (9);

$$\Gamma_{k_1} = \tfrac{1}{8}\pi^{-3}C\mu^3M^{-1}(\beta D)^{-3}\beta\epsilon_{k_1}k_1^2F(k_1^2, R)\,, \tag{10}$$

with

$$\begin{aligned}
F(k^2, R) = \tfrac{1}{6}\ln^2(\beta D k^2) - \tfrac{2}{3}[\tfrac{5}{6} - R + R^{3/2}\tan^{-1}(R^{-1/2}) \\
- \tfrac{1}{2}\ln(1+R)][\ln(\beta D k^2) - 1] + G(R)\,.
\end{aligned} \tag{11}$$

We have replaced the factor $[1 + n(\epsilon_{k_1})]^{-1}$ by $\beta\epsilon_{k_1}$ in Eq. (10) for $\beta\epsilon_{k_1} \ll 1$.

RESULT AND DISCUSSION

We rewrite Eq. (10), dropping the index 1, in the form

$$\Gamma_k = A(\beta\epsilon_k)(\beta D k^2)F(k^2, R)\,, \tag{12}$$

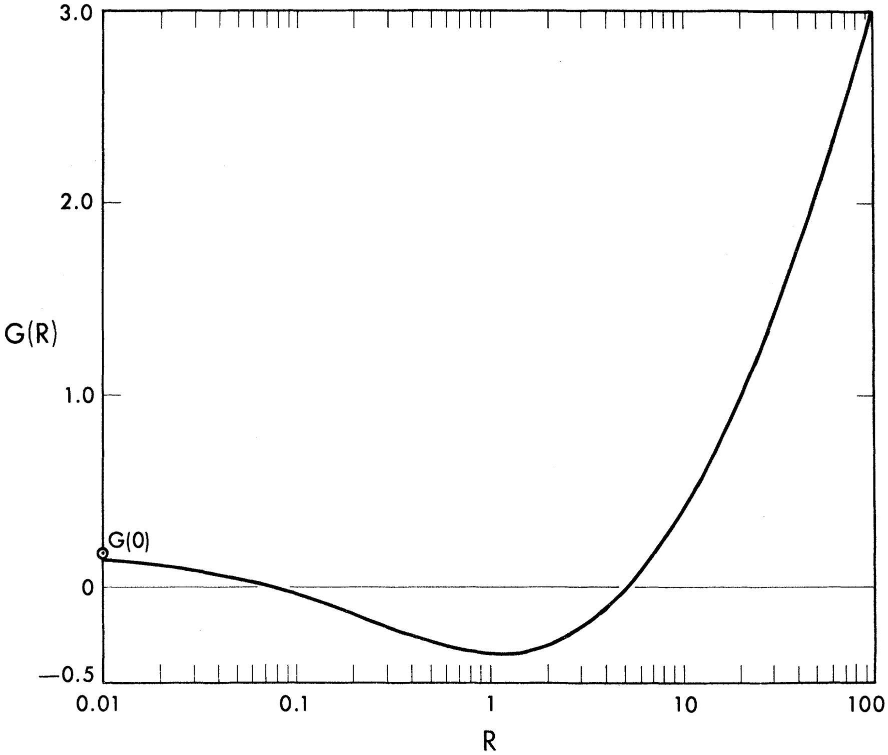

FIG. 1. Calculated factor $G(R)$ with $R=\mu H/Dk^2$ the ratio of magnetic energy to the exchange energy.

with

$$A=\tfrac{1}{16}\pi^{-3}(\beta D)^{-3}\beta^{-1}(M/\mu)^{-2}.$$

The factor ϵ_k and the factor k^2 from the exchange-interaction matrix element give the k^4 dependence for $H=0$. The function $F(k^2, R)$ is related to the available phase space for scattering.

In the absence of the magnetic field $H=0$, the ratio $R=\mu H/Dk^2$ is zero for any nonzero k. At $R=0$ we have

$$F(k^2,0)=\tfrac{1}{6}\ln^2(\beta Dk^2)-\tfrac{5}{9}\ln(\beta Dk^2)+0.731. \quad (13)$$

This is logarithmically singular at $k=0$ and agrees with the result of Harris.[2,9]

In the presence of a magnetic field, no matter how weak, the limit $k\to 0$ corresponds to the limit $R\to\infty$. With Eq. (7) for $G(R\gg 1)$, we have, for $R \gg 1$,

$$F(k^2,R)\cong\tfrac{1}{6}[1-\ln(\beta\mu H)]^2+\tfrac{1}{6}-\tfrac{1}{18}\pi^2 + R^{-1}[-0.373+\tfrac{1}{5}\ln(\beta\mu H)]. \quad (14)$$

This is finite for $k=0$ for any nonzero magnetic field. Thus the logarithmic singularities of the phase-space factor disappear in the presence of a magnetic field. The magnetic field gives the system a preferred direction in space which the exchange interaction alone cannot determine. The energy gap μH associated with the anisotropy

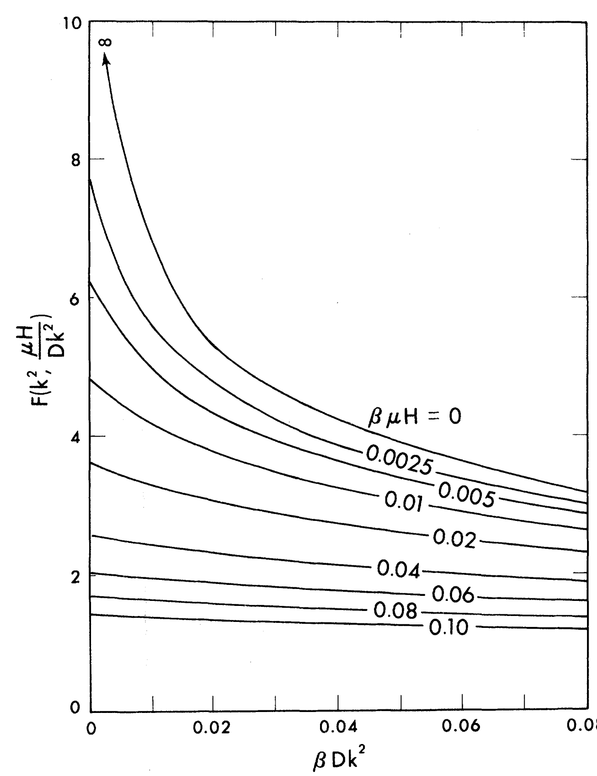

FIG. 2. Phase-space factor $F(k^2, \mu H/Dk^2)$ is plotted as a function of βDk^2 for several values of $\beta\mu H$. $F(k^2, \mu H/Dk^2)$ is related to $G(R)$ of Fig. 1 by Eq. (11) in the text.

introduced by the magnetic field removes the logarithmic singularities. We expect the same regular behavior of the phase-space factor for any anisotropic system, provided that the anisotropic energy term does not affect the four-magnon exchange scattering processes and the three-magnon decay processes can be ignored.

The phase-space factor $F(k^2, R)$ and the dimensionless energy width Γ_k/A were calculated numerically as a function of the exchange energy for several values of the magnetic field energy. The results are plotted in Figs. 2 and 3, respectively. The phase-space factor $F(k^2, R)$ decreases as $\beta\mu H$ increases, as one expects from the physical consideration that the magnetic field drifts the spins along the field direction and decreases the occupation number n of the magnons. The factor $\beta\epsilon_k$ increases as $\beta\mu H$ increases. In the neighborhood of $k=0$ the factor $\beta\epsilon_k$ in Eq. (12) has the larger effect, and we have the result that the larger the magnetic field, the larger the energy width of the magnons. As the exchange energy becomes larger, the phase-space factor becomes important, and we have the opposite dependence. It would be interesting to observe this change experimentally.

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APPENDIX: EVALUATION OF I_a

From Eq. (5), we have

$$I_a = \tfrac{1}{8}(\beta D)^{-3}\int_0^{\xi} dq\, q(1+q^2)^{5/2}(q^2+R)^{-1}(q^2+1+2R)^{-1} \times\left[J\left(\frac{2q}{1+q^2}\right) - J\left(\frac{-2q}{1+q^2}\right)\right], \quad \text{(A1)}$$

with

$$J(y)=\int dy\, y^2(1+y)^{-1/2} \times \ln\left(\frac{2R(1+q^2)^{-1}+1+(1-y^2)^{1/2}}{2R(1+q^2)^{-1}+1-(1-y^2)^{1/2}}\right).$$

The y integration can be carried out. For $q<1$, we have

$$J\left(\frac{2q}{1+q^2}\right) - J\left(\frac{-2q}{1+q^2}\right) = \tfrac{16}{15}(1+q^2)^{-5/2}\Delta J(q,R), \quad \text{(A2)}$$

with

$$\Delta J(q,R) = q^3(2q^2+5)\ln\left(\frac{1+R}{q^2+R}\right) - 2q(q^2+2)(q^2+1+2R) + (q^2+1+R)^{3/2}(2q^2+2-3R) \times \ln\left(\frac{(q^2+1+R)^{1/2}+q}{(q^2+1+R)^{1/2}-q}\right) + 2R^{3/2}(5q^2+5+3R)\tan^{-1}(qR^{-1/2}). \quad \text{(A3)}$$

We separate the q integration into two parts: $I_a^{(1)}$ from 0 to 1 and $I_a^{(2)}$ from 1 to ξ. In $I_a^{(2)}$, change variable q to $1/q$, the corresponding expression for $J[2q/(1+q^2)] - J[-2q/(1+q^2)]$ is similar to Eq. (A2) with R replaced by Rq^2. Thus

$$I_a^{(1)} = \tfrac{2}{15}(\beta D)^{-3}\int_0^1 dq\, q(q^2+R)^{-1}(q^2+1+2R)^{-1}\Delta J(q,R) \quad \text{(A4)}$$

and

$$I_a^{(2)} = \tfrac{2}{15}(\beta D)^{-3}\int_{1/\xi}^1 dq\, q^{-4}(Rq^2+1)^{-1} \times[(1+2R)q^2+1]^{-1}\Delta J(q,Rq^2). \quad \text{(A5)}$$

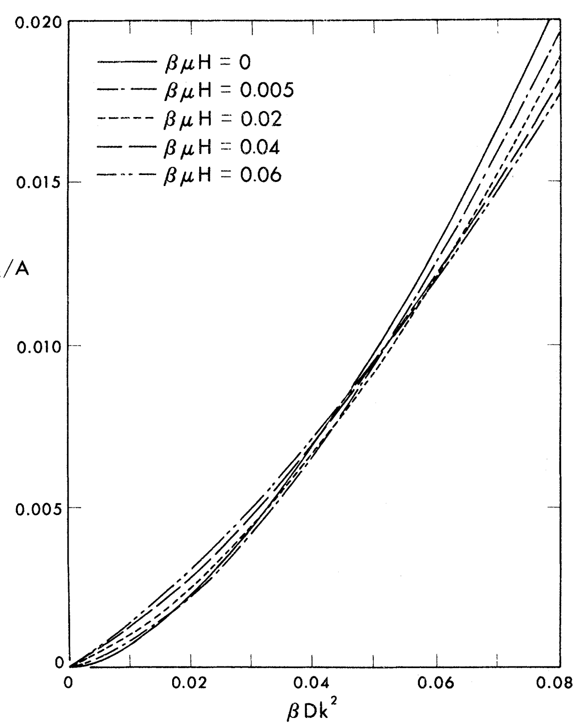

FIG. 3. Dimensionless energy width Γ_k/A is plotted as a function of βDk^2 for $\beta\mu H = 0.0$, 0.005, 0.02, 0.04, 0.06. Using Eq. (12) in the text and Fig. 2, the Γ_k/A for other values of $\beta\mu H$ is easily plotted. Note the reverse of dependence of Γ_k/A on $\beta\mu H$ as βDk^2 increases.

The integrand of $I_a^{(1)}$ is regular in the interval (0, 1), and we can integrate numerically. For $R \gg 1$, we have

$$\tilde{I}_a^{(1)} \equiv (\beta D)^3 I_a^{(1)} \cong \tfrac{104}{945} R^{-3} . \quad \text{(A6)}$$

For $I_a^{(2)}$ there are terms singular in ξ for $\xi \gg 1$. We expand the integrand and single out the singular terms and integrate to give

$$\int_{1/\xi}^{1} dq \{-10q^{-1} \ln q + 10[\tfrac{1}{3} - R + R^{3/2} \tan^{-1}(R^{-1/2}) - \tfrac{1}{2}\ln(1+R)] q^{-1}\}$$
$$= 5 \ln^2\xi + 10[\tfrac{1}{3} - R + R^{3/2}\tan^{-1}(R^{-1/2}) - \tfrac{1}{2}\ln(1+R)] \ln\xi . \quad \text{(A7)}$$

The regular part with the integrand of Eq. (A5) minus the integrand of (A7) is integrated numerically from 0 to 1. Properly expanded,[10] we have for $R \gg 1$

$$\tilde{I}_a^{(2)} \equiv (\beta D)^3 I_a^{(2)} - \tfrac{2}{3}\ln^2\xi - \tfrac{4}{3}[\tfrac{1}{3} - R + R^{3/2}\tan^{-1}(R^{-1/2}) - \tfrac{1}{2}\ln(1+R)] \ln\xi$$
$$\cong \tfrac{1}{6}\ln^2 R - \tfrac{1}{18}\pi^2 + \tfrac{1}{5}R^{-1}\ln R + (\tfrac{22}{75} + \tfrac{7}{180}\pi^2 - \tfrac{92}{75}\ln 2) R^{-1} . \quad \text{(A8)}$$

Thus we have the result for $I_a = I_a^{(1)} + I_a^{(2)}$

$$I_a = \tfrac{2}{3}(\beta D)^{-3}\{\ln^2\xi + 2[\tfrac{1}{3} - R + R^{3/2}\tan^{-1}(R^{-1/2}) - \tfrac{1}{2}\ln(1+R)]\ln\xi + \tfrac{3}{2}G(R)\}, \quad \text{(A9)}$$

with

$$G(R) = \tilde{I}_a^{(1)} + \tilde{I}_a^{(2)} , \quad \text{(A10)}$$

where $\tilde{I}_a^{(1)}$ and $\tilde{I}_a^{(2)}$ and their expansions for large R are given in Eqs. (A6) and (A8), respectively. The numerically calculated $G(R)$ is plotted in Fig. 1. For $R = 0$, we have

$$G(0) = 0.1755 . \quad \text{(A11)}$$

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[9]With the numerical correction of multiplying $\frac{4}{15}$ on the two integrals in Eq. (B7) of Ref. 2, the result of Ref. 2 is Eq. (13).

[10]It is not appropriate to treat Rq^2 as large before integrated. Thus we use $\ln[(1+x)/(1-x)] = 2(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots)$, etc., to expand the integrand, integrate term by term, then make the expansion in R.