Perfect Undetectable Acoustic Device from Fabry-Pérot Resonances

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Transformation acoustics is a method to design novel acoustic devices, while the complexity of the material parameters hinders its progress. In this paper, we analytically present a three-dimensional perfect undetectable acoustic device from Fabry-Pérot resonances and confirm its functionality from Mie theory. Such a mechanism goes beyond the traditional transformation acoustics. In addition, such a reduced version can be realized by holey-structured metamaterials. Our theory paves a way to the implementation of three-dimensional transformation acoustic devices.

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I. INTRODUCTION

Transformation acoustics and acoustic cloaks [1,2] have become a hot topic in the acoustic community following the pioneering work in transformation optics [3,4]. In the beginning, a two-dimensional (2D) acoustic cloak was proposed by making equivalence between the acoustic wave equation and the 2D Maxwell's equations [5]. Later, the form invariance of the acoustic wave equation was found, and the equivalence of acoustic material parameters and space-time was unveiled [6]. As a result, a three-dimensional (3D) acoustic cloak was then proposed [6], which was also confirmed by Mie theory [7]. Many experiments, such as those on a 2D acoustic carpet cloak [8] and a 3D acoustic carpet cloak [9], have been conducted, making transformation acoustics a very active field. At the same time, the complexity of the material parameters of transformation acoustic devices hinders the progress of experiments, similar to those in transformation optics [2]. Therefore, there is a great need to further simplify the material parameters so that normal materials or their simple compositions can be used to implement the devices.

In this paper, we start from a simple example of acoustic concentrators and derive a perfect undetectable acoustic device from Fabry-Pérot (FP) resonances. Such a version can be simply realized by holey-structured metamaterials, which makes experiments easier, and, hence has important applications in acoustic designs.

II. PERFECT ACOUSTIC CONCENTRATOR FROM FABRY-PÉROT RESONANCES

Let us start from the linear mapping of the field concentrator [10]

$$r' = f(r) = \begin{cases} \frac{c}{a} r, & 0 \le r \le a, \\ \frac{b-c}{b-a} r + \frac{c-a}{b-a} b, & a < r \le b, \\ r, & r > b, \end{cases} \tag{1}$$

which compresses the region $0 \le r' \le c$ in virtual space into that in physical space $0 \le r \le a$, while the region $c \le r' \le b$ in virtual space is transformed into $a \le r \le b$ (see a schematic plot in Fig. 1). Suppose the virtual space is filled with a homogeneous background medium. From the theory of transformation acoustics [6], we can easily derive that for the inner core $0 \le r \le a$, the density and bulk modulus are transformed from the background into

$$\rho_1 = \frac{a}{c}\rho_0, \qquad \kappa_1 = \frac{a^3}{c^3}\kappa_0, \tag{2}$$

where ρ_0, κ_0 are the density and bulk modulus of the background (the relative refractive index is thereby $[(c^2)/(a^2)]$). While for the region of $a \le r \le b$, the density becomes a tensor with

$$\rho_r = \frac{b-c}{b-a}\left(\frac{(b-a)r}{(b-c)r+(c-a)b}\right)^2 \rho_0,$$
$$\rho_\theta = \rho_\phi = \frac{b-a}{b-c}\rho_0, \tag{3}$$

and the bulk modulus becomes inhomogeneous

$$\kappa(r) = \frac{b-a}{b-c}\left(\frac{(b-a)r}{(b-c)r+(c-a)b}\right)^2 \kappa_0. \tag{4}$$

Such a device is a perfect undetectable device, as it is transformed directly from the background medium. It

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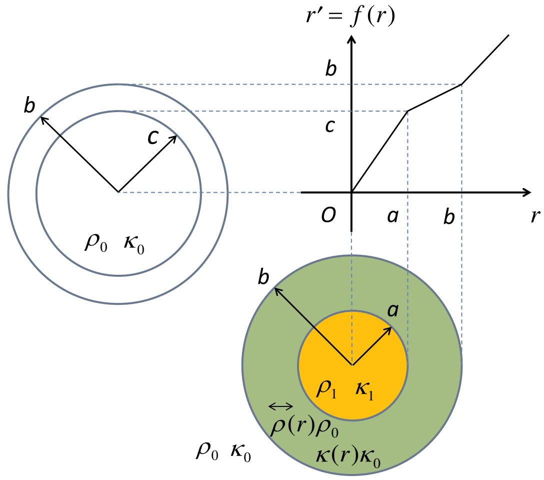

FIG. 1. The mapping of the field concentrator.

should be easy to design the inner core, but it is quite a challenge to get the required material parameters for $a \le r \le b$. If we take a limit of $c \to b$, $\rho_r \to 0$, $\rho_\theta = \rho_\phi \to \infty$, and $\kappa(r) \to \infty$, such a device is also not feasible. However, let us replace the density tensor by

$$\rho_r = \rho(r)\rho_0, \qquad \rho_\theta = \rho_\phi \to \infty, \tag{5}$$

and allow the bulk modulus profile to change arbitrarily as $\kappa(r)\kappa_0$. Such a device can then be implemented by using the holey-structured metamaterials as demonstrated in Ref. [11] or asymptotic models of dilute composites [12]. The holes should be along the radial directions with various filling ratios at different r positions. With the above conditions of $\rho_\theta = \rho_\phi \to \infty$, the acoustic equations in the region of $a \le r \le b$ can be written as [13]

$$\frac{\partial P}{\partial r} = -\omega^2 \rho(r)\rho_0 u_r, \qquad P = \kappa(r)\kappa_0\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r)\right), \tag{6}$$

where ω is the angular frequency, P is the pressure field, and u_r is the displacement along the r direction (there is no displacement along the θ and ϕ directions due to the related hard walls or the conditions of $\rho_\theta = \rho_\phi \to \infty$). In the Appendix, we show that it is suitable to choose the following material parameters for a perfect undetectable acoustic device:

$$\rho(r) = \frac{r}{b}, \qquad \kappa(r) = \frac{r^3}{b^3}, \qquad n(r) = \frac{b}{r}. \tag{7}$$

III. ANALYTICAL PROOF OF PERFECT FUNCTIONALITY

Next, we prove analytically from Mie theory that the above form is another alternative perfect undetectable version that goes beyond the traditional transformation acoustics.

Suppose the incident wave is

$$P_0(r,\theta,\phi) = \sum_{lm} P^0_{lm} j_l(k_0 r) Y^m_l(\theta,\phi). \tag{8}$$

The scattered wave is

$$P^+(r,\theta,\phi) = \sum_{lm} P^+_{lm} h^+_l(k_0 r) Y^m_l(\theta,\phi), \tag{9}$$

where P^+_{lm} are coefficients to be determined, $j_l(x)$ and $h^+_l(x)$ are the lth-order spherical Bessel functions and Hankel functions of the first kind, and $Y^m_l(\theta,\phi)$ are the spherical harmonics. Therefore, the total pressure field for $r > b$ is

$$P_3(r,\theta,\phi) = \sum_{lm} [P^0_{lm} j_l(k_0 r) + P^+_{lm} h^+_l(k_0 r)] Y^m_l(\theta,\phi). \tag{10}$$

From the above analysis, the pressure field for $a \le r \le b$ is

$$P_2(r,\theta,\phi) = \sum_{lm} (A_{lm} e^{iS(r)} + B_{lm} e^{-iS(r)}) Y^m_l(\theta,\phi), \tag{11}$$

where A_{lm} and B_{lm} are coefficients to be determined. As $\{[\partial S(r)]/(\partial r)\} = k_0 n(r) = k_0(b/r)$ [Eq. (A5)], we shall have $S(r) = k_0 b \ln r + S_0$; therefore, Eq. (11) becomes

$$P_2(r,\theta,\phi) = \sum_{lm} (A_{lm} e^{ik_0 b \ln r} + B_{lm} e^{-ik_0 b \ln r}) Y^m_l(\theta,\phi). \tag{12}$$

For the inner core $r \le a$, the pressure field is

$$P_1(r,\theta,\phi) = \sum_{lm} P^1_{lm} j_l\left(k_0 \frac{b}{a} r\right) Y^m_l(\theta,\phi), \tag{13}$$

where P^1_{lm} are coefficients to be determined [the inner core has a relative refractive index of b/a from Eq. (2)].

By matching the continual boundary conditions of P and $(1/\rho_r)[(\partial P)/(\partial r)]$ at $r = a$ and $r = b$, we shall have

$$P^0_{lm} j_l(k_0 b) + P^+_{lm} h^+_l(k_0 b) = A_{lm} e^{ik_0 b \ln b} + B_{lm} e^{-ik_0 b \ln b}, \tag{14}$$

$$P^0_{lm} j_l'(k_0 b) + P^+_{lm} h^{+\prime}_l(k_0 b) = iA_{lm} e^{ik_0 b \ln b} - iB_{lm} e^{-ik_0 b \ln b}, \tag{15}$$

$$P^1_{lm} j_l\left(k_0 \frac{b}{a} a\right) = A_{lm} e^{ik_0 b \ln a} + B_{lm} e^{-ik_0 b \ln a}, \tag{16}$$

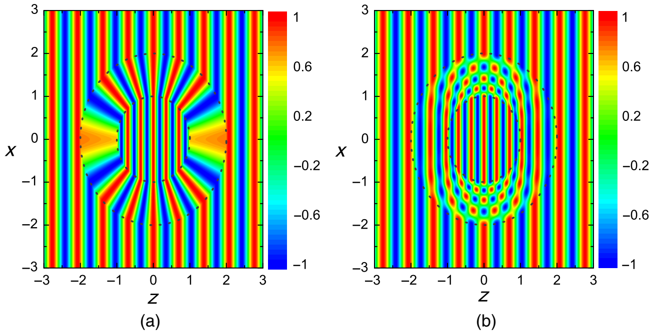

FIG. 2. The pressure field patterns of (a) a transformation acoustic version and (b) the current FP version at the plane at $y = 0$. $a = 1$ (A.U.), $b = 2$, $\lambda = [(4\ln 2)/m]$ with $m = 4$ for the FP resonances, $P^0 = 1$.

$$P^1_{lm} j_l{}'\left(k_0 \frac{b}{a} a\right) = iA_{lm} e^{ik_0 b \ln a} - iB_{lm} e^{-ik_0 b \ln a}. \tag{17}$$

If $e^{ik_0 b \ln b} = \pm e^{ik_0 b \ln a}$, $P^1_{lm} = \pm P^0_{lm}$, and $P^+_{lm} = 0$, then a perfect inaudibility is achieved. Such a condition can be further written as

$$e^{ik_0 b \ln\frac{b}{a}} = e^{im\pi} \tag{18}$$

or

$$k_0 b \ln\frac{b}{a} = \int_a^b k_0 n(r) dr = m\pi, \tag{19}$$

which are simply the FP resonance conditions along the radial direction [numerical and experimental confirmations for a 2D electromagnetic (EM) version can be found in Ref. [14]].

If the incident wave is a plane wave with

$$P_0(r,\theta,\phi) = P^0 e^{ik_0 r\cos\theta}, \tag{20}$$

the field in the inner core should be

$$P_1(r,\theta,\phi) = \pm P^0 e^{ik_0 \frac{b}{a} r\cos\theta}, \tag{21}$$

where "+/−" means the phase change is $2\pi/\pi$ for the inner and outer boundaries. The field for $a \le r \le b$ can be easily derived as

$$P_2(r,\theta,\phi) = P^0 e^{ik_0 b\cos\theta}\Big(\cos\Big(k_0 b \ln\frac{r}{b}\Big) + i\cos\theta \sin\Big(k_0 b\ln\frac{r}{b}\Big)\Big). \tag{22}$$

Therefore, if the incident wave is known, the pressure field of the inaudible devices can be analytically achieved.

For instance, we plot the field patterns for both the transformation acoustic version and the current FP version. Figure 2(a) is the perfect inaudibility field pattern for $c = 0.99b$. It simply follows the form of $P^0 e^{ik_0 f(r)\cos\theta}$, where $f(r)$ is from Eq. (1). Figure 2(b) is the perfect inaudibility field pattern for current design [Eq. (7)]. The field pattern of $a \le r \le b$ can change a lot while the perfect inaudibility is still maintained.

IV. CONCLUSION

In conclusion, we analytically find a perfect inaudible 3D device that goes beyond traditional transformation acoustics. Such a version is easier in implementation. In addition, the theory can also be used to design a 2D inaudible device or 2D invisible device (for EM waves) [14,15]. Therefore, our method goes beyond transformation acoustics and transformation optics. Such a version can also be utilized for illusion acoustics or illusion optics [16] if the inner core is replaced by other material compositions.

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APPENDIX: MATERIAL PARAMETERS OF THE PERFECT CONCENTRATOR

In this appendix, we show why Eq. (7) is a suitable set of material parameters for the perfect concentrator. Let us substitute a test solution of $P = p(r)e^{iS(r)}$ into Eq. (6),

$$\frac{\partial}{\partial r}\left\{\alpha(r)\frac{\partial}{\partial r}[p(r)e^{iS(r)}]\right\} = -k_0^2\beta(r)p(r)e^{iS(r)}, \tag{A1}$$

where $\alpha(r) = \{(r^2)/[\rho(r)]\}$, $\beta(r) = \{(r^2)/[\kappa(r)]\}$, and $k_0^2 = [(\rho_0\omega^2)/(\kappa_0)]$ is the wave vector in the background. After some calculations, we have

$$\frac{\partial}{\partial r}\left(\alpha(r)\frac{\partial p(r)}{\partial r}\right) + p(r)\left[k_0^2\beta(r) - \alpha(r)\left(\frac{\partial S(r)}{\partial r}\right)^2\right] = 0, \tag{A2}$$

$$p(r)\frac{\partial}{\partial r}\left(\alpha(r)\frac{\partial S(r)}{\partial r}\right) + 2\alpha(r)\frac{\partial S(r)}{\partial r}\frac{\partial p(r)}{\partial r} = 0. \tag{A3}$$

It is easy to get a special solution such that

$$\alpha(r)\frac{\partial p(r)}{\partial r} = C\ (\text{a constant}), \tag{A4}$$

$$\left(\frac{\partial S(r)}{\partial r}\right)^2 = k_0^2\frac{\beta(r)}{\alpha(r)} = k_0^2 n^2(r), \tag{A5}$$

$$p(r)\frac{\partial}{\partial r}\left(\alpha(r)\frac{\partial S(r)}{\partial r}\right) + 2C\frac{\partial S(r)}{\partial r} = 0. \tag{A6}$$

If $C = 0$, we shall have $p(r) = P_0$ and $\alpha(r)\{[\partial S(r)]/(\partial r)\} = D$ with both P_0 and D as constants. Suppose $\alpha(r) = Gr^g$ and $\beta(r) = Hr^h$ [or $n(r) = \sqrt{(H/G)}r^{(h-g)/2}$],

$$\alpha(r)\frac{\partial S(r)}{\partial r} = \pm k_0 n(r)\alpha(r) = \pm k_0\sqrt{GH}r^{\frac{g+h}{2}} = D. \quad (A7)$$

Therefore, $g+h=0$. As we have $n(a) = \sqrt{(H/G)}a^{(h-g)/2} = (b/a)$ [from Eq. (2), a power function of a, thereby we shall use power functions as test functions], $(h-g)/2 = -1$. Then we get $g = 1$ and $h = -1$.

If $C \neq 0$,

$$\frac{\partial p(r)}{\partial r} = \frac{C}{\alpha(r)}, \qquad p(r) = \frac{-2Cn(r)}{\frac{\partial}{\partial r}[\alpha(r)n(r)]} \quad (A8)$$

or

$$\frac{\partial}{\partial r}\left(\frac{2n(r)}{\frac{\partial}{\partial r}[\alpha(r)n(r)]}\right) = -\frac{1}{\alpha(r)}. \quad (A9)$$

Again, let us suppose $\alpha(r) = Gr^g$ and $\beta(r) = Hr^h$ [or $n(r) = \sqrt{(H/G)}r^{(h-g)/2}$]. After substituting them into Eq. (A9), we shall have $3g = h + 4$. Hence, $n(r) = \sqrt{(H/G)}r^{g-2}$ with $n(a) = \sqrt{(H/G)}a^{g-2} = (b/a)$; therefore, $g = 1$ and $h = -1$, which goes back to the previous solution and confirms again that $C = 0$.

Therefore, we have $n(r) = (b/r)$, $\alpha(r) = Gr$, and $\beta(r) = (H/r)$, and finally,

$$\rho(r) = \frac{r}{b}, \qquad \kappa(r) = \frac{r^3}{b^3}. \quad (A10)$$

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