Coulomb scattering within a spherical beam bunch in a high current linear accelerator

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Beam halo formation issues are important for the design of high current linear ion accelerators. Various mechanisms can potentially cause beam halo. Some recent studies suggested that Coulomb collisions in the beam bunch can contribute significantly to beam bunch growth and halo development in linear accelerators. Despite the general belief that collisions are not important, it is clear that a rigorous treatment of this question is needed. In an effort to explore this issue in detail we have undertaken an analysis of the effects of Coulomb scattering between ions in a self-consistent spherical bunch. [S1098-4402(99)00041-5]

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I. INTRODUCTION

During the last several years, interest has grown in the design of high current linear ion accelerators for a variety of important applications. Among these are the heavy ion fusion linac, the Accelerator for the Production of Tritium (APT), the Accelerator for the Transmutation of Waste (ATW), and the Spallation Neutron Source (SNS). This has focused attention on the phenomenon of halo formation, which can lead to unacceptably large radioactivation of the linacs.

Our recent contributions to these efforts have been the construction of a model which identifies a major mechanism for transverse halo formation [1], followed by the construction of a self-consistent 6D phase space distribution for studies of halo formation in spheroidal bunches in a linear external confining field [2]. What we found [1] is that halo formation appears to arise from a parametric resonant coupling of individual particle oscillations with collective oscillation of the charged bunch, primarily those describing the breathing modes. We explored the dependence of the halo properties (extent, rate of growth, intensity, etc.) on the longitudinal and transverse rms tune depressions and mismatches [2]. These studies automatically assumed equipartition of kinetic energy between the longitudinal and two transverse directions. We then performed numerical studies with an rms matched, but otherwise non-self-consistent 6D distribution and found a rather rapid transition to a more or less self-consistent distribution with a corresponding small mismatch [3]. In these studies we found that starting with a non-self-consistent distribution altered the parameters for halo formation obtained for the self-consistent distribution only slightly.

Our objective in this paper is to explore the effect of Coulomb collisions in the beam bunch and to see if this process contributes significantly to beam emittance growth and halo development in linear accelerators. Since the beam bunch spends a very short time in a linac, compared with a circular machine or a storage ring, the general expectation is that collisions between individual ions will take place on too long a time scale to be important.

In the process known as the Touschek effect [4] in circular machines, collisions between particles with relatively large transverse momentum scatter these particles into the longitudinal direction, allowing them to either escape from the beam bunch longitudinally or spread the bunch longitudinally. The effect is important there because the transverse beam "temperature" is much larger than the longitudinal beam "temperature" (see, for example, [5]). In the case of a linac with short bunches [6], the temperatures are comparable; in fact, they are the same for our family of stationary distributions [2]. It is therefore expected that for the beams of current interest [6] one can ignore very large angle scattering; i.e., the Touschek effect is negligible.

Treatment of small angle Coulomb scattering should distinguish between the single scattering effect and the effect of multiple collisions. The scattering by a relatively large angle in a single encounter is often less likely than a net large angle deflection due to the cumulative effect of many small angle scatterings. It is therefore important to understand the collisional diffusion associated with multiple scattering, but it is also important to understand whether rapid particle loss due to single encounters is possible. Single scattering is usually neglected due to the very small probability of the process. However, from the kinematics, it is clear that such a scattering can quickly fill a relatively large shell around the beam bunch which would resemble our earlier description of a halo [1,2]. Thus, in the present paper we are mainly concerned with single deflections, and not the diffusion process associated with multiple small angle scattering. We note that our analysis can be extended to the multiple scattering effect using the standard statistical averaging approach [7–9]. Some crude estimates of the effect of single scattering were done by Carlsten [10]. He related the rate for a particle to scatter into the shell around the bunch to the admittance of the accelerator. His conclusion was that the effect is important if the acceptance of the machine is small, and negligible only for accelerators with very large acceptance, such as the APT [6]. Also there have been some recent numerical studies which suggest that small angle single Coulomb scattering may not be negligible for spheroidal bunches in a linac [11]. Thus, the concern has arisen [12] with regard to the possibility that Coulomb collisions between ions may contribute significantly to emittance growth and/or halo formation. It is clear that, because of the importance of this question for the design of high current linear ion accelerators, a more rigorous treatment of the effect of single Coulomb collisions is needed. In an effort to explore this issue in detail we have undertaken an analysis of the effects of Coulomb scattering between ions in a self-consistent spherical bunch.

In this paper we examine the effect of binary Coulomb collisions on our understanding of halo formation. Clearly these collisions are capable of causing the beam bunch to grow in size. But the general expectation is that, for a linac, the rate at which this happens will be negligibly small compared with the rate of diffusion caused by collective oscillations within the bunch.

In the next section, we address general considerations regarding the effect of binary Coulomb collisions. In particular, we relate our perspective to previous studies of the Touschek effect, which is important in circular accelerators where the transverse energy of the particles within the bunch is much greater than the longitudinal energy, allowing particles to escape from the longitudinal bucket. And we also try to relate our perspective to diffusion caused by many small angle Coulomb collisions as treated by the Fokker-Planck equation.

II. GENERAL CONSIDERATIONS

In our present understanding of halo formation in linacs, we consider a 3D beam bunch which may or may not be equipartitioned, stationary, or even rms matched. As previously discussed [3], there is a rapid redistribution in 6D phase space which leads to a stationary beam bunch, with a relatively small oscillating mismatch which is sometimes capable of generating a small halo. The usual halo formation is then caused by a relatively large additional mismatch caused by sudden changes in the frequency, transverse focusing system, accelerating gradient, synchronous phase, etc., or by the accumulation of many small errors in the focusing magnets, rf parameters, etc. And our quantitative study [3] shows that the parameters of halo formation are similar to those caused by mismatches of an otherwise self-consistent phase space distribution.

We start with our family of self-consistent, equipartitioned phase space distributions

$$f(\boldsymbol{r},\boldsymbol{v}) = \begin{cases} N(H_0 - H)^n = N[G(r) - mv^2/2]^n, & H < H_0, \\ 0, & H > H_0, \end{cases} \tag{2.1}$$

where the Hamiltonian is

$$H(\boldsymbol{r},\boldsymbol{v}) = mv^2/2 + kr^2/2 + e\Phi_{\rm sc}(r), \tag{2.2}$$

and

$$G(r) \equiv H_0 - kr^2/2 - e\Phi_{\rm sc}(r). \tag{2.3}$$

We choose the space charge potential $\Phi_{\rm sc}(r)$ which vanishes at $r = \infty$, H_0 is a constant, the external forces are linear, and $f(\boldsymbol{r},\boldsymbol{v})$ is normalized such that

$$\rho(\boldsymbol{r}) = Q\int d\boldsymbol{v}\, f(\boldsymbol{r},\boldsymbol{v}), \qquad \int d\boldsymbol{r}\, \rho(\boldsymbol{r}) = Q, \tag{2.4}$$

where Q is the total bunch charge.

The probability per unit time (in the coordinate system of the bunch) for a Coulomb collision between ions with velocity $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ is then

$$\frac{dP}{dt} = \int d\boldsymbol{r}\int d\boldsymbol{v}_1\, f(\boldsymbol{r},\boldsymbol{v}_1) \times \int d\boldsymbol{v}_2\, f(\boldsymbol{r},\boldsymbol{v}_2)\,|\boldsymbol{v}_1 - \boldsymbol{v}_2|\, d\Omega_s\, \frac{d\sigma}{d\Omega_s}, \tag{2.5}$$

where $d\sigma/d\Omega_s$ is the classical differential Rutherford cross section. The heart of our calculation is the evaluation in Sec. IV of the 11-dimensional integral in Eq. (2.5) over the range of $\boldsymbol{r}$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, Ω_s which enables particles to leave the finite bunch defined in Eqs. (2.1) and (2.4) and enter the halo region. Before doing that, however, we perform an order of magnitude estimate of Eq. (2.5), assuming a spherical bunch of radius a. We use $|\boldsymbol{v}_1| \sim |\boldsymbol{v}_2| \sim v$ and $\int d\boldsymbol{v}\, f(\boldsymbol{r},\boldsymbol{v}) \sim 1/a^3$ and assume all integrals are finite to obtain the estimate

$$\frac{dP}{cdt} \sim \frac{r_p^2}{\epsilon_N^3}, \tag{2.6}$$

where the classical radius of the ion is given by

$$r_p = \frac{e^2}{4\pi\epsilon_0 mc^2}, \tag{2.7}$$

and where the (projected) normalized emittance of the bunch is

$$\epsilon_N \simeq av/c. \tag{2.8}$$

(The usual units of ϵ_N are mm mrad.) Note that we work in the frame of reference of the bunch, and use the speed of light c in Eq. (2.6) only for dimensional purposes. In Eq. (2.8), v stands for the maximum velocity in the projected direction. We have dropped the dependence of the normalized emittance on the relativistic factor γ, since it is of order 1–2 for the linac examples described in the Introduction.

For a proton bunch with $r_p = 1.5 \times 10^{-18}$ m and $\epsilon_N \simeq 1 \times 10^{-6}$ m rad, Eq. (2.6) yields

$$\frac{dP}{cdt} \sim 10^{-15}/\text{km}\,, \tag{2.9}$$

clearly negligible for a linac.

The problem with the foregoing estimate is that it ignores the divergence of the Coulomb cross section at $\theta_s = 0$, as well as the possible divergence of the distribution $f(\boldsymbol{r}, \boldsymbol{v})$ near $mv^2/2 = G(\boldsymbol{r})$ for values of $n < 0$ in Eq. (2.1). We shall address these issues in Sec. IV. Even there, it is not obvious whether one should restrict the scattering to impact parameters typical of interparticle spacing or the Debye length dictated by collective response of nearby ions (see Appendix D for details).

Another shortcoming of this approach is that it does not take into account the effect of a large number of small scattering angle Coulomb collisions. For this purpose we examine the evolution of the phase space distribution in time due to Coulomb collisions by starting with the Boltzmann equation for an otherwise equipartitioned beam bunch. Specifically, in Sec. V, we start with

$$\frac{\partial f(\boldsymbol{v}_1)}{dt} = n(\boldsymbol{r}) \int d\boldsymbol{v}_2 |\boldsymbol{v}_1 - \boldsymbol{v}_2| \int d\Omega_s' \left[f(\boldsymbol{v}_1')f(\boldsymbol{v}_2') \frac{d\sigma}{d\Omega_s'} (\Omega_s' \to \Omega_s) - f(\boldsymbol{v}_1)f(\boldsymbol{v}_2) \frac{d\sigma}{d\Omega_s'} (\Omega_s \to \Omega_s') \right], \tag{2.10}$$

where $f(\boldsymbol{v})$ is now the velocity distribution at $\boldsymbol{r}$, which takes into account those ions entering and leaving the velocity distribution at $\boldsymbol{v}_1$. In particular, we obtain the rate at which $\langle v_1^2 \rangle$ and $\langle v_1^4 \rangle$ increase in time as a way of estimating the rate at which the bunch diffuses. Our approach differs from the usual treatment leading to the Fokker-Planck equation because the derivatives of $f(\boldsymbol{v})$ in Eq. (2.1) with respect to $\boldsymbol{v}$ can be infinite for $n < 1$.

III. SELF-CONSISTENT PHASE SPACE DISTRIBUTION

We start with a phase space distribution for a spherical bunch which is a function of the individual ion Hamiltonian, and which is therefore self-consistent. Using Eqs. (2.1) and (2.3) we find

$$\rho(r) = QN \int_0^{v_0} 4\pi v^2 \, dv [G(r) - mv^2/2]^n, \tag{3.1}$$

where $v_0^2 = 2G(r)/m$. This leads to

$$\rho(r) = \frac{Q[G(r)]^{n+3/2}}{4\pi \int_0^a r^2 \, dr [G(r)]^{n+3/2}}\,, \tag{3.2}$$

and

$$N = \frac{\Gamma(n + 5/2)\,(m/2\pi)^{3/2}}{4\pi \int_0^a r^2 \, dr [G(r)]^{n+3/2} \Gamma(n + 1)}\,. \tag{3.3}$$

Here a is the outer radius of the spherical bunch, corresponding, for $n > -1$, to the location where $G(a) = 0$. (For $n \leq -1$, the distribution is not normalizable.)

The differential equation satisfied by $G(r)$ is

$$\nabla^2 G(r) = \frac{1}{r} \frac{d^2}{dr^2} [rG(r)] = -3k + \frac{e}{\epsilon_0} \rho(r)\,, \tag{3.4}$$

where Φ_{sc} satisfied the Poisson equation $\nabla^2 \Phi_{\text{sc}} = -e\rho(r)/\epsilon_0$. Using Eq. (3.2), we find

$$\nabla^2 G(r) = -3k + \frac{eQ}{4\pi\epsilon_0} \frac{[G(r)]^{n+3/2}}{\int_0^a r^2 \, dr [G(r)]^{n+3/2}}\,. \tag{3.5}$$

Once $G(r)$ is obtained (by numerical integration, except for the simple case $n = -1/2$), one can write the space charge potential as

$$e\Phi_{\text{sc}}(r) = \begin{cases} eQ/4\pi\epsilon_0 a + k(a^2 - r^2)/2 - G(r), & r \leq a\,, \\ eQ/4\pi\epsilon_0 r, & r \geq a\,. \end{cases} \tag{3.6}$$

Continuity of the potential at $r = a$ is assured because $G(a) = 0$. Continuity of the electric field at $r = a$ requires

$$G'(a) = \frac{eQ}{4\pi\epsilon_0 a^2} - ka\,, \tag{3.7}$$

which can also be confirmed by multiplying Eq. (3.5) by $r^2 dr$ and integrating from $r = 0$ to a.

The rms tune depression can be obtained indirectly from the Vlasov equation

$$\frac{\partial f}{\partial t} = -\sum_i \dot{x}_i \frac{\partial f}{\partial x_i} - \sum_i \ddot{x}_i \frac{\partial f}{\partial \dot{x}_i}\,. \tag{3.8}$$

Using the equation of motion for an ion in the spherical beam bunch

$$\ddot{x} = -\frac{k}{m} x + \frac{eE_x}{m}\,, \tag{3.9}$$

we find for the averages of $x\dot{x}$ and x^2 over the distribution

$$\frac{d}{dt} \langle x\dot{x} \rangle = \langle \dot{x}^2 \rangle - k\langle x^2 \rangle + \frac{e}{m} \langle xE_x \rangle\,, \tag{3.10}$$

$$\frac{d}{dt} \langle x^2 \rangle = 2\langle x\dot{x} \rangle\,. \tag{3.11}$$

If we define the rms size of the distribution as

$$X = [\langle x^2 \rangle]^{1/2}, \tag{3.12}$$

Eqs. (3.10) and (3.11) lead us to

$$\frac{d^2X}{dt^2} = -\frac{kX}{m} + \frac{e}{m} \frac{\langle xE_x \rangle}{X} + \frac{\epsilon_x^2}{X^3}\,, \tag{3.13}$$

where the rms emittance ϵ_x is defined as

$$\epsilon_x = [\langle x^2\rangle\langle \dot{x}^2\rangle - \langle x\dot{x}\rangle^2]^{1/2}. \tag{3.14}$$

Equation (3.13) is clearly an rms envelope equation. This allows us to define the rms tune depression for a matched beam ($d^2X/dt^2 = 0$) as

$$\eta_{\mathrm{rms}}^2 = 1 - \frac{e\langle xE_x\rangle}{k\langle x^2\rangle}. \tag{3.15}$$

Using Eq. (3.6),

$$eE_x = -\frac{ex}{r}\frac{\partial\Phi_{\mathrm{sc}}}{\partial r} = kx + \frac{x}{r}G'(r), \tag{3.16}$$

and $\langle x^2G'(r)/r\rangle = \langle y^2G'(r)/r\rangle = \langle z^2G'(r)/r\rangle$, we find

$$\eta_{\mathrm{rms}}^2 = -\frac{1}{3}\frac{\langle (x^2+y^2+z^2)G'(r)/r\rangle}{k\langle x^2\rangle} = -\frac{1}{3}\frac{\langle rG'(r)\rangle}{k\langle x^2\rangle}. \tag{3.17}$$

Noting that $\langle x^2\rangle = \langle r^2\rangle/3$ for a spherical bunch and that the charge density is proportional to $[G(r)]^{n+3/2}$, we obtain

$$\eta_{\mathrm{rms}}^2 = \frac{3}{(n+5/2)}\frac{\int_0^a r^2\,dr[G(r)]^{n+5/2}}{k\int_0^a r^4\,dr[G(r)]^{n+3/2}}. \tag{3.18}$$

The final item in this section is an examination of the kinematics of the Coulomb collision. Clearly the ions which travel to the largest radius before turning back are the ones which have the largest possible velocity at a given r and where, after the collision, one is left at rest and the other travels radially with all the kinetic energy. Such a collision can only conserve energy and momentum when the colliding ion velocities are at 90° to one another. With the maximum velocity given before the collision by $mv^2/2 = G(r)$ in Eq. (3.1), the maximum energy after the collision is

$$\frac{mv_{\max}^2}{2} = 2G(r). \tag{3.19}$$

The maximum radius R then satisfies

$$2G(r) + \frac{kr^2}{2} + e\Phi_{\mathrm{sc}}(r) = \frac{kR^2}{2} + e\Phi_{\mathrm{sc}}(R). \tag{3.20}$$

Using Eq. (3.6), where $r < a$, $R > a$, we have

$$\frac{k(R^2-a^2)}{2} - \frac{eQ}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{R}\right) = G(r). \tag{3.21}$$

We proceed further in this analysis for $n = -1/2$, where

$$G(r) = \frac{3k}{\kappa^2}\left[1 - \frac{i_0(\kappa r)}{i_0(\kappa a)}\right]. \tag{3.22}$$

Here

$$\kappa^2 = \frac{eQ}{4\pi\epsilon_0\int_0^a r^2\,dr\,G(r)}, \tag{3.23}$$

and

$$i_0(w) = \frac{\sinh w}{w}. \tag{3.24}$$

From Eq. (3.21) it is clear that the outermost particles are those which start near the bunch center. We then rewrite Eq. (3.21) as

$$u^3 + \left[-\frac{12}{\kappa^2a^2} + \frac{6}{\kappa a}\frac{(1+c)}{s} - 3\right]u + 6\left(1 - \kappa a\frac{c}{s}\right)\Big/\kappa^2a^2 + 2 = 0, \tag{3.25}$$

where $u = R/a$, $s = \sinh(\kappa a)$, $c = \cosh(\kappa a)$. We now use Eq. (3.18) to obtain

$$\frac{4}{15}\eta_{\mathrm{rms}}^2 = \frac{\kappa^2a^2(5-3\xi^2) - 9\kappa a\xi + 12}{\kappa^4a^4 - 5\kappa^3a^3\xi + 15\kappa^2a^2 - 30\kappa a\xi + 30}, \tag{3.26}$$

where $\xi = \coth(\kappa a)$. For large κa, $\xi \cong 1$ and Eq. (3.26) can be approximated by

$$\eta_{\mathrm{rms}}^2 \simeq \frac{15}{2\kappa^2a^2}, \tag{3.27}$$

which has been generalized to

$$\eta_{\mathrm{rms}}^2 = \frac{15}{15 + 2\kappa^2a^2} \tag{3.28}$$

in order to have $\eta_{\mathrm{rms}} = 1$ when $\kappa a = 0$ (no space charge). For large κa, Eq. (3.25) can be easily solved, and we obtain

$$\frac{R-a}{a} \simeq \frac{\sqrt{3}-1}{\sqrt{\kappa^2a^2 + 15/2}} \to \frac{\sqrt{2}(\sqrt{3}-1)}{\sqrt{15}}\eta_{\mathrm{rms}} \cong 0.27\eta_{\mathrm{rms}}. \tag{3.29}$$

An interesting consequence of Eq. (3.29) is that the thickness of the shell populated by the scattered ions decreases as the beam becomes more space charge dominated. We remind the reader that Eq. (3.29) was obtained for the self-consistent equipartitioned distribution with $n = -1/2$. In his report [13] on numerical studies for distributions which were not self-consistent, Pichoff observed a similar dependence on tune depression.

Single scattered ions can occupy a spherical shell of relatively small thickness. Multiple ion collisions are needed for the distribution to extend beyond this radius. At first glance this thin shell does not have an extent comparable to the halo in our earlier description [1,2]. However, the shell thickness given by Eq. (3.29) was obtained for an equipartitioned beam. We now use Pichoff's definition of the equipartitioning factor $\chi = v_z/v_x$. The maximum possible velocity after the collision can be rewritten as $mv_{\max}^2/2 = (1+\chi^2)G$, and we obtain for the shell thickness

$$\frac{R-a}{a} \simeq \frac{\sqrt{1+2\chi^2}-1}{\sqrt{15/2}}\eta_{\mathrm{rms}}. \tag{3.30}$$

Clearly, when the beam is nonequipartitioned or the beam with the stationary distribution is rms mismatched, the thickness of the shell can be significantly larger, depending on the equipartitioning factor. Such an increased shell thickness can then easily resemble the typical halo extent [1,2]. Thus, in order to understand whether scattering can contribute to the halo formation we need to calculate the rate at which this shell becomes populated, which is our next step in Sec. IV.

We note that solutions given by Eqs. (3.29) and (3.30) can be used with good accuracy only for large values of κa ($\kappa a > 4$) corresponding to tune depressions $\eta < 0.6$, which covers our range of interest [6]. For $\eta \geq 0.6$ a better solution of Eq. (3.25) should be used. For completeness, we present below some values based on the numerical solution of Eq. (3.25). For the equipartitioned beam ($\chi = 1$), the shell extent becomes $R/a = 1.41$, 1.37, 1.32, 1.27, 1.22 for $\eta = 1$, 0.9, 0.8, 0.7, 0.6, respectively. Similar values for the shell thickness were obtained by Pichoff for the distribution functions used in [13]. In fact, in the limit of zero space charge, the shell thickness becomes independent of the distribution, and, from Eq. (3.20), is simply given by $R/a = \sqrt{1 + \chi^2}$ (see also [13]).

IV. COULOMB SCATTERING RATE

A. Order of magnitude estimate

Consider an ion at location $\boldsymbol{r}$ with velocity $\boldsymbol{v}_1$ colliding with an ion with velocity $\boldsymbol{v}_2$ at that same location. The probability rate for scattering into the solid angle Ω_s, averaged over the distribution, is given by

$$\frac{dP}{dt} = \int d\boldsymbol{r} \int d\Omega_s \int d\boldsymbol{v}_1 f(\boldsymbol{r}, \boldsymbol{v}_1) \times \int d\boldsymbol{v}_2 f(\boldsymbol{r}, \boldsymbol{v}_2) |\boldsymbol{v}_1 - \boldsymbol{v}_2| \frac{d\sigma}{d\Omega_s}, \quad (4.1)$$

where $f(\boldsymbol{r}, \boldsymbol{v})$ is given by Eq. (2.1), and the classical nonrelativistic Coulomb collision cross section is given by

$$\frac{d\sigma}{d\Omega_s} = \left(\frac{e^2}{4\pi m \epsilon_0 |\boldsymbol{v}_1 - \boldsymbol{v}_2|^2} \right)^2 \frac{1}{(1 - \cos\theta_s)^2} \quad (4.2)$$

for a scattering angle θ_s. Note that in quantum mechanics the fact that colliding particles are identical leads to the exchange process which modifies the cross section at large scattering angles [for example, for $\theta_s = 90°$ there is a factor of 4 difference with the classical formula given by Eq. (4.2)]. In the nonrelativistic limit the cross section was first obtained by Mott [14], and then generalized for arbitrary velocities by Möller [15]. For small scattering angles, the Möller formula [7,9] gives the classical Rutherford cross section. The Möller formula is usually used when one is mainly concerned with large angle scattering, as for the Touschek effect [7,9]. However, when the effect is dominated by small angle scattering, the Rutherford formula should give sufficiently accurate results.

Our task is to find the rate at which the scattering event described in Eq. (4.1) causes the trajectory of ion #1 (or ion #2) to go beyond $r = a$. Clearly this will be only a portion of the 11-dimensional integration space in Eq. (4.1).

Making an order of magnitude estimate for a beam bunch, similar to Sec. II, and using $r_p = 1.5 \times 10^{-18}$ m and $\epsilon_N \simeq 10^{-6}$ m rad, we find

$$\frac{1}{c}\frac{dP}{dt} \sim 10^{-18}/\text{m} = 10^{-15}/\text{km}, \quad (4.3)$$

clearly negligible for a linac whose length is of the order of a few km. In fact, the effective length of acceleration is the actual linac length multiplied by the average value of $1/\beta\gamma$ for the linac, which we approximate by 1 for our applications. Note that in the estimates, leading to Eq. (4.3), no constraint has yet been made on the ion to leave the bunch. Thus, Eq. (4.3) describes the probability of the collision as does Eq. (2.6). In Sec. IV C we discuss the necessary constraint on the region of integration which is needed to estimate the probability that an ion leaves the bunch.

The above estimate will be confirmed in later sections for values $n > 0$. But for values of n between -1 and 0, we will see that a cutoff is needed for small angle Coulomb scattering.

B. Rate of escape from the bunch

We now proceed to the analysis of the rate of escape from the bunch for our self-consistent distribution in Eq. (2.1). We normalize all velocities as

$$v^2 = [2G(r)/m]u^2, \quad (4.4)$$

and obtain

$$\frac{dP}{dt} = K \int d\boldsymbol{r}[G(\boldsymbol{r})]^{2n+3/2} \int d\boldsymbol{u}_1 \int d\boldsymbol{u}_2 \times \int \frac{d\Omega_s}{(1 - \cos\theta_s)^2} \frac{(1 - u_1^2)^n (1 - u_2^2)^n}{|\boldsymbol{u}_1 - \boldsymbol{u}_2|^3}, \quad (4.5)$$

where

$$K = \frac{r_p^2 c (mc^2)^{3/2}}{2^{3/2}\pi^3} \frac{\Gamma^2(n + 5/2)}{\Gamma^2(n + 1)} \frac{1}{[\int d\boldsymbol{r}\, G^{n+3/2}(\boldsymbol{r})]^2}. \quad (4.6)$$

Here $r_p = e^2/4\pi\epsilon mc^2$ is the "classical" radius of an ion of mass m.

We now transform $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ according to

$$\boldsymbol{p} = (\boldsymbol{u}_1 + \boldsymbol{u}_2)/2, \qquad \boldsymbol{q} = \boldsymbol{u}_1 - \boldsymbol{u}_2, \qquad \boldsymbol{u}_1 = \boldsymbol{p} + \boldsymbol{q}/2, \qquad \boldsymbol{u}_2 = \boldsymbol{p} - \boldsymbol{q}/2, \quad (4.7)$$

where $\boldsymbol{p}$ is the center of mass motion and $\boldsymbol{q}$ is the relative motion of ions #1 and #2. After the Coulomb collision, we have

$$\boldsymbol{u}_1' = \boldsymbol{p} + \boldsymbol{q}'/2, \qquad \boldsymbol{u}_2' = \boldsymbol{p} - \boldsymbol{q}'/2, \tag{4.8}$$

with $q' = q$ and $\cos\theta_s = \boldsymbol{q}\boldsymbol{q}'/q^2$. We select our coordinate system for the integration over scattering angle such that

$$\begin{aligned} &\boldsymbol{p} \text{ lies along the polar axis } (\theta_p = 0), \\ &\boldsymbol{q} \text{ lies in the } x\text{-}z \text{ plane } (\theta_q = \psi, \phi_q = 0), \\ &\boldsymbol{q}' \text{ has polar angles } (\theta_{q'} \equiv \beta, \phi_{q'} \equiv \gamma). \end{aligned} \tag{4.9}$$

We then have

$$\cos\theta_s = \cos\beta\cos\psi + \sin\beta\sin\psi\cos\gamma, \tag{4.10}$$

$$u_{1,2}^2 = p^2 \pm pq\cos\psi + q^2/4. \tag{4.11}$$

We now explore the kinematics of the collision which allows us to exclude all scattering events in which neither ion is able to leave the bunch, and proceed with the integration.

C. Kinematics of the collision

If the scattered particle does not move along a radius it may not be able to escape from the bunch since its velocity cannot go to zero (if it went to zero we could not satisfy the conservation of angular momentum). This needs to be taken into account if we are interested in the situation where the particle can leave the bunch.

Let us consider a Coulomb collision at $\boldsymbol{r} = \boldsymbol{r}_0$ in which ion #1 is scattered from velocity $\boldsymbol{v}_1$ to velocity $\boldsymbol{v}_1'$. Since the $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ distributions are isotropic, all final directions of $\boldsymbol{v}_1'$ are equally likely; we take the direction of $\boldsymbol{v}_1'$ to be at an angle θ_0 to $\boldsymbol{r}_0$ with likelihood

$$\frac{1}{2}\int d\cos\theta_0, \tag{4.12}$$

where the integral is taken over those values of $\cos\theta_0$ which permit ion #1 to travel beyond the bunch boundary at $r = a$. This condition, according to Eq. (2.2), is when

$$\frac{mv_1'^2}{2} + \frac{kr_0^2}{2} + e\Phi_{\mathrm{sc}}(r_0) > \frac{mv_a^2}{2} + \frac{ka^2}{2} + e\Phi_{\mathrm{sc}}(a), \tag{4.13}$$

where v_a is the minimum possible value of v_1 at $r = a$ necessary to satisfy conservation of angular momentum. Clearly this occurs when $\boldsymbol{v}_a$ has only a tangential component, for which angular momentum conservation requires that

$$mv_1' r_0 \sin\theta_0 = mv_a a, \tag{4.14}$$

in which case Eq. (4.13) leads to

$$\frac{mv_1'^2}{2}\left(1 - \frac{r_0^2}{a^2}\sin^2\theta_0\right) > \frac{k(a^2 - r_0^2)}{2} + e\Phi_{\mathrm{sc}}(a) - e\Phi_{\mathrm{sc}}(r_0). \tag{4.15}$$

Using Eq. (3.6), Eq. (4.15) can be rewritten as

$$\frac{mv_1'^2}{2} > \sigma G(r_0), \tag{4.16}$$

where

$$\frac{1}{\sigma} = 1 - \frac{r_0^2}{a^2}\sin^2\theta_0. \tag{4.17}$$

The constraint on $\boldsymbol{u}_1'$ which allows ion #1 to reach beyond $r = a$ is

$$\boldsymbol{u}_1'^2 > \sigma. \tag{4.18}$$

According to Eq. (4.8) and the choice of axes in Eq. (4.9), this requires

$$p^2 + \boldsymbol{p}\boldsymbol{q}' + \frac{q'^2}{4} = p^2 + \frac{q^2}{4} + pq\cos\beta > \sigma, \tag{4.19}$$

a condition *independent* of the azimuthal scattering angle γ.

The integral over $d\Omega_s$ in Eq. (4.5) can now be reduced to

$$\begin{aligned} A &\equiv \int \frac{d\Omega_s}{(1 - \cos\theta_s)^2} \\ &= \int d\cos\beta \\ &\quad \times \int_0^{2\pi} \frac{d\gamma}{[1 - (\cos\beta\cos\psi + \sin\beta\sin\psi\cos\gamma)]^2} \\ &= 2\pi \int d\cos\beta \frac{(1 - \cos\beta\cos\psi)}{|\cos\beta - \cos\psi|^3}, \end{aligned} \tag{4.20}$$

where the region of integration over $\cos\beta$ is constrained by Eq. (4.19). Specifically, the range of integration over $\cos\beta$ determined by Eq. (4.19) is

$$1 \geq \cos\beta > \frac{\sigma - p^2 - q^2/4}{pq} \equiv t_0. \tag{4.21}$$

Using Eq. (4.11), we find

$$\begin{aligned} \cos\beta - \cos\psi = t_0 - \cos\psi &> \frac{\sigma - p^2 - q^2/4}{pq} \\ &\quad - \frac{u_1^2 - p^2 - q^2/4}{pq} \\ &= \frac{\sigma - u_1^2}{pq}. \end{aligned} \tag{4.22}$$

Because $\sigma > 1$ and $u_1 \leq 1$, $\cos\beta - \cos\psi$ is always positive, and we can remove the absolute value signs in Eq. (4.20).

We now integrate over $t \equiv \cos\beta$ in Eq. (4.20) to obtain

$$A = 2\pi\int_{t_0}^1 \frac{dt(1 - t\cos\psi)}{(t - \cos\psi)^3} = \pi\frac{(1 - t_0^2)}{(t_0 - \cos\psi)^2}. \tag{4.23}$$

Using Eqs. (4.21) and (4.22), we can write

$$A = \pi \frac{p^2 q^2 - (\sigma - p^2 - q^2/4)^2}{[\sigma - (p^2 + q^2/4 + pq\cos\psi)]^2}. \tag{4.24}$$

If the angle between $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ is denoted by θ_{12}, we can define $\lambda \equiv \cos\theta_{12}$ and write from Eq. (4.7)

$$\begin{aligned} 4p^2 &= u_1^2 + u_2^2 + 2\lambda u_1 u_2, \\ q^2 &= u_1^2 + u_2^2 - 2\lambda u_1 u_2. \end{aligned} \tag{4.25}$$

Substituting these expressions in Eq. (4.24) and using Eq. (4.11) for $\cos\psi$, we obtain

$$A = \pi \frac{[\sigma(u_1^2 + u_2^2 - \sigma) - \lambda^2 u_1^2 u_2^2]}{(\sigma - u_1^2)^2}. \tag{4.26}$$

We complete the evaluation of the rate of escape from the bunch in the Appendices. In Appendix D we discuss the question of singularities and the required cutoff associated with small scattering angles at large impact parameters in the ion-ion collision. We finally find that the fraction of ions which leaves the beam (and form a spherical layer around the bunch) per unit length is

$$\frac{dP}{cdt} \sim \begin{cases} r_p^2/\epsilon_N^3, & n > 0, \\ (r_p^2/\epsilon_N^3)\ln(\epsilon_N^2/r_p a), & n = 0, \\ (r_p^2/\epsilon_N^3)(\epsilon_N^2/r_p a)^{-n}, & 0 < -n < 1, \end{cases} \tag{4.27}$$

where, for the distributions with $0 \geq n > -1$, we assume that the Coulomb force between ions is screened at the Debye length λ_D (see Appendix D). Using $r_p = 1.5 \times 10^{-18}$ m, $\epsilon_N \cong 10^{-6}$ m rad, and $a \cong 10^{-2}$ m, we find

$$\frac{dP}{cdt} \sim \begin{cases} 10^{-15}/\text{km}, & n > 0, \\ 10^{-14}/\text{km}, & n = 0, \\ 10^{-11}/\text{km}, & n = -0.5, \\ 10^{-8}/\text{km}, & n = -0.9, \end{cases} \tag{4.28}$$

At this time it is useful to point out that the singular behavior for $n < 0$ is related to the singular behavior of the distribution in Eq. (2.1) for the highest velocity particles within the bunch.

V. STRUCTURE OF THE SHELL AND EFFECT OF MULTIPLE COLLISIONS

In plasma physics, when considering the Boltzmann collision integral or Fokker-Planck equation, the interest is usually associated with the multiple scattering effects, and therefore with transport quantities. The definition of transport quantities rests on averaging the action of a large number of particles. Similarly, in the accelerator community, the studies of multiple intrabeam scattering are based on statistical averaging [7–9]. As a result, the transport cross section diverges only logarithmically. The factor which describes this divergence is usually referred to as the Coulomb logarithm.

We now explore the structure of the shell described in Eq. (3.29) for an equipartitioned bunch. Since the cutoff for $n < 0$ is related to those ions which, when scattered, reach a radius just beyond $r = a$, we can model the cutoff behavior by an ion density

$$\tilde{n}(r) = \frac{[r - a + (R - a)\theta_{\min}]^{n-1}(R - r)}{(R - a)^{n+1}}, \tag{5.1}$$

where $\theta_{\min} = r_p a/\epsilon_N^2 \ll 1$ is defined in Eq. (B20) and R is the outer radius of the shell. Equation (5.1) is constructed so that $\tilde{n}(R) = 0$ and satisfies

$$\frac{dp}{cdt} \equiv \int_a^R \tilde{n}(r)dr \cong \begin{cases} [1/n(n+1)], & \text{for } n > 0, \\ \ln(1/\theta_{\min}) - 1, & \text{for } n = 0, \\ \theta_{\min}^n/|n|, & \text{for } n < 0, \end{cases} \tag{5.2}$$

Thus dP/cdt in Eq. (4.27) can be generalized by

$$\frac{dP}{cdt} \sim \frac{r_p^2}{\epsilon_N^3}\frac{dp}{cdt}. \tag{5.3}$$

In an effort to understand the structure of the shell in greater detail, we examine the scattering in velocity space by starting with the Boltzmann equation for the phase space density. For our self-consistent distribution, the Boltzmann equation, which accounts for the scattering of particles into and out of regions of velocity space, can be written as

$$\begin{aligned} \frac{\partial f(\boldsymbol{u}_1)}{\partial t} &= K n_D \int d\boldsymbol{u}_2 |\boldsymbol{u}_1 - \boldsymbol{u}_2| \\ &\times \int d\Omega_s \bigg[\frac{d\sigma}{d\Omega_s}(\boldsymbol{u}_1', \boldsymbol{u}_2' \to \boldsymbol{u}_1, \boldsymbol{u}_2) f(\boldsymbol{u}_1') f(\boldsymbol{u}_2') \\ &\quad - \frac{d\sigma}{d\Omega_s}(\boldsymbol{u}_1, \boldsymbol{u}_2 \to \boldsymbol{u}_1', \boldsymbol{u}_2') f(\boldsymbol{u}_1) f(\boldsymbol{u}_2) \bigg]. \end{aligned} \tag{5.4}$$

Here n_D is the ion particle density and $d\sigma/d\Omega_s$ is the Coulomb scattering cross section for the initial and final states. Apart from constants, which are absorbed in K, $f(\boldsymbol{u})$ is again $(1 - u^2)^n$, as used in Eq. (4.5).

The Rutherford cross section, apart from constants, can be written in the form

$$\frac{d\sigma}{d\Omega_s} d\Omega_s = \frac{2d\boldsymbol{q}'}{q}\frac{\delta((q')^2 - q^2)}{[(\boldsymbol{q}' - \boldsymbol{q})^2 + \theta_m^2 q^2]^2}, \tag{5.5}$$

which incorporates the necessary cutoff angle and the fact that $q' = q$ in the center of mass system of the colliding ions. Note that the two forms of $d\sigma/d\Omega_s$ are the same in Eq. (5.4). We now change the integration variable from $\boldsymbol{u}_2$ to $\boldsymbol{q} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, expressing all quantities in terms of $\boldsymbol{u}_1$, $\boldsymbol{q}$, and $\boldsymbol{q}'$. For example

$$\begin{aligned} \boldsymbol{u}_2 = \boldsymbol{u}_1 - \boldsymbol{q}, \qquad \boldsymbol{u}_1' &= \boldsymbol{u}_1 + (\boldsymbol{q}' - \boldsymbol{q})/2, \\ \boldsymbol{u}_2' = \boldsymbol{u}_2 - (\boldsymbol{q}' - \boldsymbol{q})/2 &= \boldsymbol{u}_1 - (\boldsymbol{q}' + \boldsymbol{q})/2. \end{aligned} \tag{5.6}$$

Finally we change from $\boldsymbol{q}$ and $\boldsymbol{q}'$ to

$$\boldsymbol{Q} = \boldsymbol{q}' - \boldsymbol{q}, \qquad \boldsymbol{s} = (\boldsymbol{q}' + \boldsymbol{q})/2, \\ d\boldsymbol{q}d\boldsymbol{q}' = d\boldsymbol{s}d\boldsymbol{Q}, \tag{5.7}$$

so that

$$(q')^2 - q^2 = 2\boldsymbol{Q}\boldsymbol{s}. \tag{5.8}$$

This leads to

$$\frac{\partial f(\boldsymbol{u}_1)}{\partial t} = 2Kn_D \int d\boldsymbol{s} \int \frac{d\boldsymbol{Q}\,\delta(2\boldsymbol{Q}\boldsymbol{s})}{(Q^2 + q^2\theta_{\min}^2)^2} \times \{[(1 - (\boldsymbol{u}_1 + \boldsymbol{Q}/2)^2)(1 - (\boldsymbol{u}_1 - \boldsymbol{s})^2)]^n - [(1 - \boldsymbol{u}_1^2)(1 - (\boldsymbol{u}_1 - \boldsymbol{s} + \boldsymbol{Q}/2)^2)]^n\}. \tag{5.9}$$

It should be noted that we are using the stationary distributions on the right-hand side of Eq. (5.9) and not their evolution in time. For this reason our calculation of $\partial f/\partial t$ will give only the initial rate of change of f with time.

We now calculate the rate of change of $\langle u_1^2\rangle$ and $\langle u_1^4\rangle$ and find (see Appendix C for details) that $d\langle u_1^2\rangle/dt = 0$ and that the lowest nonvanishing power occurs for $d\langle u_1^4\rangle/dt$,

$$\frac{d\langle u_1^4\rangle}{dt} = \frac{4\pi^2}{18} Kn_D\left[\ln\left(\frac{1}{\theta_{\min}}\right) - 1\right]. \tag{5.10}$$

It is possible to calculate the rate of change of the expectation value of $(1 - u_1^2)^n$ for all values of n, with the same result as in Eq. (5.10), except for a numerical factor of order 1. In fact, the results obtained suggest the expected rounding of the $n = 0$ distribution near $u = 1$.

Similar logarithmic divergence occurs when one studies transport quantities, or the effect of multiple collisions based on statistical averaging. We therefore expect that multiple scattering simply leads to a generalization of our previous result in Sec. IV with logarithmic behavior

$$\frac{dP}{cdt} \sim \frac{r_p^2}{\epsilon_N^3} \ln\left(\frac{1}{\theta_D}\right), \tag{5.11}$$

where θ_D is the minimum angle corresponding to the Debye length impact parameter. If so, the rate of scattering due to multiple collisions is only slightly higher than the rate for single encounters for the distributions with $n > 0$. For the same parameters as those used in Sec. IV the rate becomes clearly negligible with $dP/cdt \sim 10^{-14}$/km.

At this point we note that the Boltzmann form for the collision integral is based on an assumption that the duration of a collision is much less than the time between collisions. In our case, for the parameters similar to those in [6], the duration of a collision is the transit time for the particles to travel a Debye length, as in plasma. Because of the large number of particles inside the Debye sphere (see Appendix D for details), the interaction is by no means binary, and the ion simultaneously interacts with all the other particles in the Debye sphere. To properly estimate the effect of multiple collisions, a better model for the collision operator is needed. The Fokker-Planck approximation seems to be a good tool for implementing such a model. This approximation is justified as long as the Coulomb logarithm $\Lambda_c \gg 1$.

We have examined the derivation of the damping and diffusion coefficients in the Fokker-Planck formulation [16,17] in order to estimate dP/dt for multiple collisions. The details depend on the particular distribution $f(\boldsymbol{x}, \boldsymbol{v})$ used, but, apart from a numerical constant depending on the distribution, we find that Eq. (5.11) is still valid. Thus we expect that Eq. (5.11) gives the correct order of magnitude estimate for multiple collisions. The Fokker-Planck approximation and its numerical implementation is currently under consideration at LANL [18].

VI. SUMMARY AND CONCLUSIONS

In Secs. III and IV we have calculated the effect of single Coulomb scattering of a self-consistent 6D distribution for a spherical beam bunch. In this calculation we find:

(i) Single collisions are capable of populating a thin spherical shell around the beam bunch. This result is for the stationary phase space distribution with $n = -1/2$. Numerical studies for the nonstationary distributions by Pichoff [13] indicate that the shell extent is insensitive to the distribution. We therefore believe that our result for $n = -1/2$ is probably quantitatively similar for other higher values of n.

(ii) When the beam is nonequipartitioned or the beam with the stationary distribution is rms mismatched, the thickness of the shell can be significantly larger, depending on the equipartitioning factor.

(iii) For the relatively singular distribution with $n = -1/2$, a proton bunch with a normalized emittance $\epsilon_N \sim 10^{-6}$ m rad and a radius of 1 cm will populate the shell with a probability of 10^{-11} per kilometer of linac. Note that for the singular distribution the rate would be significantly larger if one considered only the radial scattering. However, since $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are isotropic, the velocities after collision are also isotropic. As a result, we have to integrate over all scattering directions as well, confining our integration space only to the regions where one of the ions is able to travel beyond the spherical bunch boundary. The correct treatment provides the small scattering rate for the singular distribution given above.

(iv) For distributions with $n > 0$, this rate of population is further reduced by a factor of 10^{-4}.

Our conclusion is that the effect of single Coulomb collisions on halo development in high current ion linear accelerators is not important. A similar analysis for nonstationary distributions was performed by Pichoff [13], who arrived at the same conclusion.

In Sec. V we related our analysis to diffusion caused by many small angle Coulomb collisions, with the conclusion that the effect of multiple Coulomb collisions in halo development in high current ion accelerators is also not expected to be important.

Another effect, associated with collisions, which can contribute to halo formation, is the interaction of beam particles with the residual gas. This effect was recently studied by Pichoff *et al.* [19]. A rough estimate suggests that in high current linear accelerators such a scattering process is also insignificant for typical residual gas pressure 10^{-10} atm [6].

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APPENDIX A

In Eq. (4.5) we have written the rate of escape of ion #1 from the bunch as

$$\frac{dP}{dt} = 4\pi K \int_0^a r^2\, dr [G(r)]^{2n+3/2} \times \int d\boldsymbol{u}_1 (1 - u_1^2)^n \int \frac{d\boldsymbol{u}_2 (1 - u_2^2)^n}{|\boldsymbol{u}_1 - \boldsymbol{u}_2|^3} A\,, \tag{A1}$$

where A is obtained using Eq. (4.26). Here σ is defined in Eq. (4.17) as

$$\sigma = \left[1 - \frac{r^2}{a^2}\cos^2\theta\right]^{-1} \geq 1\,, \tag{A2}$$

where we have dropped the subscript on both $\boldsymbol{r}_0$ and θ_0. We define the parameter $\lambda = \cos\theta_{12}$, where θ_{12} is the angle between $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. As a final step we will later integrate over $\cos\theta$ according to the constraint in Eq. (4.21).

We first integrate over the directions of $\boldsymbol{u}_2$ with respect to $\boldsymbol{u}_1$. Using $|\boldsymbol{u}_1 - \boldsymbol{u}_2|^3 = (u_1^2 + u_2^2 - 2u_1u_2\lambda)^{3/2}$, we have

$$\begin{aligned} B &= \int d\boldsymbol{u}_1 (1 - u_1^2)^n \int \frac{d\boldsymbol{u}_2 (1 - u_2^2)^n}{|\boldsymbol{u}_1 - \boldsymbol{u}_2|^3} A \\ &= 8\pi^3 \int_0^1 \frac{u_1^2 du_1 (1 - u_1^2)^n}{(\sigma - u_1^2)^2} \int_0^1 u_2^2 du_2 (1 - u_2^2)^n C\,, \end{aligned} \tag{A3}$$

where

$$C = \int_{\lambda_{\min}}^{\lambda_{\max}} \frac{d\lambda[\sigma(u_1^2 + u_2^2 - \sigma) - \lambda^2 u_1^2 u_2^2]}{[u_1^2 + u_2^2 - 2\lambda u_1 u_2]^{3/2}}\,. \tag{A4}$$

The limits on λ are, in principle, -1 to 1. But we must *exclude* those values of λ for which $t_0^2 = \cos^2\beta_{\max,\min} > 1$ in Eq. (4.23), for which A must be set $= 0$. This means that the integration must be confined to the region where

$$p^2 q^2 \leq \left(\sigma - p^2 - \frac{q^2}{4}\right)^2, \tag{A5}$$

consistent with $A \geq 0$ in Eqs. (4.23) and (4.24). Thus

$$\lambda_{\max,\min} = \pm \frac{\sqrt{\sigma}\sqrt{u_1^2 + u_2^2 - \sigma}}{u_1 u_2}\,. \tag{A6}$$

We now change the integration variable in Eq. (A3) from λ to w, where

$$u_1^2 + u_2^2 - 2\lambda u_1 u_2 = w^2, \qquad u_1 u_2 d\lambda = -w dw\,, \tag{A7}$$

with

$$w_{\max,\min} = \sqrt{\sigma} \pm \sqrt{u_1^2 + u_2^2 - \sigma}\,. \tag{A8}$$

Finally, Eqs. (A7) and (A8) lead to

$$C = \frac{4}{3u_1u_2}(u_1^2 + u_2^2 - \sigma)^{3/2}. \tag{A9}$$

We now proceed to the integrals over u_1 and u_2. Using Eqs. (A3) and (A9), with $u_1^2 = x$, $u_2^2 = y$, we obtain

$$B = \frac{8\pi^3}{3} \int_{x\leq 1} dx \int_{y \leq 1, x+y \geq \sigma} dy \left[\frac{(1 - x)^n (1 - y)^n (x + y - \sigma)^{3/2}}{(\sigma - x)^2}\right]. \tag{A10}$$

Note that the region of integration in the x, y space is the triangle bounded by $x = 1$, $y = 1$, $x + y = \sigma$. We therefore can write

$$\begin{aligned} B &= \frac{8\pi^3}{3} \int_{\sigma-1}^1 dx \frac{(1 - x)^n}{(\sigma - x)^2} \int_{\sigma - x}^1 dy [y - (\sigma - x)]^{3/2} (1 - y)^n \\ &= \frac{8\pi^3}{3} \frac{\Gamma(n + 1)\Gamma(5/2)}{\Gamma(n + 7/2)} \int_{\sigma-1}^1 dx \frac{(1 - x)^n (x + 1 - \sigma)^{n+5/2}}{(\sigma - x)^2}\,, \end{aligned} \tag{A11}$$

where we used the substitution $1 - y = (x + 1 - \sigma)\eta$ to evaluate the integral over y. With the substitution $x = 1 - (2 - \sigma)\xi$, Eq. (A11) becomes

$$B = \frac{8\pi^3}{3} \frac{\Gamma(n+1)\Gamma(5/2)}{\Gamma(n+7/2)} F(\sigma), \tag{A12}$$

where

$$F(\sigma) = (2-\sigma)^{2n+3/2} \int_0^1 \frac{d\xi\, \xi^n (1-\xi)^{n+5/2}}{[\xi + (\sigma-1)/(2-\sigma)]^2}. \tag{A13}$$

Our result for the rate of escape of *either* ion #1 or #2 from the bunch is therefore

$$\frac{dP}{dt} = \frac{16\pi^{9/2} K\Gamma(n+1)}{\Gamma(n+7/2)} \int_0^a r^2\, dr [G(r)]^{n+3/2} F(\sigma), \tag{A14}$$

where the result has been doubled because of the identity of ions #1 and #2.

Finally, we average over the direction θ, as indicated in Eq. (4.12). Since

$$\cos\theta = \frac{a}{r}\left[\frac{1}{\sigma} - \left(1 - \frac{r^2}{a^2}\right)\right]^{1/2}, \tag{A15}$$

we can change variables from $\cos\theta$ to σ by writing

$$d\cos\theta = -\frac{a}{2r} \frac{d\sigma}{\sigma^{3/2}[1 - \sigma(a^2 - r^2)/a^2]^{1/2}}. \tag{A16}$$

We then obtain for the rate of escape of either ion, averaged over the angle θ,

$$\left\langle \frac{dP}{dt} \right\rangle_\theta = \frac{8\pi^{9/2} Ka\Gamma(n+1)}{\Gamma(n+7/2)} \int_0^a r\, dr [G(r)]^{2n+3/2} \int_1^{\sigma_{\max}} \frac{d\sigma F(\sigma)}{\sigma^{3/2}[1 - \sigma(a^2-r^2)/a^2]^{1/2}}, \tag{A17}$$

where $\sigma_{\max}$ is the smaller of 2 or $a^2/(a^2 - r^2)$. One can change the order of integration to write

$$\frac{dP}{dt} = \frac{8\pi^{9/2} Ka\Gamma(n+1)}{\Gamma(n+7/2)} \int_1^2 \frac{d\sigma F(\sigma)}{\sigma^{3/2}} \int_{a\sqrt{(\sigma-1)/\sigma}} \left[\frac{r\, dr [G(r)]^{2n+3/2}}{(r^2\sigma/a^2 + 1 - \sigma)^{1/2}}\right], \tag{A18}$$

where we suppress the symbol $\langle\,\rangle_\theta$ for later use.

We continue our analysis for different values of n in Appendix B.

APPENDIX B

For $n = -1/2$, we have for $F(\sigma)$ in Eq. (A13),

$$F(\sigma) = (2-\sigma)^{1/2} \int_0^1 \frac{d\xi\, \xi^{-1/2}(1-\xi)^2}{(\xi+\mu)^2}, \tag{B1}$$

where

$$\mu = \frac{\sigma - 1}{2-\sigma}, \qquad 2 - \sigma = \frac{1}{1+\mu}. \tag{B2}$$

The integral in Eq. (B1) can be readily evaluated and is

$$F(\sigma) = \frac{\pi}{2(1+\mu)^{1/2}}\left[\frac{1}{\mu^{3/2}} - g(\mu)\right], \tag{B3}$$

where

$$g(\mu) = \frac{2}{\mu^{1/2}} + 3\mu^{1/2} - \frac{2}{\pi}\left(\frac{1+3\mu}{\mu} - \frac{(1+\mu)(1-3\mu)}{\mu^{3/2}}\tan^{-1}\sqrt{\mu}\right) \tag{B4}$$

is $O(\mu^{-1/2})$ for small μ (σ near 1). As can be seen in Eq. (A18), the integration over σ is therefore divergent at $\sigma = 1$ for the $F(\sigma)$ in Eq. (B3). This singularity involves the region where the scattering angle is small and where a cutoff is needed for small scattering angle (large impact parameter in the ion-ion collision).

We must incorporate a cutoff into the divergent term in Eq. (B3). Specifically, the use of a cutoff implies the modification

$$1 - \cos\theta_s \to 1 - \cos\theta_s + \theta_m^2/2 \tag{B5}$$

in Eq. (4.5), where θ_m is the "cutoff angle." This translates to the change

$$1 - (\cos\beta\cos\psi + \sin\beta\sin\psi\cos\gamma) + \theta_m^2/2 \tag{B6}$$

in Eq. (4.20), with the result that $(\cos\beta - \cos\psi)^3$ is to be replaced by

$$[(\cos\beta - \cos\psi)^2 + \theta_m^2(1 - \cos\beta\cos\psi)]^{3/2}. \tag{B7}$$

It is clear that the singular behavior in Eq. (4.23) comes from the region near $t = \cos\beta = \cos\psi$, so that Eq. (4.23) can be replaced by

$$A \cong \frac{\pi(1 - t_0^2)}{(t_0 - \cos\psi)^2 + \theta_m^2 \sin^2\psi}. \tag{B8}$$

As a result, $C/(\sigma - u_1^2)^2$ in Eqs. (A3) and (A4) is replaced by

$$\frac{C}{(\sigma - u_1^2)^2} \to \int_{\lambda_{\min}}^{\lambda_{\max}} \frac{d\lambda[\sigma(u_1^2 + u_2^2 - \sigma) - \lambda^2 u_1^2 u_2^2]}{[u_1^2 + u_2^2 - 2\lambda u_1 u_2]^{3/2}[(\sigma - u_1^2)^2 + \theta_m^2 u_2^2 u_1^2(1-\lambda^2)]}. \tag{B9}$$

We now note that the singular behavior in Eq. (A10) occurs near $\sigma = 1$ and $x = u_1^2 = 1$. But the numerator of the integrand vanishes at

$$\lambda^2 = \lambda_{\min}^2 = \lambda_{\max}^2 = \frac{\sigma(u_1^2 + u_2^2 - \sigma)}{u_1^2 u_2^2}, \quad \text{(B10)}$$

as seen in Eq. (A5). We therefore replace $\theta_m^2 u_1^2 u_2^2 (1 - \lambda^2)$ in the denominator of the integrand in Eq. (B9) by some sort of average value over the region of integration, that is, by $\alpha^2 \theta_m^2$ where α is of order 1. As a result, C in Eq. (A9) is unchanged, but B in Eq. (A11) is replaced by

$$B = \pi^4 \int_{\sigma - 1}^{1} \frac{dx(1 - x)^{-1/2}(x + 1 - \sigma)^2}{(\sigma - x)^2 + \alpha^2 \theta_m^2}. \quad \text{(B11)}$$

Finally, the singular behavior for small $\sigma - 1$ can be estimated by dropping the term in θ_m^2, but not allowing $\sigma - 1$ to become smaller than $\alpha\theta_m$. This leads to

$$F(\sigma) \simeq \frac{\pi}{2} \begin{cases} (\sigma - 1)^{-3/2}, & \sigma - 1 \geq \alpha\theta_m, \\ (\alpha\theta_m)^{-3/2}, & \sigma - 1 \leq \alpha\theta_m, \end{cases} \quad \text{(B12)}$$

in Eq. (B3) where we have set $\sigma = 1$ and $\mu = 0$ in all terms except the $\mu^{-3/2}$ term. The net result for the escape rate in Eq. (A18) for $n = -1/2$ is then

$$\frac{dP}{dt} \simeq 4\pi^5 K a^2 \int_0^a dr[G(r)]^{1/2} \int_1^\infty d\sigma\, F(\sigma), \quad \text{(B13)}$$

where we have set $\sigma = 1$ in all nonsingular terms. Using Eq. (B12), this becomes

$$\frac{dP}{dt} \simeq 6\pi^6 K a^2 (\alpha\theta_m)^{-1/2} \int_0^a dr[G(r)]^{1/2}, \quad \text{(B14)}$$

where

$$K = \frac{r_p^2 c(mc^2)^{3/2}}{2^{3/2}\pi^4} \frac{1}{16\pi^2} \frac{1}{[\int_0^a r^2\, dr\, G(r)]^2}. \quad \text{(B15)}$$

We then have

$$\frac{dP}{dt} \simeq \frac{3 r_p^2 c(mc^2)^{3/2}}{16\sqrt{2}} \frac{(\alpha\theta_m)^{-1/2} a^2 \int_0^a dr[G(r)]^{1/2}}{[\int_0^a r^2\, dr\, G(r)]^2}, \quad \text{(B16)}$$

where α is of order 1.

To complete our order of magnitude estimate, we consider G to be constant and of order $mv_0^2/2$. At the same time, for a reasonable tune depression (of order 0.5), we expect, from transverse energy considerations, that

$$G \sim \frac{mv_0^2}{2} \sim \frac{eQ}{4\pi\epsilon_0 a} = \frac{e^2(Q/e)}{4\pi\epsilon_0 a}, \quad \text{(B17)}$$

where Q/e is the number of ions in the spherical bunch. The net result for the rate of loss of ions in the bunch per unit length is

$$\frac{1}{c}\frac{dP}{dt} \sim \frac{1}{(\alpha\theta_m)^{1/2}} \frac{r_p^2}{\epsilon_N^3}, \quad \text{(B18)}$$

where $\epsilon_N = av_0/c$ is the projected normalized emittance and c is used for dimensional purposes.

We now need to estimate θ_m. For a maximum impact parameter b, the small angle scattering angle in the center of mass system of the colliding ions is

$$\theta_m \cong \frac{e^2}{4\pi\epsilon_0 mv^2 b}. \quad \text{(B19)}$$

We assume that the Coulomb force between ions is screened at the Debye length λ_D (see Appendix D), which, for a tune depression of order 0.5, is approximately $a/7$. We then have

$$\theta_{\min} \sim \frac{r_p a}{\epsilon_N^2}, \quad \text{(B20)}$$

and

$$\frac{1}{c}\frac{dP}{dt} = \frac{r_p^{3/2}}{\epsilon_N^2 a^{1/2}} \quad (\text{for } n = -1/2). \quad \text{(B21)}$$

For $n > 0$, the situation is different. From Eq. (A13), we see that $F(\sigma)$ will still be singular near $\sigma - 1 = 0$ for $n \leq 1$, varying as

$$F(\sigma) \sim (\sigma - 1)^{n-1}, \quad \text{(B22)}$$

but the subsequent integral in Eq. (A17) or (A18) will converge without a cutoff. While it may be possible to obtain $G(r)$ numerically for $n > 0$ and to perform the integrations in Eqs. (A13) and (A17) or (A18) numerically, we note that $F(\sigma)$ will be of order 1. This permits us to make an order of magnitude estimate using the same guidelines as for $n = -1/2$ to approximate G. The final result is similar to Eq. (B18), but without the cutoff angle θ_m. Specifically,

$$\frac{1}{c}\frac{dP}{dt} \sim \frac{r_p^2}{\epsilon_N^3} \quad (\text{for } n > 0), \quad \text{(B23)}$$

where we neglect any additional n-dependent factor.

APPENDIX C

We now calculate the rate of change of $\langle u_1^2 \rangle$ and $\langle u_1^4 \rangle$. Using Eq. (5.9) we have

$$\frac{d}{dt}\langle u_1^2 \rangle = 2Kn_D \int d\boldsymbol{s} \int \frac{d\boldsymbol{Q}\, \delta(2\boldsymbol{Q}\boldsymbol{s})}{(Q^2 + q^2\theta_{\min}^2)^2} (I_1 - I_2), \quad \text{(C1)}$$

where

$$I_1 = \int d\boldsymbol{u}_1 [(1 - (\boldsymbol{u}_1 + \boldsymbol{Q}/2)^2)(1 - (\boldsymbol{u}_1 - \boldsymbol{s})^2)]^n u_1^2, \quad \text{(C2)}$$

$$I_2 = \int d\boldsymbol{u}_1 [(1 - \boldsymbol{u}_1^2)(1 - (\boldsymbol{u}_1 - \boldsymbol{s} + \boldsymbol{Q}/2)^2)]^n u_1^2, \quad \text{(C3)}$$

and

$$\frac{d\langle u_1^4 \rangle}{dt} = 2Kn_D \int d\boldsymbol{s} \int \frac{d\boldsymbol{Q}}{(Q^2 + q^2\theta_{\min}^2)^2} (J_1 - J_2), \quad \text{(C4)}$$

with

$$J_1 = \int d\boldsymbol{u}_1[(1 - (\boldsymbol{u}_1 + \boldsymbol{Q}/2)^2)(1 - (\boldsymbol{u}_1 - \boldsymbol{s})^2)]^n u_1^4, \tag{C5}$$

$$J_2 = \int d\boldsymbol{u}_1[(1 - \boldsymbol{u}_1^2)(1 - (\boldsymbol{u}_1 - \boldsymbol{s} + \boldsymbol{Q}/2))^2]^n u_1^4. \tag{C6}$$

Finally, we change variables in Eqs. (C2) and (C5) from $\boldsymbol{u}_1$ to

$$\begin{aligned} \boldsymbol{u}_1 + \boldsymbol{Q}/2 &= \boldsymbol{w} + \boldsymbol{s}/2 + \boldsymbol{Q}/4, \\ \boldsymbol{u}_1 - \boldsymbol{s} &= \boldsymbol{w} - \boldsymbol{s}/2 - \boldsymbol{Q}/4, \end{aligned} \tag{C7}$$

and in Eqs. (C3) and (C6) from $\boldsymbol{u}_1$ to

$$\boldsymbol{u}_1 = \boldsymbol{w} + \boldsymbol{q}/2, \qquad \boldsymbol{u}_2 = \boldsymbol{w} - \boldsymbol{q}/2 \tag{C8}$$

to obtain

$$I_1 = \int d\boldsymbol{w}[1 - (\boldsymbol{w} + \boldsymbol{q}'/2)^2]^n \times [1 - (\boldsymbol{w} - \boldsymbol{q}'/2)^2]^n(\boldsymbol{w} - \boldsymbol{q}/2)^2, \tag{C9}$$

$$I_2 = \int d\boldsymbol{w}[1 - (\boldsymbol{w} + \boldsymbol{q}/2)^2]^n \times [1 - (\boldsymbol{w} - \boldsymbol{q}/2)^2]^n(\boldsymbol{w} - \boldsymbol{q}/2)^2, \tag{C10}$$

and

$$J_1 = \int d\boldsymbol{w}[1 - (\boldsymbol{w} + \boldsymbol{q}'/2)^2]^n \times [1 - (\boldsymbol{w} - \boldsymbol{q}'/2)^2]^n(\boldsymbol{w} - \boldsymbol{q}/2)^4, \tag{C11}$$

$$J_2 = \int d\boldsymbol{w}[1 - (\boldsymbol{w} + \boldsymbol{q}/2)^2]^n \times [1 - (\boldsymbol{w} - \boldsymbol{q}/2)^2]^n(\boldsymbol{w} - \boldsymbol{q}/2)^4, \tag{C12}$$

where

$$\boldsymbol{q}' = \boldsymbol{s} + \boldsymbol{Q}/2, \qquad \boldsymbol{q} = \boldsymbol{s} - \boldsymbol{Q}/2. \tag{C13}$$

Clearly the overall region of integration in Eqs. (C9) and (C11) is restricted to

$$(\boldsymbol{w} \pm \boldsymbol{q}'/2)^2 \le 1, \tag{C14}$$

and in Eqs. (C10) and (C12) to

$$(\boldsymbol{w} \pm \boldsymbol{q}/2)^2 \le 1. \tag{C15}$$

To continue the evaluation of Eqs. (C9)–(C12) we treat $\boldsymbol{q}'$ as the polar axis in Eqs. (C9) and (C11) and use $(\psi, 0)$ and (θ, ϕ) as the polar azimuthal angles for $\boldsymbol{q}$ and $\boldsymbol{w}$ respectively, so that

$$\begin{aligned} \boldsymbol{w}\boldsymbol{q}'/2 &= wq\cos\theta \equiv wq\mu, \\ \boldsymbol{w}\boldsymbol{q} &= wq(\mu\cos\psi + \sqrt{1 - \mu^2}\sin\psi\cos\phi). \end{aligned} \tag{C16}$$

Thus, we can write

$$I_1 = \int w^2\,dw \int d\mu\left[\left(1 - w^2 - \frac{q^2}{4}\right)^2 - w^2q^2\mu^2\right]^n I_3, \tag{C17}$$

where

$$I_3 = \int d\phi\left[w^2 + \frac{q^2}{4} - wq(\mu\cos\psi + \sqrt{1 - \mu^2}\sin\psi\cos\phi)\right] = 2\pi\left(w^2 + \frac{q^2}{4} - w\mu q^2\cos\psi\right). \tag{C18}$$

The evaluation of I_2 is even easier, since we use $\boldsymbol{q}$ as the polar angle and have no azimuthal dependence. As a result, we find for $I_1 - I_2$

$$I_1 - I_2 = -2\pi q^2(1 - \cos\psi)\int w^2\,dw \int \mu\,d\mu\left[1 - w^2 - \frac{q^2}{4} - w^2q^2\mu^2\right]^n, \tag{C19}$$

where the region of integration is confined to

$$w^2 + \frac{q^2}{4} - w^2q^2\mu^2 \le 1, \qquad \mu^2 \le 1. \tag{C20}$$

Because the integrand is odd in μ and the limits in Eq. (C20) do not change when $\mu \to -\mu$, we see that $I_1 - I_2 = 0$. This is nothing more than conservation of kinetic energy since $\langle u_1^2 \rangle$ is directly related to the kinetic energy.

The evaluation of the fourth order moments proceeds in a similar manner, but this time

$$J_1 - J_2 = \int w^2\,dw \int d\mu\left[\left(1 - w^2 - \frac{q^2}{4}\right)^2 - w^2q^2\mu^2\right]^n (J_3 - J_4), \tag{C21}$$

where

$$J_3 = \int d\phi \left[w^2 + \frac{q^2}{4} - wq\mu \cos\psi - wq\sqrt{q - \mu^2} \sin\psi \cos\phi \right]^2$$
$$= 2\pi \left[\left(w^2 + \frac{q^2}{4} - wq\mu \cos\psi \right)^2 - \frac{w^2 q^2 (1 - \mu^2) \sin^2\psi}{2} \right]. \tag{C22}$$

For J_4 we just set $\psi = 0$ in the expression for J_3 and drop all odd powers of μ obtaining

$$J_3 - J_4 = \pi w^2 q^2 (1 - \mu^2) \sin^2\psi - w^2 q^2 \mu^2 \sin^2\psi = \pi w^2 q^2 \sin^2\psi (1 - 3\mu^2). \tag{C23}$$

Our result for Eq. (C4) is therefore

$$\frac{d\langle u_1^4 \rangle}{dt} = 2\pi n_D K \int d\boldsymbol{s} \int \frac{d\boldsymbol{Q}\, \delta(2\boldsymbol{sQ})}{(Q^2 + q^2\theta_m^2)^2} q^2 \sin^2\psi \int_0^{w_{\max}} w^4\, dw \int_{\mu_{\min}}^{\mu_{\max}} d\mu (1 - 3\mu^2)$$
$$\times [(1 - w^2 - q^2/4 - wq\mu)(1 - w^2 - q^2/4 + wq\mu)]^n, \tag{C24}$$

where we restrict the integration range to those values of w and μ which satisfy both

$$-1 \le \mu \le \frac{1 - q^2/4}{wq} - \frac{w}{q} \tag{C25}$$

and

$$\frac{w}{q} - \frac{(1 - q^2/4)}{wq} \le \mu \le 1. \tag{C26}$$

By considering only non-negative values for μ and doubling the result, we find

$$\frac{d\langle u_1^4 \rangle}{dt} = 4\pi K n_D \int s^2\, d\boldsymbol{s} \int \frac{Q^2\, d\boldsymbol{Q}\, \delta(2\boldsymbol{sQ})}{(Q^2 + q^2\theta_m^2)^2} \frac{J(q)}{q^2}, \tag{C27}$$

where we have used

$$q^2 \sin^2\psi = q^2 - \frac{(qq')^2}{q^2} = \frac{(s^2 + Q^2/4)^2 - (s^2 - Q^2/4)^2}{q^2} = \frac{s^2 Q^2}{q^2}, \tag{C28}$$

and where

$$J(q) \equiv \int_0^1 d\mu (1 - 3\mu^2) \int_0^{-q\mu/2 + \sqrt{1 - q^2 + q^2\mu^2/4}} w^4\, dw [(1 - w^2 - q^2/2)^2 - w^2 q^2 \mu^2]^n \tag{C29}$$

is only a function of q.

We can now simplify the integral over θ, the angle between $\boldsymbol{s}$ and $\boldsymbol{Q}$, by writing

$$2\pi \int_{-1}^{1} d\cos\theta\, \delta(2sQ\cos\theta) = \pi/sQ, \tag{C30}$$

leading to

$$\frac{d\langle u_1^4 \rangle}{dt} = 4\pi^2 K n_D \int s^3\, ds \int \frac{Q^3\, dQ}{(Q^2 + q^2\theta_{\min^2})^2} \frac{J(q)}{q^2}. \tag{C31}$$

Since $q^2 = s^2 + Q^2/4$, we proceed further by transforming to polar coordinates defined by

$$s = q\sin\phi, \qquad Q = 2q\cos\phi, \qquad dsdQ = 2qdqd\phi, \tag{C32}$$

obtaining

$$\frac{d\langle u_1^4 \rangle}{dt} = 2\pi^2 K n_D \int_0^{q_{\max}} q\, dq\, J(q) \int_0^{\pi/2} \frac{d\phi \sin 2\phi (1 - \cos^2 2\phi)}{(\cos 2\phi + \alpha)^2}, \tag{C33}$$

where

$$\alpha = 1 + \theta_{\min}^2/2. \tag{C34}$$

The integral over ϕ is readily performed, leading, for $\theta_{\min}^2 \ll 1$, to

$$\frac{d\langle u_1^4\rangle}{dt} = 4\pi^2 K n_D[\ln(1/\theta_{\min}) - 1]\int_0^2 q\,dq\,J(q)\,. \tag{C35}$$

The evaluation of $J(q)$ for general n must be done numerically. Even for $n = -1/2$ one runs into elliptic integrals. But for $n = 0$, one can proceed further by interchanging the order of integration over μ and w in Eq. (C29). In this case we find that

$$\begin{aligned} J(q) &= \int_{1-q/2}^{\sqrt{1-q^2/4}} w^4\,dw \int_0^{(1-q^2/4-w^2)/qw} d\mu(1 - 3\mu^2) \\ &= \int_{1-q/2}^{\sqrt{1-q^2/4}} w\,dw\left[\frac{w^2}{q}\left(1 - \frac{q^2}{4} - w^2\right) - \frac{(1 - q^2/4 - w^2)^3}{q^3}\right]. \end{aligned} \tag{C36}$$

Substituting $w^2 = 1 - q^2/4 - x$, one finds

$$J(q) = \frac{1}{2}\int_0^{q-q^2/2} dx\left[\frac{x(1 - q^2/4 - x)}{q} - \frac{x^3}{q^3}\right] = \frac{q(1 - q/2)^3}{48}(6 + q)\,. \tag{C37}$$

Finally, for $n = 0$,

$$\int_0^2 q\,dq\,J(q) = \frac{1}{18}\,, \tag{C38}$$

and

$$\frac{d\langle u_1^4\rangle}{dt} = \frac{4\pi^2}{18} K n_D\left[\ln\left(\frac{1}{\theta_{\min}}\right) - 1\right]. \tag{C39}$$

The cause of the logarithmic divergence is the parameter $\sin^2\psi$ in Eq. (C24). Similar parameter and logarithmic behavior occurs when one studies transport quantities, or the effect of multiple collisions based on statistical averaging [7–9].

APPENDIX D

This section addresses the question of the cutoff parameter required for the singular distributions. It is clear that the concept of binary collisions works only up to the mean interparticle spacing. However, extensive study of this question showed that the range of interaction should be extended further. It is now universally agreed that it extends to about 1 Debye length. Proper justification for this is found in the advanced Liouville equation theory, but heuristic reasons can be found in almost any book on plasma physics (see, for example, [20]). The general conclusion is that, if one ignores the fact that the collisions are not binary and the Debye cutoff is used, the momentum transfer in the collision is the same as that obtained by a more exact treatment. For neutral plasmas, the general agreement is that the Debye length should be used as a cutoff parameter. For non-neutral plasmas or charged-particle beams, Jansen indicates that it is more realistic to use the mean interparticle distance [16]. In the accelerator community a similar argument is often followed (see, for example, Le Duff [9] and Struckmeier [21]). We note that Jansen's arguments are valid mainly for low density beams, used in electron and ion beam lithography instruments. In the beams of current interest [6], both the Debye length and Coulomb logarithm are large, which justifies the use of the concepts from plasma physics. Studies of the non-neutral plasma by Davidson and others showed that it resembles the properties of the neutral plasma very closely, including the effect of Debye screening [22]. Therefore, Davidson suggests that the Debye length should be used as a cutoff also in non-neutral plasmas and charged particle beams [23]. We also note that the Debye length gives a scattering rate larger than the one corresponding to using the mean interparticle distance as the cutoff parameter. In order to stay on the safe side, we thus choose the Debye length as a cutoff parameter for our singular distribution with $n = -1/2$. Note that no cutoff is required for all other distributions with $n > 0$.

Note that in our case of a proton beam with normalized emittance $\epsilon_N \cong 10^{-6}$ m rad and beam radius $a \cong 10^{-2}$ m, the beam temperature is relatively high, corresponding to a large Debye length and N_D (number of particles inside the Debye sphere) $\gg$1. This validates the use of the Debye length and the concepts of plasma physics. It also ensures that the Fokker-Planck approach for beams with similar parameters would be appropriate.

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