

Analytic approximation of the head-tail phase difference from continuous transverse excitation for measuring chromaticity

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We will explore a method for measuring chromaticity by continuously kicking the beam transversely. This is called the continuous head-tail method for measuring chromaticity. The complete analytic approximation in terms of trigonometric functions is derived for zero transverse emittance beam. A simple formula for calculating chromaticity from experimental data is also shown. Finally, the theory is compared with experimental data.

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I. INTRODUCTION

Continuous chromaticity control up the ramp and through the squeeze will be vital for operating the Large Hadron Collider (CERN-LHC). It is especially important at the beginning of the ramp because of snapback where chromaticity swings are expected to drift at a rate of 0.33 s^{-1} even with preprogrammed chromaticity correction tables [1]. Furthermore, the tolerance required for nominal 7 TeV head-on collisions operations is ± 1 unit. It is because of these tight tolerances that a chromaticity feedback system has been envisioned for the LHC. However, the best method for measuring chromaticity which has a measurement rate better than 1 Hz has not yet been pinned down. During the 2006 LHC Tune Feedback Review as part of the U.S. LHC Accelerator Research Program (US-LARP), it was pointed out that there are at least five different methods for doing this. One method, however, stirred up considerable interest: the continuous head-tail measurement technique.

To start off the discussion, we will look first at the traditional head-tail method [2]. In this method, the beam is kicked once transversely and, because of chromaticity, the head and the tail dephase differently. When the phase difference between the head and the tail is measured, it can be shown that, when the phases are maximally different, this difference is linearly related to the chromaticity. However, this technique, although proven to work at both CERN and Fermilab, suffers from at least three deficiencies: (i) It causes large emittance growth because of the large kicks. (ii) The chromaticity cannot be continuously measured because after a small number of kicks (< 5), there is so much beam loss that the signal to noise (S/N) becomes really poor. (iii) It is not compatible with the phase locked loop (PLL) tune tracking method.

Some preliminary measurements at the Relativistic Heavy Ion Collider (BNL-RHIC) with a special detector called the [3] direct diode detector baseband tune (3D-BBQ) have shown a phase difference between the head and the tail which changed with chromaticity. Later work at the Super Proton Synchrotron (CERN-SPS) performed by the authors and the CERN beam instrumentation group also measured a phase difference in a few data sets, see Fig. 1. We believe that those that did not were due to saturation in the 3D-BBQ electronics because, although the 3D-BBQ electronics is extremely sensitive, it has been demonstrated at both RHIC and the Tevatron to be prone to saturation [4,5]. With these ambiguous results, it begged a theoretical understanding of whether the phase difference was real or not and thus this paper. If this method can be proven to work it has the following advantages: (i) There is already a continuous transverse kick from the tune tracker PLL. It has already been demonstrated at Fermilab, RHIC, and SPS that these small kicks ~ 1 nrad do not blow up the emittance or cause beam lifetime problems. (ii) No extra modulations are required for the chromaticity measure-

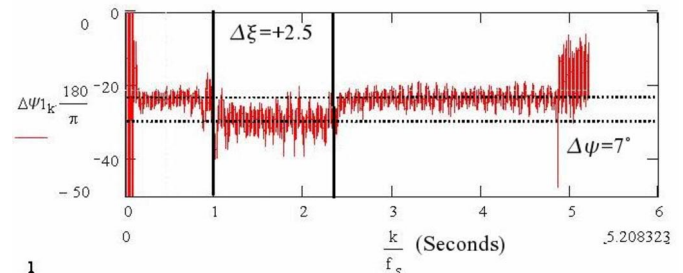


FIG. 1. (Color) Shown here is the phase difference between the head and tail with continuous kicks. The head-tail phase changed when the chromaticity was increased by 2.5 units. This data was collected by the authors and the CERN beam instrumentation group during the SPS machine studies period on 29 September 2006.

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ment. For example, the traditional method for measuring chromaticity requires changing the RF frequency to change the momentum of the beam. It is compatible with the tune tracker PLL.

Therefore, the goal of this paper is to answer the following two questions: (i) Is there a phase difference between the head and tail when the bunch is continuously kicked transversely? (ii) Is there a formula which connects chromaticity to this phase difference if the first item is true?

II. THEORY

The way in which we will find the analytic solution for the phase difference between the head and the tail of the bunch is the following: (i) We write down the ordinary differential equation (o.d.e.) which describes the dynamics of a single particle. (ii) We transform to a rotating frame which simplifies the o.d.e. In this frame, we find that it is a simple harmonic oscillator with two forcing terms: one from the transverse kick of the tune tracker PLL and the other from the radio frequency (RF) (which is the source of the synchrotron frequency). (iii) The method of averaging is used to solve the o.d.e. for two cases: (a) when the transverse kick is on the betatron frequency and (b) when the transverse kick is close to the betatron frequency. (iv) The solution in the rotating frame is transformed back to the laboratory frame. (v) The phase difference between the head and tail is calculated. (vi) The particles in phase space are projected down from momentum space onto position space so as to reflect what is seen at the output of the beam position monitors.

A. The differential equation

We can write down the transverse equation of motion for a single particle as

$$\frac{d^2 X(s)}{ds^2} + \frac{\omega_Q^2}{c^2} X(s) = 0. \quad (1)$$

Here X is the transverse position of the particles, s is the longitudinal coordinate, and ω_Q is the betatron frequency and c is the speed of light. However, if the particle resides in an RF bucket, we must consider its longitudinal motion inside the bucket and so the equation of motion becomes

$$\frac{d^2 X(s, \delta, z)}{ds^2} + \frac{\omega_Q^2(\delta)}{c^2} X(s, \delta, z) = 0. \quad (2)$$

Here z defines the longitudinal position relative to the center of the RF bucket and δ is the relative momentum difference from the “on momentum” particle. If we expand the betatron frequency to first order in δ , we obtain

$$\omega_Q(\delta) = \omega_0 Q + \xi \omega_0 \delta, \quad (3)$$

where ω_0 is the revolution frequency, $Q = \omega_Q/\omega_0$ is the betatron tune, and ξ is the linear chromaticity. Therefore Eq. (2) becomes

$$\frac{d^2 X(s, \delta, z)}{ds^2} + \omega_0^2 (Q + \xi \delta)^2 / c^2 X(s, \delta, z) = 0. \quad (4)$$

We will approximate the longitudinal motion inside the RF bucket with

$$\begin{aligned} \delta(s) &= -\frac{\omega_s}{\eta c} r \sin\left(\frac{\omega_s s}{c} + \phi\right) \\ z(s) &= r \cos\left(\frac{\omega_s s}{c} + \phi\right), \end{aligned} \quad (5)$$

where ω_s is the synchrotron frequency, (r, ϕ) is the position of the particle at $s/c = 0$, and η is the slip factor. We can write down a linear transformation with maps $[z(0), \delta(0)] \equiv (z_0, \delta_0)$ to $[z(s), \delta(s)]$:

$$\begin{pmatrix} z(s) \\ \delta(s) \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\omega_s s}{c}\right) & \frac{\eta c}{\omega_s} \sin\left(\frac{\omega_s s}{c}\right) \\ -\frac{\omega_s}{\eta c} \sin\left(\frac{\omega_s s}{c}\right) & \cos\left(\frac{\omega_s s}{c}\right) \end{pmatrix} \begin{pmatrix} z_0 \\ \delta_0 \end{pmatrix}. \quad (6)$$

We can also think of the map given by Eq. (6) as a frame that is rotating at the synchrotron frequency. Thus in this frame, Eq. (4) becomes

$$\frac{d^2 x(s, \delta_0, z_0)}{ds^2} + \omega_0^2 \left\{ Q + \xi \left[\delta_0 \cos\left(\frac{\omega_s s}{c}\right) - \frac{\omega_s \tau_0}{\eta} \sin\left(\frac{\omega_s s}{c}\right) \right]^2 \right\} x(s, \delta_0, z_0) = 0, \quad (7)$$

where $\tau_0 = z_0/c$ and we have mapped $X \rightarrow x$ to remind us that we are in the rotating frame. Next, let us change variables to use turns n rather than s , i.e.

$$n = \frac{s}{2\pi R} \Rightarrow \frac{dn}{ds} = \frac{1}{2\pi R}, \quad (8)$$

where R is the radius of the accelerator. In this variable, Eq. (7) becomes

$$\frac{d^2 x}{dn^2} + \left[2\pi Q + 2\pi \xi \left(\delta_0 \cos(2\pi Q_s n) - \frac{\omega_s \tau_0}{\eta} \sin(2\pi Q_s n) \right)^2 \right] x = 0, \quad (9)$$

where $s/c = 2\pi n/\omega_0$, $Q_s = \omega_s/\omega_0$ is the synchrotron tune.

If the weak sinusoidal kick from the tune tracker PLL is given by $\epsilon \lambda \cos(2\pi Q_k n)$ where $\epsilon \ll 1$, Q_k is the frequency of the kick in tune units and $\lambda = 1$ has the same dimensions as x to keep the dimensions of the left-hand side and right-hand side (rhs) of the ordinary differential equation correct, then Eq. (9) with this force is

$$\begin{aligned} \ddot{x} + \left[2\pi Q + 2\pi \xi \left(\delta_0 \cos(2\pi Q_s n) - \frac{\omega_s \tau_0}{\eta} \sin(2\pi Q_s n) \right)^2 \right] x &= \epsilon \lambda \cos 2\pi Q_k n, \end{aligned} \quad (10)$$

where $\frac{d^2 x}{dn^2} = \ddot{x}$.

Let us define the following new variables so that it is easier to lug Eq. (10) around:

$$\begin{aligned}\theta_Q &\equiv 2\pi Q & \theta_s &\equiv 2\pi Q_s & \theta_k &\equiv \pi Q_k \\ \nu &\equiv 2\pi\xi\delta_0 \ll 1 & \mu &\equiv -2\pi\xi\omega_s\tau_0/\eta \ll 1.\end{aligned}\quad (11)$$

Thus Eq. (10) becomes

$$\begin{aligned}\ddot{x} + (\theta_Q + \nu \cos n\theta_s + \mu \sin n\theta_s)^2 x &\equiv \ddot{x} + W(n)^2 x \\ &= \epsilon \lambda \cos n\theta_k.\end{aligned}\quad (12)$$

It is clear that (12) is Hill's equation with an external periodic forcing because $W(n) = W(n + 2\pi/\theta_s)$.

B. The single particle solution

The approach to solving Eq. (12) is to find the solution to the homogenous problem, then solve for the first order solutions with the method of variation of parameters and, finally, obtain the single particle solution by the method of averages.

1. Homogeneous solution

The o.d.e. from Eq. (12) with the rhs set to zero is

$$\ddot{y}_h + W(n)^2 y_h = 0, \quad (13)$$

where we have introduced y_h to be the solution of this equation. An approximate solution of Eq. (13) is [6]

$$y_h = \phi_0 e^{\pm i \int W(n) dn}, \quad (14)$$

where $\phi_0 \in \mathbb{C}$ is a constant determined from initial conditions. Therefore, the homogeneous solution is

$$\begin{aligned}y_h &= \phi_{01} \cos\left[n\theta_Q + \frac{\rho}{\theta_s} \sin(n\theta_s - \varphi)\right] \\ &\quad + \phi_{02} \sin\left[n\theta_Q + \frac{\rho}{\theta_s} \sin(n\theta_s - \varphi)\right] \\ &\equiv \phi_{01} y_1(n) + \phi_{02} y_2(n),\end{aligned}\quad (15)$$

where $\phi_{01}, \phi_{02} \in \mathbb{R}$ are constants to be derived from initial conditions and

$$\rho = \sqrt{\nu^2 + \mu^2} \quad \varphi = \arctan \frac{\mu}{\nu}. \quad (16)$$

2. Variation of parameters

The two relevant equations which can be written down from Eq. (15) which are prescribed by the variation of parameters method are

$$\begin{aligned}y &= z_1(n)y_1(n) + z_2(n)y_2(n) \\ \dot{y} &= z_1(n)\dot{y}_1(n) + z_2(n)\dot{y}_2(n),\end{aligned}\quad (17)$$

where z_1 and z_2 are the equations to be solved from initial conditions. The first order differential equations which when integrated give us z_1 and z_2 are

$$\begin{aligned}\dot{z}_1 &= -\frac{\epsilon \lambda \cos n\theta_k}{\theta_Q} \sin\left[n\theta_Q + \frac{\rho}{\theta_s} \sin(n\theta_s - \varphi)\right] \\ &\quad \times \left[1 - \frac{\rho}{\theta_Q} \cos(n\theta_s - \varphi)\right] \\ \dot{z}_2 &= +\frac{\epsilon \lambda \cos n\theta_k}{\theta_Q} \cos\left[n\theta_Q + \frac{\rho}{\theta_s} \sin(n\theta_s - \varphi)\right] \\ &\quad \times \left[1 - \frac{\rho}{\theta_Q} \cos(n\theta_s - \varphi)\right],\end{aligned}\quad (18)$$

if we assume that $\rho/\theta_Q \ll 1$. To integrate the above, we will apply the method of averaging.

3. Averaging

Notice that the rhs of Eq. (18) is not periodic but almost periodic because of the three tunes θ_k , θ_Q , and θ_s . The *nonlinear averaging theorems* described by Hoppensteadt [7] can be used to approximately integrate these two first order differential equations.

The definition of averaging of a function $f(t_1, t_2, \dots)$ with respect to (w.r.t.) t_1 in this theorem is

$$\bar{f}(t_2, t_3, \dots) \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sum_{n=0}^{\infty} f_n(t_2, t_3, \dots) e^{i\omega_n t_1} dt_1, \quad (19)$$

where $f_n()$ are the Fourier coefficients for the expansion of $f(t_1, t_2, \dots)$ in t_1 and ω_n . In this context, $n\theta_Q$ and $n\theta_s$ are treated as independent variables when we perform averaging. Furthermore, we will find solutions for only $|\theta_k - \theta_Q| \ll 1$ because the tune tracker PLL is always assumed to lock onto the betatron tune or close to it.

Suppose the tune tracker PLL locks close or onto θ_Q , i.e. $|\theta_k - \theta_Q| \ll 1$. Let us write

$$\theta_k = \theta_Q + \Delta\theta_Q. \quad (20)$$

When we substitute this into Eq. (18) we get

$$\begin{aligned}\dot{z}_1 &= -\frac{\epsilon \lambda}{\theta_Q} (\cos n\theta_Q \cos n\Delta\theta_Q - \sin n\theta_Q \sin n\Delta\theta_Q) \\ &\quad \times \left(\sin n\theta_Q \cos\left[\frac{\rho}{\theta_s} \sin(n\theta_s - \varphi)\right] \right. \\ &\quad \left. + \cos n\theta_Q \sin\left[\frac{\rho}{\theta_s} \sin(n\theta_s - \varphi)\right] \right) \\ \dot{z}_2 &= +\frac{\epsilon \lambda}{\theta_Q} (\cos n\theta_Q \cos n\Delta\theta_Q - \sin n\theta_Q \sin n\Delta\theta_Q) \\ &\quad \times \left(\cos n\theta_Q \cos\left[\frac{\rho}{\theta_s} \sin(n\theta_s - \varphi)\right] \right. \\ &\quad \left. - \sin n\theta_Q \sin\left[\frac{\rho}{\theta_s} \sin(n\theta_s - \varphi)\right] \right),\end{aligned}\quad (21)$$

if $\rho/\theta_Q^2 \approx 0$. When we average over $n\theta_Q$, we get

$$\begin{aligned}\dot{z}_1 &= \frac{\epsilon\lambda}{2\theta_Q} \left[\sin n\Delta\theta_Q - \frac{\rho}{\theta_s} \cos n\Delta\theta_Q \sin(n\theta_s - \varphi) \right] \\ \dot{z}_2 &= \frac{\epsilon\lambda}{2\theta_Q} \left[\cos n\Delta\theta_Q + \frac{\rho}{\theta_s} \sin n\Delta\theta_Q \sin(n\theta_s - \varphi) \right],\end{aligned}\quad (22)$$

if we assume that $\rho/\theta_s \ll 1$. Equation (22) is actually integrable and so we get

$$\begin{aligned}\bar{z}_1 &= -\frac{\epsilon\lambda}{2\theta_Q} \left(\frac{\cos n\Delta\theta_Q}{\Delta\theta_Q} - \frac{\rho}{\theta_s(\Delta\theta_Q^2 - \theta_s^2)} \right. \\ &\quad \times [\theta_s \cos n\Delta\theta_Q \cos(n\theta_s - \varphi) \\ &\quad \left. + \Delta\theta_Q \sin n\Delta\theta_Q \sin(n\theta_s - \varphi)] \right) \\ \bar{z}_2 &= +\frac{\epsilon\lambda}{2\theta_Q} \left(\frac{\sin n\Delta\theta_Q}{\Delta\theta_Q} - \frac{\rho}{\theta_s(\Delta\theta_Q^2 - \theta_s^2)} \right. \\ &\quad \times [\theta_s \sin n\Delta\theta_Q \cos(n\theta_s - \varphi) \\ &\quad \left. - \Delta\theta_Q \cos n\Delta\theta_Q \sin(n\theta_s - \varphi)] \right).\end{aligned}\quad (23)$$

As expected, as $\Delta\theta_Q \rightarrow 0$, $\cos n\Delta\theta_Q/\Delta\theta_Q \rightarrow \infty$ in the $\bar{z}_1$ equation. This singularity will be removed when the initial conditions are included in the full solution [8]. The secular term arises when $\Delta\theta_Q \rightarrow 0$, $\sin n\Delta\theta_Q/\Delta\theta_Q \rightarrow n$, which is exactly what we expect when the excitation is exactly on the betatron tune. Furthermore, we notice that there are extra singularities which occur when $\Delta\theta_Q = \pm\theta_s$.

Carrying on, we can calculate the particular integral from Eq. (17) and therefore, the complete solution to Eq. (10) when $\theta_k - \theta_Q = \Delta\theta_Q$ is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(\phi_{01} - \frac{\epsilon\lambda \cos n\Delta\theta_Q}{2\theta_Q \Delta\theta_Q} \right) \cos \left[n\theta_Q + \frac{\rho}{\theta_s} \sin(n\theta_s - \varphi) \right] + \left(\phi_{02} + \frac{\epsilon\lambda \sin n\Delta\theta_Q}{2\theta_Q \Delta\theta_Q} \right) \sin \left[n\theta_Q + \frac{\rho}{\theta_s} \sin(n\theta_s - \varphi) \right] \\ &\quad + \frac{\epsilon\lambda \rho}{4\theta_Q \theta_s} \left(\frac{1}{\Delta\theta_Q + \theta_s} \cos \left[n(\theta_Q + \Delta\theta_Q) + \frac{\rho}{\theta_s} \sin(n\theta_s - \varphi) + (n\theta_s - \varphi) \right] \right. \\ &\quad \left. - \frac{1}{\Delta\theta_Q - \theta_s} \cos \left[n(\theta_Q + \Delta\theta_Q) + \frac{\rho}{\theta_s} \sin(n\theta_s - \varphi) - (n\theta_s - \varphi) \right] \right).\end{aligned}\quad (24)$$

C. Laboratory frame

All the calculations that we have done so far are in the frame that follows the single particle. In the laboratory frame, we measure the transverse position of the beam once a turn at the longitudinal positions $\pm\tau_B$ while we continuously kick the beam. To arrive at an analytic solution for this paper, we will make the assumption that the beam is matched to the RF bucket, i.e., in the (τ, δ) plane the particle distribution is stationary [9]. This means that the particles are dense on the contour r_B/c , see Fig. 2. The series of pictures here give us a clue for how to calculate the transverse position of the beam at $(r_B/c, \phi_B)$ for all n . Note that the labels $(\hat{\tau}_B, \hat{\delta}_B)$, $(\hat{\tau}_1, \hat{\delta}_1)$, $\dots$, are the initial conditions at $n = 0$ which are used to calculate ϕ_{01} and

ϕ_{02} of Eq. (24). Notice that we have used the symbol “ $\hat{}$ ” to denote initial conditions at $n = 0$. At time $n = 0$, the point labeled $(\hat{\tau}_B, \hat{\delta}_B)$ is at the observation point. The point $(\hat{\tau}_1, \hat{\delta}_1)$ is at an angle θ_s away from $(\hat{\tau}_B, \hat{\delta}_B)$. So at time $n = 1$, $(\hat{\tau}_B, \hat{\delta}_B)$ rotates counterclockwise away by θ_s and the point $(\hat{\tau}_1, \hat{\delta}_1)$ now lands at the observation point. At $n = 2$, the point $(\hat{\tau}_1, \hat{\delta}_1)$ rotates away by θ_s and the point $(\hat{\tau}_2, \hat{\delta}_2)$ lands at the observation point. This continues *ad infinitum*.

Therefore, at the observation point, the transverse amplitude $\hat{Y}_B(n)$ [the symbols are here to remind us that the amplitude is measured at point $(r_B/c, \phi_B)$ and uses initial conditions $(\hat{\tau}_B, \hat{\delta}_B)$, $(\hat{\tau}_1, \hat{\delta}_1)$, $\dots$] is simply

$$\hat{Y}_B(n) = \begin{cases} y(0) & \text{if } n = 0 \text{ with initial conditions } (\hat{\tau}_B, \hat{\delta}_B) \\ y(1) & \text{if } n = 1 \text{ with initial conditions } (\hat{\tau}_1, \hat{\delta}_1) \\ y(2) & \text{if } n = 2 \text{ with initial conditions } (\hat{\tau}_2, \hat{\delta}_2) \\ \vdots & \vdots \end{cases}\quad (25)$$

where $y(n)$ comes from Eq. (24). For the remainder of this paper, we will use the $y(n)$ from Eq. (24). It will be more

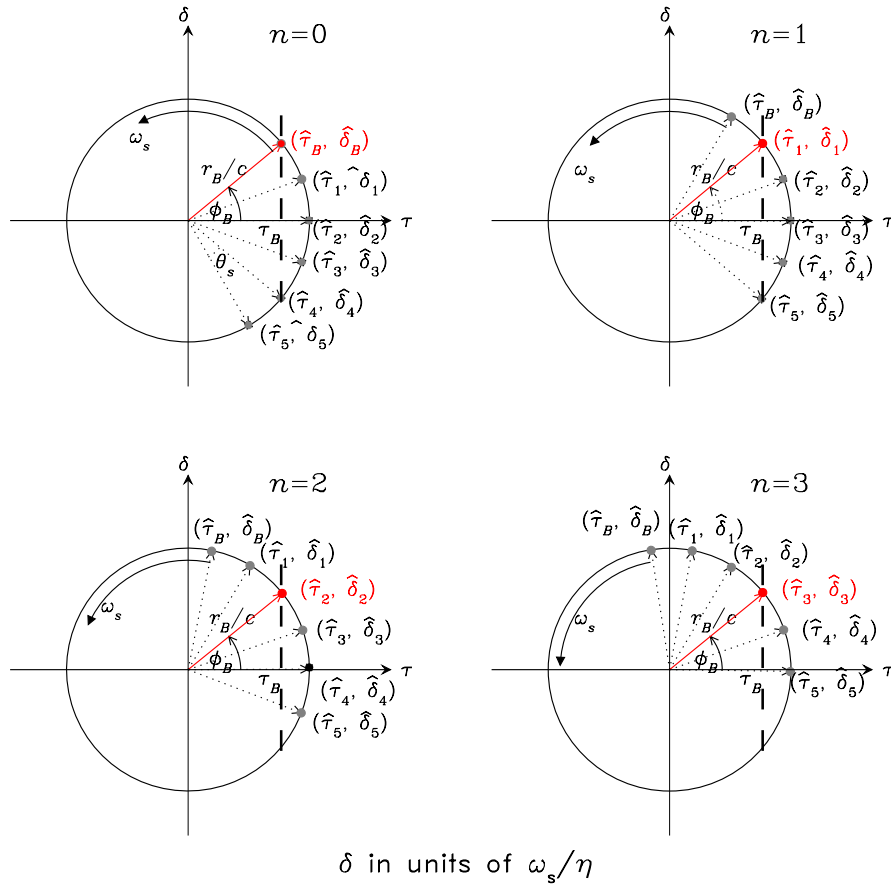


FIG. 2. (Color) These series of pictures tell us how to calculate the transverse position of the beam at $(r_B/c, \phi_B)$. The angle between each vector is θ_s . Note: (a) the labels $(\hat{\tau}_B, \hat{\delta}_B)$, $(\hat{\tau}_1, \hat{\delta}_1)$, etc., are the initial conditions at $n = 0$ which are used to calculate ϕ_{01} and ϕ_{02} of Eq. (24). (b) The circle r_B/c contains a dense set of points. We only highlight the points which are relevant for the calculation. (c) $\phi_B - n\theta_s = \hat{\phi}_B + n\theta_s$ because $\hat{\phi}_B = \pi/2 - \phi_B$.

convenient if we write the initial conditions $(\hat{\tau}_n, \hat{\delta}_n)$ in terms of $(\hat{\rho}_B, \hat{\phi}_B)$. It is easy to show from Eq. (16) and μ and ν from Eq. (11) that

$$\begin{aligned} \hat{\tau}_n &= \begin{cases} +\frac{\eta\hat{\rho}_B}{2\pi\xi\omega_s} \sin(n\theta_s + \hat{\phi}_B) & \text{if } \nu_B, \mu_B > 0 \text{ or } \nu_B < 0, \mu_B > 0 \\ -\frac{\eta\hat{\rho}_B}{2\pi\xi\omega_s} \sin(n\theta_s + \hat{\phi}_B) & \text{if } \nu_B, \mu_B < 0 \text{ or } \nu_B > 0, \mu_B < 0 \end{cases} \\ \hat{\delta}_n &= \begin{cases} -\frac{\hat{\rho}_B}{2\pi\xi} \cos(n\theta_s + \hat{\phi}_B) & \text{if } \nu_B, \mu_B > 0 \text{ or } \nu_B < 0, \mu_B > 0 \\ \frac{\hat{\rho}_B}{2\pi\xi} \cos(n\theta_s + \hat{\phi}_B) & \text{if } \nu_B, \mu_B < 0 \text{ or } \nu_B > 0, \mu_B < 0 \end{cases} \end{aligned} \quad (26)$$

and so with the above relationship, we can write $\hat{Y}_B(n)$ as

$$\hat{Y}_B(n) = \begin{cases} y(0) & \text{if } n = 0 \text{ with initial conditions } (\hat{\rho}_B, \hat{\phi}_B) \\ y(1) & \text{if } n = 1 \text{ with initial conditions } (\hat{\rho}_B, \theta_s + \hat{\phi}_B) \\ y(2) & \text{if } n = 2 \text{ with initial conditions } (\hat{\rho}_B, 2\theta_s + \hat{\phi}_B) \\ \vdots & \vdots \end{cases} \quad (27)$$

1. $\hat{Y}_B$ for zero transverse emittance case

For the zero transverse emittance case with $y(0) = \dot{y}(0) = 0$, we can show that ϕ_{01} and ϕ_{02} in terms of ρ and φ are [10]

$$\begin{aligned}
\phi_{01}(\rho, \varphi) &= \epsilon\lambda \left\{ -\frac{1}{4\theta_Q(\theta_Q + \rho \cos\varphi)} \cos\left(\frac{2\rho}{\theta_s} \sin\varphi\right) + \frac{1}{4\theta_Q\theta_s(\theta_Q + \rho \cos\varphi)} \right. \\
&\quad \times \left[\theta_s - \rho \sin\varphi \sin\left(\frac{2\rho}{\theta_s} \sin\varphi\right) \right] + \frac{1}{2\Delta\theta_Q\theta_Q} + \frac{\rho \cos\varphi}{2(\Delta\theta_Q^2 - \theta_s^2)(\theta_Q + \rho \cos\varphi)} \left(1 + \frac{\rho \cos\varphi}{\theta_Q} \right) \Big\} \\
\phi_{02}(\rho, \varphi) &= \epsilon\lambda \left\{ \frac{1}{4\theta_Q(\theta_Q + \rho \cos\varphi)} \sin\left(\frac{2\rho}{\theta_s} \sin\varphi\right) - \frac{\rho \sin\varphi}{2\theta_Q\theta_s} \left(\frac{\cos^2(\frac{\rho}{\theta_s} \sin\varphi)}{\theta_Q + \rho \cos\varphi} + \frac{\Delta\theta_Q}{\Delta\theta_Q^2 - \theta_s^2} \right) \right\},
\end{aligned} \tag{28}$$

and so from Eq. (27), $\hat{Y}_B(n)$ is easily derived by mapping $\rho \rightarrow \hat{\rho}_B$ and $\varphi \rightarrow (n\theta_s + \hat{\varphi}_B)$ of y . Thus, the analytic solution for the transverse amplitude at $(\hat{\rho}_B, \hat{\varphi}_B)$ for all n is

$$\begin{aligned}
\hat{Y}_B(n) &= \left(\phi_{01}(\hat{\rho}_B, n\theta_s + \hat{\varphi}_B) - \frac{\epsilon\lambda \cos n\Delta\theta_Q}{2\theta_Q\Delta\theta_Q} \right) \cos \left[n\theta_Q - \frac{\hat{\rho}_B}{\theta_s} \sin\hat{\varphi}_B \right] - \left(\phi_{02}(\hat{\rho}_B, n\theta_s + \hat{\varphi}_B) + \frac{\epsilon\lambda \sin n\Delta\theta_Q}{2\theta_Q\Delta\theta_Q} \right) \\
&\quad \times \sin \left[n\theta_Q - \frac{\hat{\rho}_B}{\theta_s} \sin\hat{\varphi}_B \right] + \frac{\epsilon\lambda \hat{\rho}_B}{4\theta_Q\theta_s} \left(\frac{1}{\Delta\theta_Q + \theta_s} \cos \left[n(\theta_Q + \Delta\theta_Q) - \frac{\hat{\rho}_B}{\theta_s} \sin\hat{\varphi}_B - \hat{\varphi}_B \right] \right. \\
&\quad \left. - \frac{1}{\Delta\theta_Q - \theta_s} \cos \left[n(\theta_Q + \Delta\theta_Q) - \frac{\hat{\rho}_B}{\theta_s} \sin\hat{\varphi}_B + \hat{\varphi}_B \right] \right).
\end{aligned} \tag{29}$$

It is interesting to note that the time dependent synchrotron term $n\theta_s$ is now embedded in ϕ_{01} and ϕ_{02} and removed from the terms $\cos n\theta_Q$, $\sin n\theta_Q$, $\cos n(\theta_Q + \Delta\theta_Q)$, and $\sin n(\theta_Q + \Delta\theta_Q)$. The reason why there can be any phase difference between the head and the tail originates from here. This means that, if we had ignored the initial conditions, we would have obtained an incorrect solution.

2. Changing perspective

To calculate the phase difference between the head and tail, we will first change perspective from that of a fixed betatron tune θ_Q and variable kick tune θ_k to that of a fixed θ_k and variable θ_Q because ultimately this is what is done in our experiment when we use the tune tracker PLL: we measure phase w.r.t. the frequency of the kick and not the betatron frequency. In this perspective, we have

$$\theta_Q = \theta_k - \Delta\theta_Q \tag{30}$$

from Eq. (20), and thus $\hat{Y}_B(n)$ is transformed to this perspective with a trivial change of variables, where we just replace θ_Q with $\theta_k - \Delta\theta_Q$ to become

$$\begin{aligned}
\hat{Y}_{k,B}(n) &= \left(\phi_{01}(\hat{\rho}_B, n\theta_s + \hat{\varphi}_B) - \frac{\epsilon\lambda \cos n\Delta\theta_Q}{2(\theta_k - \Delta\theta_Q)\Delta\theta_Q} \right) \cos \left[n(\theta_k - \Delta\theta_Q) - \frac{\hat{\rho}_B}{\theta_s} \sin\hat{\varphi}_B \right] \\
&\quad - \left(\phi_{02}(\hat{\rho}_B, n\theta_s + \hat{\varphi}_B) + \frac{\epsilon\lambda \sin n\Delta\theta_Q}{2(\theta_k - \Delta\theta_Q)\Delta\theta_Q} \right) \sin \left[n(\theta_k - \Delta\theta_Q) - \frac{\hat{\rho}_B}{\theta_s} \sin\hat{\varphi}_B \right] \\
&\quad + \frac{\epsilon\lambda \hat{\rho}_B}{4(\theta_k - \Delta\theta_Q)\theta_s} \left(\frac{1}{\Delta\theta_Q + \theta_s} \cos \left[n\theta_k - \frac{\hat{\rho}_B}{\theta_s} \sin\hat{\varphi}_B - \hat{\varphi}_B \right] - \frac{1}{\Delta\theta_Q - \theta_s} \cos \left[n\theta_k - \frac{\hat{\rho}_B}{\theta_s} \sin\hat{\varphi}_B + \hat{\varphi}_B \right] \right),
\end{aligned} \tag{31}$$

where we have added k to the subscript of $\hat{Y}_{k,B}(n)$ to remind ourselves that we are in the perspective of the kicker and the new functions $\phi_{k,01}()$ and $\phi_{k,02}()$ are $\phi_{01}()$ and $\phi_{02}()$ with $\theta_Q \rightarrow \theta_k - \Delta\theta_Q$ i.e.

$$\begin{aligned}
\phi_{k,01}(\hat{\rho}_B, n\theta_s + \hat{\varphi}_B) &= \epsilon\lambda \left\{ -\frac{\cos[\frac{2\hat{\rho}_B}{\theta_s} \sin(n\theta_s + \hat{\varphi}_B)]}{4(\theta_k - \Delta\theta_Q)[\theta_k - \Delta\theta_Q + \hat{\rho}_B \cos(n\theta_s + \hat{\varphi}_B)]} \right. \\
&\quad + \frac{\{\theta_s - \hat{\rho}_B \sin(n\theta_s + \hat{\varphi}_B) \sin[\frac{2\hat{\rho}_B}{\theta_s} \sin(n\theta_s + \hat{\varphi}_B)]\}}{4\theta_s(\theta_k - \Delta\theta_Q)[\theta_k - \Delta\theta_Q + \hat{\rho}_B \cos(n\theta_s + \hat{\varphi}_B)]} + \frac{1}{2(\theta_k - \Delta\theta_Q)\Delta\theta_Q} \\
&\quad \left. + \frac{\hat{\rho}_B \cos(n\theta_s + \hat{\varphi}_B)(1 + \frac{\hat{\rho}_B \cos(n\theta_s + \hat{\varphi}_B)}{\theta_k - \Delta\theta_Q})}{2(\Delta\theta_Q^2 - \theta_s^2)[\theta_k - \Delta\theta_Q + \hat{\rho}_B \cos(n\theta_s + \hat{\varphi}_B)]} \right\},
\end{aligned} \tag{32}$$

and

$$\phi_{k,02}(\hat{\rho}_B, n\theta_s + \hat{\varphi}_B) = \epsilon\lambda \left\{ \frac{\sin[\frac{2\hat{\rho}_B}{\theta_s} \sin(n\theta_s + \hat{\varphi}_B)]}{4(\theta_k - \Delta\theta_Q)[\theta_k - \Delta\theta_Q + \hat{\rho}_B \cos(n\theta_s + \hat{\varphi}_B)]} - \frac{\hat{\rho}_B \sin(n\theta_s + \hat{\varphi}_B)}{2(\theta_k - \Delta\theta_Q)\theta_s} \right. \\ \left. \times \left(\frac{\cos^2[\frac{\hat{\rho}_B}{\theta_s} \sin(n\theta_s + \hat{\varphi}_B)]}{(\theta_k - \Delta\theta_Q + \hat{\rho}_B \cos(n\theta_s + \hat{\varphi}_B))} + \frac{\Delta\theta_Q}{\Delta\theta_Q^2 - \theta_s^2} \right) \right\}. \quad (33)$$

When we look at Eqs. (31) and (33)—which are rather complicated—very carefully, we can extract out two different cases. The first is when $|\Delta\theta_Q| \ll \theta_s \ll 1$ or $\Delta\theta_Q = 0$, i.e., the kick is very close to the betatron tune but not on a synchrotron line or on the betatron tune. The other case is when $|\Delta\theta_Q| \approx \theta_s \ll 1$, i.e. when the kick is close to or on a synchrotron line. The second case will not be considered in this paper but has been discussed elsewhere [11].

3. Solution when $|\Delta\theta_Q| \ll \theta_s \ll 1$ or $\Delta\theta_Q = 0$

In this case, the dominant terms in Eq. (31) are those with coefficients which contain $1/\Delta\theta_Q$. Picking these out, we have

$$\hat{Y}_{\Delta\theta_Q,k,B}(n) = -\frac{\epsilon\lambda}{2(\theta_k - \Delta\theta_Q)\Delta\theta_Q} \left(\cos \left[n(\theta_k - \Delta\theta_Q) - n\Delta\theta_Q - \frac{\hat{\rho}_B}{\theta_s} \sin \hat{\varphi}_B \right] - \cos \left[n(\theta_k - \Delta\theta_Q) - \frac{\hat{\rho}_B}{\theta_s} \sin \hat{\varphi}_B \right] \right), \quad (34)$$

and the string of $\hat{Y}$ subscripts has increased with the addition of $\Delta\theta_Q$ which reminds us that we are only looking at terms which contain $1/\Delta\theta_Q$.

4. Averaged transverse position

To obtain an averaged transverse position $\hat{X}$, we have to integrate $\hat{Y}$ over δ [12], see Fig. 3. Note that the δ distribution is time independent because we have assumed that the (τ, δ) distribution is matched to the bucket and thus stationary. Let $\sigma(\tau_B, \delta) \equiv \sigma(\delta)$ be the density of particles in the (τ, δ) plane which is normalized, i.e.

$$\int_{-\infty}^{\infty} d\delta \sigma(\delta) = 1. \quad (35)$$

Therefore, the fraction of particles in the small segment δ is $\sigma d\delta$ which gives us the weight for calculating the average $\hat{X}$, i.e.

$$\hat{X}_{\theta_k,k,B} = \int_{-\infty}^{\infty} d\delta \sigma(\delta) \hat{Y}_{\theta_k,k,B}. \quad (36)$$

Thus, the averaged transverse position $\hat{X}$ at τ_B is

$$\hat{X}_{\Delta\theta_Q,k,B}(n) = -\frac{\epsilon\lambda}{2(\theta_k - \Delta\theta_Q)\Delta\theta_Q} \left(\cos \left[n(\theta_k - \Delta\theta_Q) - n\Delta\theta_Q + \frac{\xi\omega_0\tau_B}{\eta} \right] - \cos \left[n(\theta_k - \Delta\theta_Q) + \frac{\xi\omega_0\tau_B}{\eta} \right] \right), \quad (37)$$

which is independent of σ . If we define ψ to be the phase w.r.t. the kick $\epsilon\lambda \cos n\theta_k$, then it is easy to show that

$$\psi = \arctan \left[\frac{\cos(\frac{3}{2}n\Delta\theta_Q - \frac{\xi\omega_0\tau_B}{\eta})}{\sin(\frac{3}{2}n\Delta\theta_Q - \frac{\xi\omega_0\tau_B}{\eta})} \right] - \pi \\ = -\frac{\pi}{2} - \frac{3}{2}n\Delta\theta_Q + \frac{\xi\omega_0\tau_B}{\eta}, \quad (38)$$

because $\cot\theta = \tan(\pi/2 - \theta)$ and the $-\pi$ is from the “ $-$ ” sign in front of $\epsilon\lambda$. Equation (38) makes physical sense, because we expect $\psi = -\pi/2$ at resonance when $\tau_B = 0$. Now, from Eq. (38), we can immediately read off the phase difference $\psi(\xi, \tau_B)$ between the head at $+\tau_B$ w.r.t. the tail at $-\tau_B$ to be

$$\Delta\psi(\xi, \tau_B) = +\frac{2\xi\omega_0\tau_B}{\eta}, \quad (39)$$

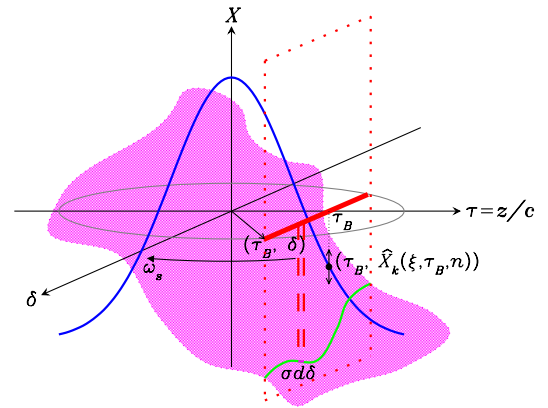


FIG. 3. (Color) The blue curve represents the bunch signal at a beam position monitor. The point $(\tau_B, \hat{X}_k(\xi, \tau_B, n))$ comes from integrating $\hat{Y}$ over all the particles in the red line and thus gives us an averaged position $\hat{X}$ at τ_B . The magenta surface is the distribution of particles in (τ, δ, X) space. We have a sheet because of the initial conditions. The green line is the intersection between the plane at constant τ_B and the magenta surface.

which is astonishingly simple for all the work that we have done. However, we must again remind ourselves that Eq. (39) is only valid if $\Delta\theta_Q \ll \theta_s \ll 1$ or $\Delta\theta_Q = 0$. Note: The head-tail phase difference has also been derived by Fartoukh [13] using a different method and agrees with our answer. We have also used numerical calculations to verify this formula as well as Eq. (29). The numerical results agree very well with the analytic solutions.

III. EXPERIMENT

In this experiment, the basic idea is for us to continuously kick the beam vertically with the AC-dipole. The phase difference between the head and tail due to this kick is then measured and plotted for different chromaticities.

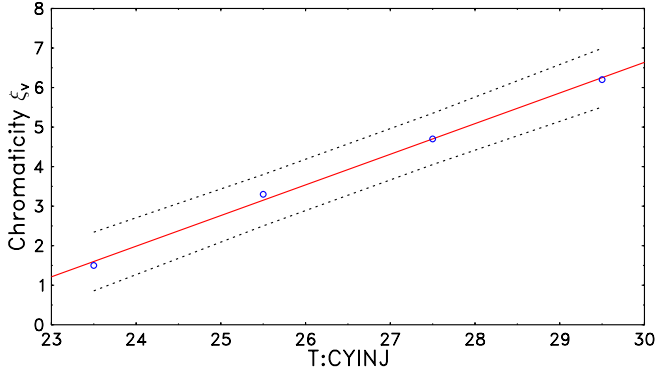


FIG. 4. (Color) The T:CYINJ knob is calibrated with chromaticity measured with uncoalesced beam instead of coalesced beam. The dotted lines are the 95% confidence intervals.

The advantage of doing it this way is because the AC-dipole kicks are big enough to be seen directly with an oscilloscope and so do not require any special high gain electronics. Furthermore, this method is independent of the tune tracker PLL electronics which gave ambiguous results (discussed in the Introduction). Unfortunately, the disadvantage is that the transverse emittance will blow up because the AC-dipole kicks are very close to the betatron tune.

Practically, we must first calibrate the knob T:CYINJ which we will use to change the vertical chromaticity. We do this by measuring the vertical chromaticity ξ_v for different T:CYINJ settings. The reason for using uncoalesced [14] rather than coalesced [15] beam is because it is much easier to measure chromaticity with uncoalesced beam with the traditional change of RF frequency method than with coalesced beam. The fit that we have found is

$$\xi_v = (0.78 \pm 0.03) \times \text{T:CYINJ} - (16.6 \pm 0.8). \quad (40)$$

From this fit, we find that the error in the value of ξ_v when we are given T:CYINJ is approximately ± 0.65 units at a confidence level of 95% for the range of T:CYINJ that we will be setting. See Fig. 4.

The experimental setup is shown in Fig. 5 [16]. We inject one coalesced bunch, which contains about 330×10^9 protons, to be used for the entire experiment. We set the chromaticity with T:CYINJ and then we kick the bunch transversely with the AC-dipole for a short period of time for each head-tail phase measurement. The reason for turning off the AC-dipole is to keep the vertical emittance growth to a minimum. The exact ramp waveform of the

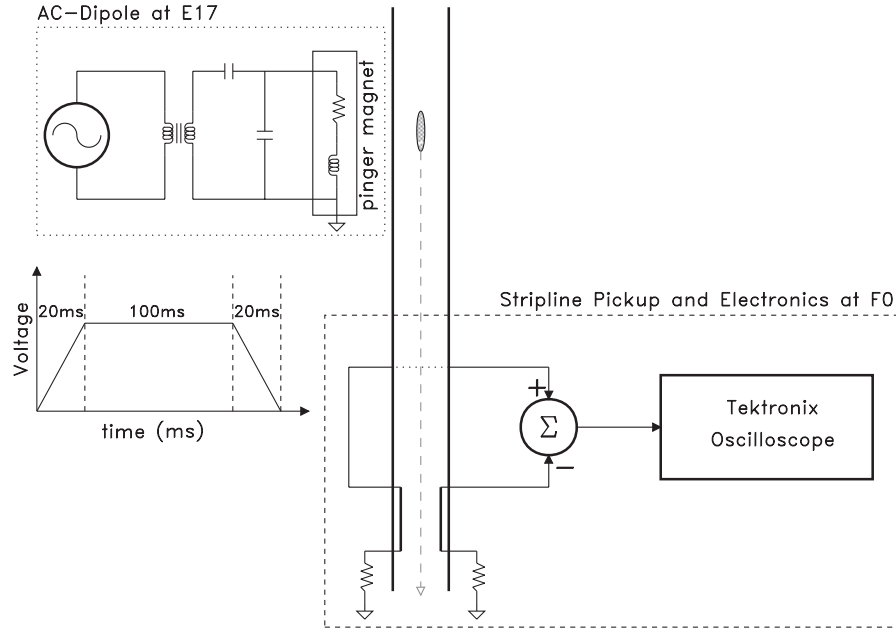


FIG. 5. The experimental setup consists of an AC-dipole at E17 and a stripline with its electronics at F0. The AC-dipole is ramped up to its set voltage in 20 ms and stays at flat top for about 100 ms (equivalent to about 10 synchrotron periods or 5000 turns) and then ramped down again in 20 ms.

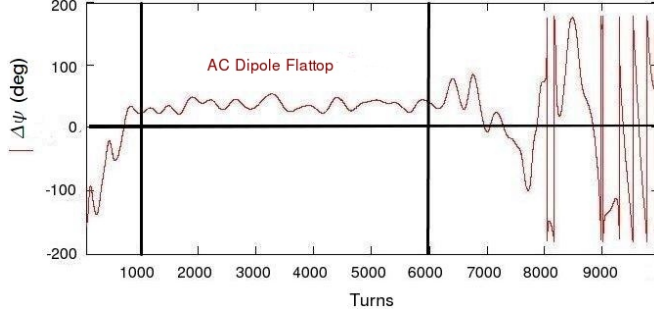


FIG. 6. (Color) This shows the calculated phase difference between the head w.r.t. tail. This plot is for $\xi_v = 4$, $\tau_B = 0.8$ ns. The phase difference is $\Delta\psi = (30 \pm 7)^\circ$ during the 100 ms (5000 turns) flattop period of the AC-dipole kicker. See Fig. 5. The ripples during this time give us the error of the phase measurement.

AC-dipole is shown in Fig. 5. The frequency of the kick is set to 0.6×10^{-3} tune units ($< Q_s = 1.7 \times 10^{-3}$) below the vertical betatron tune which is at 0.5776. The transverse position of the head and the tail are measured with a fast sampling oscilloscope and the phase difference is calculated off-line. After each change of chromaticity, we move the vertical tune if necessary to ensure that the betatron tune remains at 0.5776. Despite having set the AC-dipole kick to its minimum value, the vertical emittance measured with the flying wires system, grows from 17π mm mrad to 36π mm mrad during the experiment. The sigma bunch length σ_B did not change during this time and is 2.9 ns.

Figure 6 shows an example of the head-tail phase difference which we have analyzed. In this example $\xi_v = 4$ and $\tau_B = 0.8$ ns. From here it is unambiguous that there is a phase difference between the head w.r.t. tail.

The measured phase difference $\Delta\psi$ for three values of $\tau_B = 0.4$ ns, 0.8 ns, and 1.2 ns versus ξ_v are shown in Fig. 7. The red line is the expected $\Delta\psi$ as a function of ξ_v from Eq. (39) where we have used the Tevatron parameters $\omega_0 = 2\pi \times (47.71 \times 10^3) \text{ s}^{-1}$ and $\eta = 0.0028$. To check how good our theory fits to the data despite the nonzero vertical emittance, we have calculated χ^2 and the reduced χ^2 for these three cases which are summarized in Table I.

For all three τ_B cases, the χ^2 and the reduced χ^2 results tell us that the phase difference data does not match Eq. (39) very well and, in fact, this is rather obvious by eye from Fig. 7.

We suspect that there are nonlinear components in $\Delta\psi$ that are not taken care of in the theory. If we assume that the lowest power nonlinear term is quadratic in ξ , then we can try to fit the data with this added ξ^2 term,

$$\Delta\psi(\text{deg}) = \xi_v(a\xi_v + b), \quad (41)$$

where a and b are fit parameters. The blue curves in Fig. 7 show the quadratic fit and Table II shows the fit parameters.

It is clear from the plots that the quadratic fits are rather good despite the wildly different reduced χ^2 for each case.

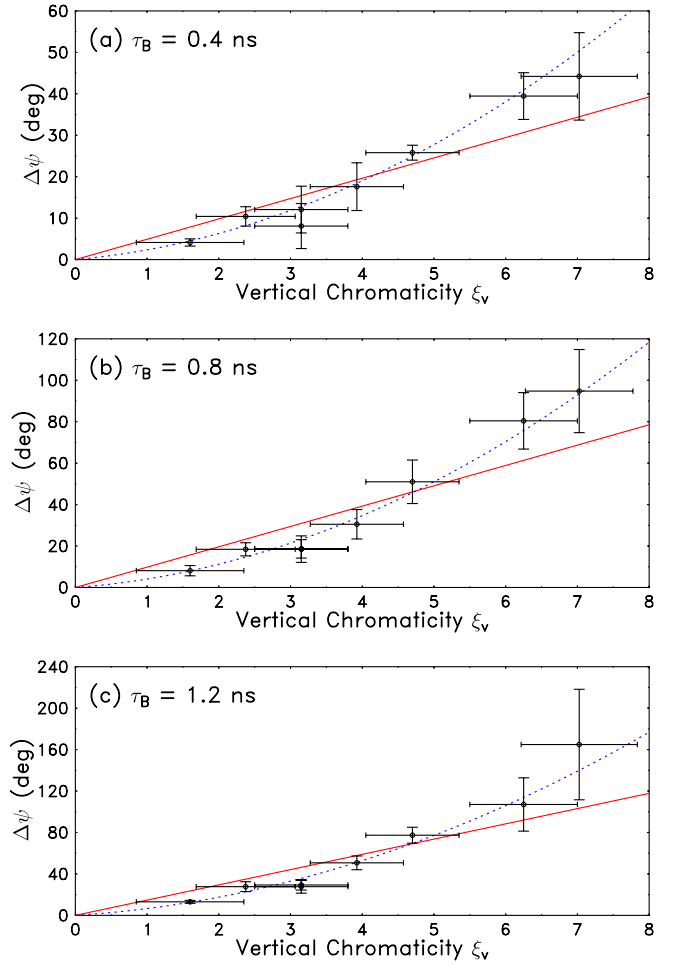


FIG. 7. (Color) These plots show the data analyzed for $\tau_B = 0.4$ ns, 0.8 ns, and 1.2 ns. The red line is calculated from Eq. (39) and the blue dotted line is a quadratic fit of the data using coefficients from Table II.

A. Sources of the nonlinearity

Equation (39) has been derived by assuming that the RF bucket is linear and that the transverse emittance is zero. The effects of nonlinearity from transverse impedance,

TABLE I. χ^2 comparison between experiment and theory.

τ_B (ns)	τ_B/σ_B	χ^2	$P(\geq \chi^2)$	Reduced χ^2
0.4	0.14	5.8	0.67	0.72
0.8	0.28	9.6	0.29	1.2
1.2	0.41	8.8	0.35	1.1

TABLE II. Quadratic fit of data.

τ_B (ns)	τ_B/σ_B	a	b	Reduced χ^2
0.4	0.14	0.8 ± 0.1	1.6 ± 0.5	0.42
0.8	0.28	1.5 ± 0.3	3 ± 1	0.59
1.2	0.41	2.2 ± 0.4	4 ± 1	0.97

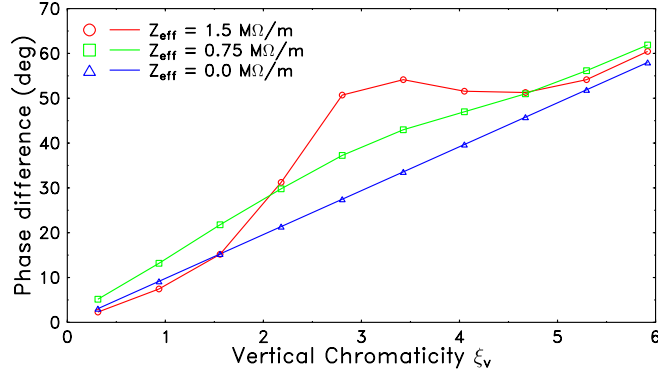


FIG. 8. (Color) The effect of resistive wall wakefield on the measured phase difference between the head slice (+ 0.4 ns from bucket center) and the tail slice (- 0.4 ns from bucket center) for a single kick.

second order chromaticity as well as amplitude driven effects have not been included. A full accounting of all these effects is beyond the scope of this paper and would require a complete numerical simulation where the beam is driven resonantly.

However, some preliminary insight can be gained by considering a *single* kick computer simulation which includes the effects of transverse impedance from short range resistive wall wakefields, linear and second order chromaticity. These simulations seem to indicate that neither second order chromaticity by itself nor resistive wall impedance by itself can account for the observed phase structure. This is because the phase effects of 2nd order chromaticity by itself cancel when the phase difference between the head and the tail is calculated. The effect of the impedance of the beam pipe can introduce a nonlinear phase response which can be seen in Fig. 8. It is clear from this figure that the nonlinear effect becomes more pronounced as the effective impedance Z_{eff} is increased from zero to 1.5 MΩ/m at 100 MHz. Note: 1.5 MΩ/m at 100 MHz is a realistic value for the Tevatron from previous analysis. However, resistive wall impedance by itself fails to fully capture the observed phase response [17]. When both 2nd order and resistive wall impedance effects are included, a complex interplay between these effects is observed which can be seen in Fig. 9. In this simulation three different values of second order chromaticity are used to illustrate the behavior of the head-tail phase difference as a function of linear chromaticity. For reference, the Tevatron 2nd order chromaticity is ~ 1000 . Notice that, as the second order chromaticity is increased, the maximum phase difference moves towards higher chromaticity and the curvature of the phase difference before the maxima takes on a similar look to the measured data shown in Fig. 7. However, it must be emphasized that these simulations have been performed with a *single* kick instead of a resonantly driven system and so a direct comparison cannot be made between these two results.

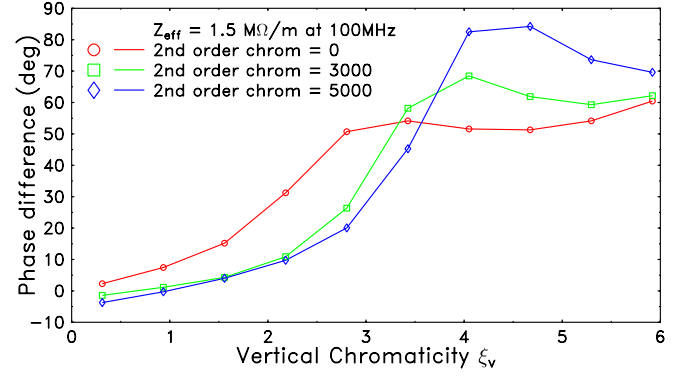


FIG. 9. (Color) The effect of resistive wall wakefield and 2nd order chromaticity on the measured phase difference between the head slice (+ 0.4 ns from bucket center) and the tail slice (- 0.4 ns from bucket center) for a single kick. In the presence of wakefields, 2nd order chromaticity can now contribute to the phase response, and as it increases, the maximum phase difference shifts to higher chromaticity. The curvature of the phase difference before the peak has a similar flavor to the measured phase difference shown in Fig. 7.

IV. CONCLUSION

We have found an analytic approximation for the weakly forced Hill's equation which describes the transverse motion of the beam in the presence of chromaticity for a single particle. We have checked that our analytic approximation matches very closely to the numerically integrated solution. From here, we have derived an analytic approximation which shows that there is a phase difference between the head and tail for a zero transverse emittance but non-zero longitudinal bunch length. A simple formula for this phase difference has been derived when the kick tune is very close to the betatron tune.

For the experiment, we have used the AC-dipole to excite the beam transversely and then measured the head-tail phase of the bunch. We have shown that there is a phase difference between the head and tail and thus have removed any lingering doubts as to whether the effect is real or not. From the measured data, we have found that there are nonlinear terms which are not taken care of by the theory. There are several possible sources for these terms, with short range wakefields coupling with second order chromaticity being the best candidate. The exact nature of this coupling will have to be resolved with computer simulations in the future.

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- [1] S. Fartoukh and J. Koutchouk, Technical Report, CERN LHC-B-ES-0004 rev 2.0, 2004.
 - [2] D. Cocq *et al.*, in *Beam Instrumentation Workshop*, AIP Conf. Proc. No. 451 (AIP, New York, 1998), pp. 281–288.
 - [3] M. Gasior and R. Jones, in *DIPAC* (JACoW, Lyon, France, 2005), pp. 312–314.
 - [4] C. Tan, Technical Report, CARE Workshop, LARP-doc-696-v1, 2007.
 - [5] P. Cameron, Technical Report, LARP Collaboration Meeting, Berkeley, LARP-doc-199-v1, 2006.
 - [6] S. Fartoukh and R. Jones, Technical Report, CERN-LHC-PROJ. REP.-602, 2002.
 - [7] F. Hoppensteadt, *Analysis and Simulation of Chaotic Systems* (Springer-Verlag, Berlin, 1993), pp. 194–198.
 - [8] A. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators* (Wiley, New York, 1993), pp. 218–233.
 - [9] C. Tan, Technical Report, Fermilab Technical Memo, TM-2155, 2001.
 - [10] These algebraic solutions have been verified with MATHEMATICA.
 - [11] C. Tan and V. Ranjbar, Technical Report, Fermilab Technical Memo, TM-2376, 2007.
 - [12] In general, this integral is 2-dimensional with $\sigma(\tau_B, \delta, X)$, however, because of the initial condition that $y(0) = \dot{y}(0) = 0$, we have a sheet in the (τ, δ, X) space and not a volume.
 - [13] S. Fartoukh, Technical Report, CERN-LHC-PROJ. REP.-986, 2007.
 - [14] Uncoalesced beam in the Tevatron means that there are 30 bunches in adjacent buckets.
 - [15] Coalesced beam means that 7 bunches from the main injector are “coalesced” into 1 high current bunch and then injected into the Tevatron.
 - [16] Note that there is no 3D-BBQ electronics (which is used in the PLL tune tracker electronics at SPS and RHIC) in the setup so that there cannot be any arguments as to whether the phase difference effect is an artifact of the 3D-BBQ.
 - [17] V. Ranjbar and P. Ivanov, Technical Report, Fermilab PUB-07-448-AD, 2007.