Qubit-oscillator-based gate implementations for approximate Gottesman-Kitaev-Preskill codes

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We consider hybrid qubit-oscillator systems together with Gaussian, multiqubit, as well as qubit-controlled Gaussian unitaries. We propose implementations of logical gates for approximate Gottesman-Kitaev-Preskill codes in this model using two oscillators and three qubits. We show that these gate implementations become exact in the limit of large squeezing: The logical gate error is upper-bounded by a linear function of the squeezing parameter and depends polynomially on the number of encoded qubits. For a subset of Clifford gates, our constructions overcome a fundamental drawback of well-known Gaussian implementations, which—when executed without subsequent error correction—suffer from a constant logical gate error. In contrast, our fully unitary circuit construction achieves an asymptotically vanishing logical gate error rate even without applying a recovery map, i.e., performing syndrome measurements and adaptive corrections.

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I. INTRODUCTION

Linear optics—and more generally, Gaussian operations—offer an efficient and experimentally accessible framework for continuous-variable quantum information, yet its Gaussian structure imposes fundamental limits: it cannot realize error correction, entanglement distillation or universal fault-tolerant computation. Gottesman-Kitaev-Preskill (GKP) codes appear to circumvent these obstacles, but only in their idealized unphysical form. For realistic approximate GKP codes, Gaussian implementations of key logical gates incur constant error, making them unsuitable for fault-tolerant use [1]. This no-go result motivates our proposal: The use of non-Gaussian resources in hybrid qubit-oscillator systems to achieve accurate implementations of logical operators for physically realistic GKP codes.

Gaussian operations and their limitations. Gaussian operations provide a versatile toolbox for continuous-variable (CV) quantum information processing and are a fertile playground for both theoretical and experimental studies. Mathematically, the corresponding states and operations have efficient descriptions, enabling the analytical study of a diverse range of quantum phenomena and processes ranging from entanglement and Bell inequality violations [2–4] to measurement-based quantum information processing [5,6]. Physically, the corresponding operations are typically easy to realize in the lab. In particular, the subgroup of Gaussian unitaries is generated by a number of elementary building blocks, each of which has a physical realization in terms of a linear optics element such as a mirror.

Let us briefly specify these operations in more detail. A system made of n harmonic oscillators (or modes) is associated with the Hilbert space $L^2(\mathbb{R})^{\otimes n}$. The collection of canonical mode operators will be denoted as $R = (Q_1, P_1, \ldots, Q_n, P_n)$. They satisfy the canonical commutation relations

$$[R_k, R_\ell] = iJ_{k,\ell} I_{\mathcal{H}_n} \quad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\oplus n}.$$

Of particular interest are the Gaussian (Weyl) displacement operators

$$W(\xi) = e^{-i\xi \cdot JR} \quad \text{where } \xi \in \mathbb{R}^{2n}. \tag{1}$$

We call such a displacement of bounded strength if $\|\xi\|$ is bounded by a constant (independent of, i.e., the number n of modes). A displacement operator $W(\xi)$ is a one-mode operator if and only if $\xi \cdot JR$ is a linear combination of the mode operators Q_j and P_j only, for some $j \in \{1, \ldots, n\}$. We also consider Gaussian unitaries of the form

$$U(A) = e^{iR \cdot AR} \quad \text{where } A = A^T \in \mathsf{Mat}_{2n\times 2n}(\mathbb{R}), \tag{2}$$

i.e., unitaries generated by Hamiltonians which are quadratic in the mode operators. Again, we say that $U(A)$ is bounded-strength if the operator norm $\|A\|$ is bounded by a constant, and $U(A)$ is called a one-mode (two-mode) operation if the quadratic form $R \cdot AR$ only involves operators Q_j, P_j for a single mode $j \in \{1, \ldots, n\}$ (Q_j, P_j, Q_k, P_k for two modes $j, k \in \{1, \ldots, n\}$).

The set of Gaussian operations consists of

(i) the preparation of the single-mode vacuum state $|\mathsf{vac}\rangle \in L^2(\mathbb{R})$ defined as $\mathsf{vac}(x) = \pi^{-1/4} e^{-x^2/2}$ on any mode,

(ii) the application of any one- or two-mode Gaussian unitary of bounded strength, and

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(iii) single-mode heterodyne or homodyne detection, i.e., measurements corresponding to projections onto coherent states and measurement of the quadrature operators.

Any dynamics consisting of a sequence of polynomially-many basic Gaussian operations of the type (i)–(iii) has an efficient description in the covariance matrix formalism (see, e.g., Ref. [7]), and is typically easily realizable in practice. Unfortunately, however, this set of operations is fundamentally limited. In particular, it was shown early on that Gaussian operations cannot protect encoded information even against relatively simple Gaussian noise [8,9]. In other words, suitable quantum error-correcting codes for CV systems need to involve non-Gaussian states. In addition, it was demonstrated that it is not possible to perform entanglement distillation from Gaussian states and entanglement swapping by means of Gaussian operations [10–12].

Ideal Gottesman-Kitaev-Preskill codes and Gaussian operations. First introduced in Ref. [13], GKP codes are a CV analog of stabilizer codes for qubits and are in many ways the *de facto* standard for quantum error correction using CV systems. The ideal (single-mode) GKP code encodes a qudit into formal linear combinations of position-eigenstates of an oscillator, i.e., Dirac-δ distributions. In more detail, let (Q, P) denote the canonical position and momentum operators on $L^2(\mathbb{R})$. Let $d \geqslant 2$ be an integer. The codewords $\{\mathsf{GKP}(j)_d\}_{j\in\{0,\dots,d-1\}}$ of the ideal GKP code are then defined in position space as

$$\mathsf{GKP}(j)_d(x) \propto \sum_{s\in\mathbb{Z}} \delta\left(x - \sqrt{\frac{2\pi}{d}}(j + sd)\right) \quad \text{for } x \in \mathbb{R}$$
$$\text{where } j \in \{0, \dots, d-1\}. \tag{3}$$

The states (distributions) $\{\mathsf{GKP}(j)_d\}_{j\in\{0,\dots,d-1\}}$ form a basis of the (ideal) GKP code $\mathcal{GKP}[d]$. It is the space stabilized by the two stabilizer generators

$$S_d^{(1)} = e^{i\sqrt{2\pi d}Q} \quad \text{and} \quad S_d^{(2)} = e^{-i\sqrt{2\pi d}P},$$

which are both displacement operators. This fact means, for example, that syndrome extraction circuits can be realized by Gaussian circuits (see Ref. [13]). Furthermore, as discussed in Ref. [13], there is a set of logical Clifford group generators which can be implemented by Gaussian unitaries. Thus, although the GKP "states" defined by Eq. (3) themselves are non-Gaussian, it seems that many operations of interest associated with fault-tolerant computation with GKP codes can be realized using Gaussian operations.

Approximate symmetrically squeezed Gottesman-Kitaev-Preskill codes. Any physically realistic implementation of GKP codes must use approximate, i.e., finitely squeezed and normalizable, states instead of the formal superpositions given in Eq. (3). In our analysis, we use what we call ε-truncated approximate GKP states with squeezing parameters $\kappa, \Delta > 0$. The corresponding family of states (normalized functions) $\{\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(j)_d\}_{j\in\{0,\dots,d-1\}} \subset L^2(\mathbb{R})$ is defined as

$$\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(j)_d(x) \propto \begin{cases} \sum_{s\in\mathbb{Z}} e^{-\kappa^2 s^2/2} e^{-(x-\sqrt{2\pi/d}\,j)^2/(4\pi d\Delta^2)} & \text{if } \mathsf{dist}\left(\frac{x-\sqrt{2\pi/d}\,j}{\sqrt{2\pi d}}, \mathbb{Z}\right) \leqslant \varepsilon \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

for $x \in \mathbb{R}$ and $j \in \{0, \dots, d-1\}$. Here the truncation parameter $\varepsilon > 0$ ensures that each of these functions only has support on intervals of width $2\sqrt{2\pi d}\varepsilon$ centered around integer multiples of $\sqrt{2\pi d}$ (shifted by $\sqrt{2\pi/d}\,j$). In particular, for any truncation parameter $\varepsilon > 0$ satisfying

$$\varepsilon \leqslant 1/(2d),$$

the states $\{\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(j)_d\}_{j\in\{0,\dots,d-1\}}$ are pairwise orthogonal. We typically choose $\varepsilon = \varepsilon_d = 1/(2d)$ and call this the optimal truncation parameter. As discussed in Ref. [1], this choice guarantees orthogonality while also ensuring that these states are close to the untruncated approximate GKP states commonly considered for large squeezing (see Ref. [13]), a two-parameter family $\{\mathsf{GKP}_{\kappa,\Delta}(j)_d\}_{j\in\{0,\dots,d-1\}}$ of states only depending on the squeezing parameters (κ, Δ). In other words, the truncation introduced in Eq. (4) should be considered as a technically convenient proxy for the actual approximate GKP states used in any protocol.

In addition to working with truncated approximate GKP states with the optimal truncation parameter ε_d, we choose the following linear dependence between the parameters κ (determining the envelope) and Δ (determining the individual local maxima): We set

$$\Delta = \kappa/(2\pi d). \tag{5}$$

The special choice made in Eq. (5) is motivated by the fact that this provides a particularly simple linear optics implementation of the Fourier transform (see Ref. [1], Theorem 6.1). We call the corresponding code (which now only depends on κ) the symmetrically squeezed GKP code with squeezing parameter κ, and denote the code space as

$$\mathcal{GKP}^{\star}_{\kappa}[d] := \mathsf{span}\left\{\mathsf{GKP}^{1/(2d)}_{\kappa,\kappa/(2\pi d)}(j)_d\right\}_{j\in\{0,\dots,d-1\}}. \tag{6}$$

We note that for a fixed code space dimension $d \geqslant 2$, this defines a one-parameter family $\{\mathcal{GKP}^{\star}_{\kappa}[d]\}_{\kappa>0}$ of symmetrically squeezed approximate GKP codes.

Approximate GKP codes and limitations of linear optics. In recent work [1], we introduced the (composable) logical gate error $\mathsf{err}_{\mathcal{L}}(\mathcal{W}, U)$, which quantifies the error of an implementation given by a completely positive and trace-preserving (CPTP) map $\mathcal{W} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ (acting on a physical Hilbert space $\mathcal{H}$) of a logical gate U on a code space $\mathcal{L} \subset \mathcal{H}$. It is defined as

$$\mathsf{err}_{\mathcal{L}}(\mathcal{W}, U) = \|(\mathcal{W} - \boldsymbol{\mathcal{U}}) \circ \Pi_{\mathcal{L}}\|_{\diamond},$$

where $\Pi_{\mathcal{L}}(\rho) = \pi_{\mathcal{L}}\rho\pi_{\mathcal{L}}^{\dagger}$ is the CP map applying the orthogonal projection $\pi_{\mathcal{L}} : \mathcal{H} \to \mathcal{L}$ and $\boldsymbol{\mathcal{U}} : \mathcal{B}(\mathcal{L}) \to \mathcal{B}(\mathcal{L})$ is the CPTP map associated with the logical unitary U on the code space [see Eq. (A1)]. We refer to Appendix A for more details on this definition and a summary of the properties of this quantity.

We note that the definition of the logical gate error in Eq. (9) is not restricted to unitary physical implementations. Typical examples of such nonunitary implementations are unitary Gaussians followed by syndrome measurements and adaptive correction operations, e.g., Knill-type GKP error correction schemes as proposed in Refs. [14,15] or passive teleportation-based GKP error correction worked out in Ref. [16]. Different quantities are used to quantify the accuracy

of such implementations: The analysis in Ref. [14] is based on the stabilizer subsystem decomposition (see Ref. [17]) in combination with the energy-constrained diamond norm, see Ref. [18]. In Ref. [16], the quality of an implementation is quantified using the average gate fidelity, which is also estimated using the stabilizer subsystem decomposition. In Ref. [15], the so-called logical fidelity is used to evaluate how well gate implementations match the target logical gate. In more detail, the logical fidelity compares the action of a physical implementation and a logical gate by tracing out auxiliary (gauge) degrees of freedom arising from the modular subsystem decomposition [19].

Here we focus on unitary (nonadaptive) gate implementations, i.e., we consider CPTP maps of the form $\mathcal{W}(\rho) = W\rho W^\dagger$ for a unitary $W : \mathcal{H} \to \mathcal{H}$. In this case, we write the logical gate error as

$$\mathsf{err}_{\mathcal{L}}(W, U) := \mathsf{err}_{\mathcal{L}}(\mathcal{W}, U).$$

In Ref. [1], we have shown that using unitary Gaussian operations to perform logical gates in approximate GKP codes does not work as may be expected. Specifically, we have considered the logical phase gate P, a diagonal Clifford acting on computational basis states as

$$\mathsf{P}|j\rangle = e^{i\pi j(j+c_d)/d}|j\rangle \quad \text{where}$$
$$c_d = d \mod 2 \quad \text{for } j \in \{0, \ldots, d-1\}. \tag{7}$$

We have shown that the linear optics implementation (proposed in Ref. [13] for the ideal GKP code)

$$W_{\mathsf{P}} = e^{i(Q^2 + c_d\sqrt{2\pi/d}Q)/2} \tag{8}$$

of this gate has a constant logical gate error lower bounded by (see Ref. [1], Result 2)

$$\mathsf{err}_{\mathcal{GKP}^\star_\kappa[d]}(W_{\mathsf{P}}, \mathsf{P}) > 3/100 \tag{9}$$

for all $\kappa < 1/250$ when considering the symmetrically squeezed approximate GKP code $\mathcal{GKP}^\star_\kappa[d]$. Equation (9) means that the implementation (8) is unsuitable for use with the approximate GKP code $\mathcal{GKP}^\star_\kappa[d]$ because it introduces a constant logical gate error even in the absence of noise. We emphasize that this result is not an artifact of the mathematical structure of truncated GKP states, whose discontinuities imply that their energy is unbounded. An analogous statement also holds for nontruncated approximate GKP states. This follows from continuity bounds on the logical-gate error, see Ref. [1, Lemma E.1]. More precisely, consider an encoding defined by two parametrized families of linearly independent states $\{\Phi^{(s)}_\kappa(j)\}_{j=0}^{d-1}$ for $\kappa > 0$ and $s \in \{1, 2\}$, satisfying

$$\lim_{\kappa\to 0} \max_{j\in\{0,\ldots,d-1\}} \left\| \left|\Phi^{(1)}_\kappa(j)\right\rangle\!\left\langle\Phi^{(1)}_\kappa(j)\right| - \left|\Phi^{(2)}_\kappa(j)\right\rangle\!\left\langle\Phi^{(2)}_\kappa(j)\right| \right\|_1 = 0.$$

Then the logical-gate error of any fixed physical implementation vanishes in the limit $\kappa \to 0$ for one of these encodings if and only if it vanishes for both.

Beyond Gaussian operations. Motivated by the no-go result for unitary Gaussian implementations expressed by Eq. (9), we ask if the use of alternative (non-Gaussian) resources can lead to logical gate implementations which are exact in the limit $\kappa \to 0$ of large squeezing. We take an approach similar to that of Ref. [20], where the authors propose implementing the logical two-qubit gate CZ in approximate GKP codes on a trapped-ion platform by coupling an additional three-level system manifested in the electronic level structure of the ion. The authors show that, for this specific gate, such an auxiliary discrete-variable system can mitigate the shortcomings of a purely Gaussian implementation.

Concretely, in this work we focus on the qubit-oscillator model, where in addition to the n oscillators and the Gaussian operations (i)–(iii), we add m qubits and the following capabilities:

(iv) one- and two-qubit gates, single-qubit state preparation of the computational basis state $|0\rangle$, computational qubit basis measurements and

(v) qubit-controlled single-mode phase-space displacements, i.e., unitaries of the form

$$\mathsf{ctrl}_j W(\xi) = |0\rangle\langle 0|_j \otimes I_{L^2(\mathbb{R})^{\otimes n}} + |1\rangle\langle 1|_j \otimes W(\xi),$$

where $\xi \in \mathbb{R}^{2n}$ and $W(\xi) = e^{-i\xi\cdot JR}$ is a displacement operator, and where $|0\rangle\langle 0|_j$ and $|1\rangle\langle 1|_j$ are the projections onto the computational basis states of the jth qubit. We assume that ξ is such that $W(\xi)$ is a single-mode bounded strength phase-space displacement.

The additional operations (iv) and (v) can be used to introduce non-Gaussianity. In particular, they can be leveraged to design protocols preparing GKP states, see, e.g., Ref. [21]. We refer to Ref. [22] for a recent review of this hybrid qubit-oscillator model, as well as a discussion of associated experimental realizations.

We refer to each of the operations (i)–(v) as an elementary operation in the hybrid qubit-oscillator model, and use the number of such operations as a measure of the (circuit) complexity of an implementation. Moreover, we denote the set of elementary unitary operations in (ii)–(v) acting on the system $L^2(\mathbb{R})^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes m}$ by $\mathcal{U}^{n,m}_{\mathsf{elem}}$. We note that $\mathcal{U}^{n,m}_{\mathsf{elem}}$ includes the set of passive (single- and two-mode) Gaussian unitaries, i.e., the energy-preserving Gaussian unitaries, or equivalently the unitaries $U(A)$ defined in Eq. (2) such that e^{JA} is an orthogonal matrix. This is because for every passive Gaussian unitary U there is a representative $U = U(A)$ with $\|A\| \leqslant \pi$. However, our concrete implementations do not make use of passive Gaussian unitaries.

Occasionally, it will be useful to work with concrete bounds on the squeezing and displacement operators, which are not energy-preserving. For this purpose, we define the squeezing operator $M_\beta = e^{-i\ln\beta(QP+PQ)/2}$ for $\beta > 0$ and the set

$$\mathcal{U}^{n,m}_{\mathsf{SD}}(\alpha, \zeta) := \{\mathsf{ctrl}_a e^{-itP_j}, \mathsf{ctrl}_a e^{itQ_j}, e^{-itP_j}, e^{itQ_j}, \quad (M_\beta)_j\}_{t\in(-\zeta,\zeta),\beta\in(\alpha^{-1},\alpha),\, j\in\{1,\ldots,m\},\, a\in\{1,\ldots,n\}}$$
$$\cup \quad \{U_a, U_{a,b} \mid U_a, U_{a,b} \text{ one- or two-qubit unitary}\}_{a,\, b\in\{1,\ldots,n\}} \tag{10}$$

of unitary operations on $L^2(\mathbb{R})^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes m}$ of strength bounded by parameters $\alpha > 1$ and $\zeta \geqslant 0$ for squeezing and displacement, respectively. We emphasize that $\mathcal{U}_{\mathsf{SD}}^{n,m}(\alpha, \zeta)$ is not necessarily contained in the set $\mathcal{U}_{\mathsf{elem}}^{n,m}$ since we do not require that α or ζ are constantly bounded, but we allow these parameters to depend, e.g., on the dimension of an encoded qudit system. Nonetheless, we can simply decompose any squeezing and displacement operation into constant strength versions thereof.

A. Our contribution

We propose new qubit-oscillator-based implementations of logical gates or unitaries for ℓ logical qubits encoded in an (single-mode) approximate symmetrically squeezed GKP code $\mathcal{GKP}_\kappa^\star[2^\ell]$ with squeezing parameter $\kappa > 0$. Our main result is the following:

Theorem 1. Let $T, \ell \in \mathbb{N}$. Let $U = U_T \cdots U_1 : (\mathbb{C}^2)^{\otimes \ell} \to (\mathbb{C}^2)^{\otimes \ell}$ be an ℓ-qubit unitary circuit which is the composition of T two-qubit gates $U_1, \ldots, U_T$. Then there is a unitary circuit

$$W_U = W_{T'} \cdots W_1 : L^2(\mathbb{R})^{\otimes 2} \otimes (\mathbb{C}^2)^{\otimes 3} \to L^2(\mathbb{R})^{\otimes 2} \otimes (\mathbb{C}^2)^{\otimes 3}$$

consisting of

$$T' = O(\ell^2 T)$$

elementary unitary operations $W_1, \ldots, W_{T'} \in \mathcal{U}_{\mathsf{elem}}^{2,3}$ on two oscillators and three qubits such that

$$\mathsf{err}_{\mathcal{L}_\kappa}(W_U, U) \leqslant O(\ell T \kappa).$$

Here, the code space $\mathcal{L}_\kappa \cong (\mathbb{C}^2)^{\otimes \ell}$ is

$$\mathcal{L}_\kappa = \mathcal{GKP}_\kappa^\star[2^\ell] \otimes \mathbb{C}(|\Psi\rangle) \subset L^2(\mathbb{R})^{\otimes 2} \otimes (\mathbb{C}^2)^{\otimes 3},$$

where $\mathcal{GKP}_\kappa^\star[2^\ell]$ is the symmetrically squeezed GKP code with parameter $\kappa > 0$ encoding a qudit of dimension 2^ℓ [see Eq. (6)], and where $\Psi \in L^2(\mathbb{R}) \otimes (\mathbb{C}^2)^{\otimes 3}$ is the state

$$|\Psi\rangle = \left|\mathsf{GKP}_\kappa^{2^{-(\ell+1)}}(0)_2\right\rangle \otimes |0\rangle^{\otimes 3}.$$

Put more succinctly, Theorem 1 demonstrates that any multiqubit circuit U can approximately be recompiled (with linear overhead in the number of elementary gates as a function of the size of U) into a circuit W_U on two oscillators and three qubits. Furthermore, the logical gate error of the implementation vanishes in the limit $\kappa \to 0$ of large squeezing. In particular, in this limit, the elementary operations $\mathcal{U}_{\mathsf{elem}}^{2,3}$ together with the GKP resource state $|\mathsf{GKP}_\kappa^{2^{-(\ell+1)}}(0)_2\rangle$ can be used to generate a dense subgroup of ℓ-qubit unitaries on an appropriate subspace of $L^2(\mathbb{R})^{\otimes 2} \otimes (\mathbb{C}^2)^{\otimes 3}$. This means that a hybrid qubit-oscillator system consisting of two oscillators and three qubits gives a universal platform for realizing any ℓ-qubit (logical) unitary. We note that GKP states can be prepared with the set of all (including nonunitary) elementary operations (i)–(v) using one additional auxiliary mode and one qubit, see Ref. [21].

While we mainly focus on gate implementations using single-mode GKP codes $\mathcal{GKP}_\kappa^\star[2^\ell]$ as logical information carriers, our constructions extend straightforwardly to the multimode setting. Theorem 1 can be generalized to implement logical unitaries acting on multiple qudits, each encoded in a separate approximate GKP code space $\mathcal{GKP}_\kappa^\star[2^\ell]$ of dimension 2^ℓ. Importantly, the number of elementary operations as well as the auxiliary system does not change compared with the single-mode case, i.e., we still only need one additional oscillator and three qubits as an auxiliary system. We refer to Appendix E for details.

Implementations of Cliffords. As an example, we apply Theorem 1 to obtain accurate implementations of the qudit Cliffords $\mathsf{F}, \mathsf{P}, \mathsf{CZ}$ for (encoded) qudits of dimension $d = 2^\ell$, where $\ell \in \mathbb{N}$. Here F is the Fourier transform acting as

$$\mathsf{F}|x\rangle = 2^{-\ell/2} \sum_{y \in \mathbb{Z}_{2^\ell}} e^{2\pi i xy/2^\ell} |y\rangle \quad \text{for} \quad x \in \mathbb{Z}_{2^\ell} := \{0, \ldots, 2^\ell - 1\}, \tag{11}$$

the unitary P is the one-qudit phase gate defined by Eq. (7), and CZ is the two-qudit controlled phase gate acting as

$$\mathsf{CZ}(|x\rangle \otimes |y\rangle) = e^{2\pi i xy/2^\ell}(|x\rangle \otimes |y\rangle) \quad \text{for} \quad x, y \in \mathbb{Z}_{2^\ell}. \tag{12}$$

By a simple application of Theorem 1, we demonstrate that each unitary $U \in \{\mathsf{F}, \mathsf{P}, \mathsf{CZ}\}$ has an implementation $W_U = W_{T_U} \cdots W_1$ using $T_U = O(\ell^3)$ elementary operations $W_1, \ldots, W_{T_U} \in \mathcal{U}_{\mathsf{elem}}^{2,3}$ where the qudit is encoded in a symmetrically squeezed GKP code $\mathcal{GKP}_\kappa^\star[2^\ell]$, with a gate error which scales as $O(\ell^2 \kappa)$.

Combined with the results from Ref. [1] (which show that qudit Paulis X, Z have accurate implementations by Gaussian unitaries in symmetrically squeezed GKP codes), we conclude the following: There is a complete set of generators of the (logical) qudit Clifford group on $\mathbb{C}^{2^\ell}$, where each generator has an efficient implementation [i.e., one which uses $\mathsf{poly}(\ell)$ elementary operations], and each implementation has a logical error vanishing linearly in κ with a polynomial prefactor (in ℓ). In particular, these implementations become exact in the limit $\kappa \to 0$ of infinite squeezing.

Noisy implementations. We note that our implementations are also robust to noise: The logical gate error of a noisy implementation is bounded in terms of a computable function of the underlying noise channel $\mathcal{N}$.

In more detail, consider the ideal implementation $W_U = W_{T'} \cdots W_1$ of a logical unitary $U = U_{T'} \cdots U_1$ (see Theorem 1). The corresponding noisy implementation

$$\widetilde{\mathcal{W}}_U = (\mathcal{N} \circ \mathcal{W}_{T'}) \circ \cdots \circ (\mathcal{N} \circ \mathcal{W}_1)$$

with noise channel $\mathcal{N}$ satisfies

$$\mathsf{err}_{\mathcal{L}_\kappa}(\widetilde{\mathcal{W}}_U, U) \leqslant \mathsf{err}_{\mathcal{L}_\kappa}(\mathcal{W}_U, U) + \sum_{t=1}^{T'} \mathsf{err}_{\mathcal{L}_\kappa^{(t)}}\big(\mathcal{N}, I_{\mathcal{L}_\kappa^{(t)}}\big), \tag{13}$$

where we write $\mathcal{W}_t(\rho) = W_t \rho W_t^\dagger$, $\mathcal{L}_\kappa^{(t)} = W_{t-1} \cdots W_1 \mathcal{L}_\kappa \subset \mathcal{H}$ for $t \in \{1, \ldots, T'\}$ and we use $I_{\mathcal{L}_\kappa^{(t)}}$ to denote the identity on the subspace $\mathcal{L}_\kappa^{(t)}$. Equation (13) is an immediate consequence of the triangle inequality and the monotonicity of the diamond norm.

This implies, for example, that logical Clifford group generators of an encoded 2^ℓ-qudit system satisfy the following: Associated noisy implementations have a logical gate error with a component that scales with $O(\ell T \kappa)$ plus a term that depends on the noise.

B. Outline

This work is structured as follows: In Sec. II we construct a general bit-transfer unitary and demonstrate how it can be used to build implementations of multiqubit gates. In Sec. III we show how these operations can be implemented physically in ideal GKP codes using the elementary operations (i)–(v). In Sec. IV we prove the main result, i.e., Theorem 1, and derive bounds on implementations of Cliffords in hybrid qubit-oscillator systems. We further discuss our results and raise open questions in Sec. V

The appendixes are structured as follows: Appendix A recalls the definition and properties of the logical gate error. Appendix B computes matrix elements of hybrid qubit-oscillator implementations of bit-manipulation maps. In Appendix C, we bound the logical gate error of the implementations of basic bit-manipulation maps proposed in Sec. III. In Appendix D, we generalize the results obtained to the case of rectangular-envelope (instead of Gaussian-envelope) GKP codes. Appendix E generalizes the construction of qubit-oscillator implementations from one- to two-mode encodings.

II. CONSTRUCTION OF BIT-BASED UNITARIES

In this section, we define multiple operations for manipulating individual bits of ℓ-bit strings encoded into a 2^ℓ-dimensional space without a natural tensor product structure. We first introduce these operations on a logical level as a preparation for Sec. III, where we turn to physical implementations. In Sec. III the 2^ℓ-dimensional space used here will correspond to a GKP code space; that is, a code space embedded in a single oscillator.

In Sec. II A, we define what we call basic bit-manipulation maps. In Sec. II B we then show how these basic bit-manipulation maps can be composed to obtain unitary circuits achieving what we call bit-transfer. In Sec. II C we translate unitaries on ℓ qubits to unitary circuits on a 2^ℓ-dimensional space using bit-transfer unitaries.

A. Basic bit-manipulation maps

We define linear maps involving a qudit of dimension 2^ℓ (with orthonormal basis $\{|x\rangle\}_{x\in\mathbb{Z}_{2^\ell}}$, $\mathbb{Z}_{2^\ell}=\{0,\dots,2^\ell-1\}$) and a single qubit; that is, unitaries on $\mathbb{C}^{2^\ell}\otimes\mathbb{C}^2$. These correspond to associated bit manipulations. We represent elements of $\mathbb{Z}_{2^\ell}$ by ℓ-bit strings as

$$x=[x_{\ell-1},\dots,x_0]\quad\text{where } x=\sum_{j=0}^{\ell-1}2^j x_j\in\mathbb{Z}_{2^\ell}.$$

The qubit-controlled modular shift $\mathsf{CX}_\ell:\mathbb{C}^{2^\ell}\otimes\mathbb{C}^2\to\mathbb{C}^{2^\ell}\otimes\mathbb{C}^2$ is defined as the unitary that acts as

$$\mathsf{CX}_\ell(|x\rangle\otimes|b\rangle)=|x\oplus b\rangle\otimes|b\rangle\quad\text{for all}\quad x\in\mathbb{Z}_{2^\ell}\text{ and }b\in\{0,1\}$$

(here $\oplus$ denotes addition modulo 2^ℓ), i.e., it is given by

$$\mathsf{CX}_\ell=I\otimes|0\rangle\langle 0|+X\otimes|1\rangle\langle 1|,$$

where $X:\mathbb{C}^{2^\ell}\to\mathbb{C}^{2^\ell}$ is defined as

$$X|x\rangle=|x\oplus 1\rangle\quad\text{for}\quad x\in\mathbb{Z}_{2^\ell}.$$

Next, we define a unitary LSB_ℓ on $\mathbb{C}^{2^\ell}\otimes\mathbb{C}^2$ called the least significant bit gate which extracts the least significant bit x_0 of x, copying it onto a qubit. Concretely, we use the orthonormal basis $\{|x\rangle\otimes|b\rangle\}_{x\in\mathbb{Z}_{2^\ell},\,b\in\{0,1\}}$ of $\mathbb{C}^{2^\ell}\otimes\mathbb{C}^2$. Then we set

$$\mathsf{LSB}_\ell(|x\rangle\otimes|b\rangle)=|x\rangle\otimes|b\oplus x_0\rangle\text{ for }x=[x_{\ell-1},\dots,x_0]\text{ and }b\in\{0,1\},$$

where $\oplus$ denotes addition modulo 2.

We need one more map. It is an isometry which embeds a 2^ℓ-dimensional space ($\ell\in\mathbb{N}$) into a $2^{\ell+1}$-dimensional space. To define it, consider the injective map

$$\begin{array}{rccc}\mathsf{Embed}_\ell: & \mathbb{Z}_{2^\ell} & \to & \mathbb{Z}_{2^{\ell+1}},\\ & x & \mapsto & 2x.\end{array}$$

When applied to computational basis states, it gives rise to an isometry, i.e., a norm-preserving linear map, we denote as

$$\begin{array}{rccc}\mathsf{Embed}_\ell: & \mathbb{C}^{2^\ell} & \to & \mathbb{C}^{2^{\ell+1}},\\ & |x\rangle & \mapsto & |2x\rangle,\end{array}\tag{14}$$

with a slight abuse of notation. We refer to (14) as an "embedding map." We note that—when expressed in binary—this operation appends a 0 as the least significant bit, i.e., it acts as

$$\mathsf{Embed}_\ell|[x_{\ell-1},\dots,x_0]\rangle=|[x_{\ell-1},\dots,x_0,0]\rangle\quad\text{for}\quad x=[x_{\ell-1},\dots,x_0]\in\mathbb{Z}_{2^\ell}.$$

Even though the map $\mathsf{Embed}_\ell:\mathbb{C}^{2^\ell}\to\mathbb{C}^{2^{\ell+1}}$ is not a unitary, we can easily extend it by introducing an additional two-dimensional space. Concretely, we can define the unitary extension as

$$\begin{array}{rccc}\widetilde{\mathsf{Embed}}_\ell: & \mathbb{C}^{2^\ell}\otimes\mathbb{C}^2 & \to & \mathbb{C}^{2^{\ell+1}},\\ & |[x_{\ell-1},\dots,x_0]\rangle\otimes|y_0\rangle & \mapsto & |[x_{\ell-1},\dots,x_0,y_0]\rangle.\end{array}$$

Clearly, we have $\mathsf{Embed}_\ell(|x\rangle)=\widetilde{\mathsf{Embed}}_\ell(|x\rangle\otimes|0\rangle)$ for all $x\in\mathbb{Z}_{2^\ell}$.

See Table I for an illustration of these (unitary, respectively, isometric) maps, which we call basic bit-manipulation maps. We denote them by

$$\mathcal{G}_\ell=\{\mathsf{CX}_\ell,\mathsf{LSB}_\ell,\mathsf{Embed}_\ell\}.\tag{15}$$

We also write

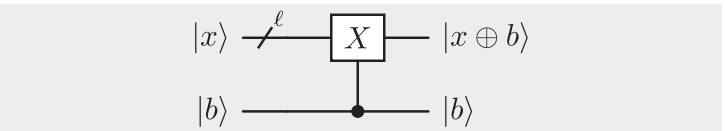
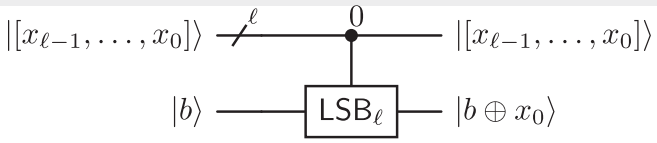
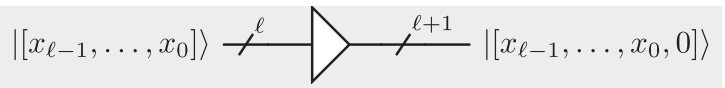
$$\mathcal{G}(\ell)=\bigcup_{k=1}^{\ell}(\mathcal{G}_k\cup\mathcal{G}_k^\dagger)\tag{16}$$

for the collection of all basic bit-manipulation maps and their adjoints up to dimension 2^ℓ. All the maps belonging to $\mathcal{G}(\ell)$ map between spaces of the form $\mathbb{C}^{2^k}\otimes\mathbb{C}^2$ for $k\in\{1,\dots,\ell+1\}$.

B. Circuits for bit-transfer unitaries in $\mathbb{C}^{2^\ell}\otimes\mathbb{C}^2$

The basic bit-manipulation maps [i.e., operations belonging to $\mathcal{G}(\ell)$, see Eq. (16)] are particularly powerful when used in a modular fashion. By composing these basic operations, one can construct more complex functionalities represented by their action on basis states.

TABLE I. The set $\mathcal{G}_\ell$ of basic bit-manipulation maps: The qubit-controlled modular shift unitary $\mathsf{C}X_\ell$ and the least significant bit-unitary LSB_ℓ on $\mathbb{C}^{2^\ell} \otimes \mathbb{C}^2$, as well as the embedding isometry $\mathsf{Embed}_\ell : \mathbb{C}^{2^\ell} \to \mathbb{C}^{2^{\ell+1}}$ defined by Eq. (14).

diagram	gate	type
	qubit-controlled shift $\mathsf{C}X_\ell$	unitary
	least significant bit LSB_ℓ	unitary
	embedding map Embed_ℓ	isometry

We first consider what we call bit-transfer unitaries. Generalizing the LSB_ℓ gate, we may consider unitaries which extract any individual bit x_j, $j \in \{0, \ldots, \ell-1\}$ in the binary representation of $x \in \mathbb{Z}_{2^\ell}$, effectively cutting the jth bit from the register and pasting it onto a qubit.

Concretely, consider again a Hilbert space of the form $\mathbb{C}^{2^\ell} \otimes \mathbb{C}^2$ with orthonormal basis $\{|x\rangle \otimes |b\rangle\}_{x \in \mathbb{Z}_{2^\ell},\, b \in \{0,1\}}$. For every $j \in \{0, \ldots, \ell-1\}$, we then define the bit-transfer gate

$$\mathsf{Transf}_\ell^j : \mathbb{C}^{2^\ell} \otimes \mathbb{C}^2 \to \mathbb{C}^{2^\ell} \otimes \mathbb{C}^2$$

by

$$\mathsf{Transf}_\ell^j(|x\rangle \otimes |b\rangle) = |x - 2^j(b \oplus x_j)\rangle \otimes |b \oplus x_j\rangle \quad \text{for} \quad x = [x_{\ell-1}, \ldots, x_0] \in \mathbb{Z}_{2^\ell} \text{ and } b \in \{0,1\},$$

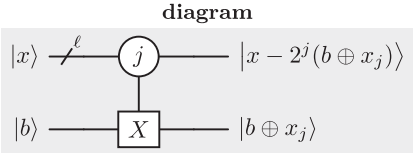
where the term $x - 2^j(b \oplus x_j)$ is to be understood modulo 2^ℓ, see Table II for an illustration. In particular, it follows that

$$\mathsf{Transf}_\ell^j(|[x_{\ell-1}, \ldots, x_0]\rangle \otimes |0\rangle) = |[x_{\ell-1}, \ldots, x_{j+1}, 0, x_{j-1}, \ldots, x_0]\rangle \otimes |x_j\rangle \tag{17}$$

for any $[x_{\ell-1}, \ldots, x_0] \in \mathbb{Z}_{2^\ell}$, i.e., Transf_ℓ^j extracts the bit x_j. Correspondingly, the adjoint map $(\mathsf{Transf}_\ell^j)^\dagger$ satisfies

$$\left(\mathsf{Transf}_\ell^j\right)^\dagger |[x_{\ell-1}, \ldots, x_{j+1}, 0, x_{j-1}, \ldots, x_0]\rangle \otimes |x_j\rangle = |[x_{\ell-1}, \ldots, x_0]\rangle \otimes |0\rangle. \tag{18}$$

TABLE II. The bit-transfer unitary Transf_ℓ^j. The adjoint bit-transfer unitary $(\mathsf{Transf}_\ell^j)^\dagger$ is represented by a similar diagram with $X^\dagger$ instead of X.

diagram	gate	type
	Transf_ℓ^j	unitary

The following lemma shows that bit-transfer unitaries can be realized using basic bit-manipulation maps, i.e., operations belonging to the set $\mathcal{G}(\ell)$ [see Eq. (16)]. The construction involves two auxiliary qubits, which are in the state $|0\rangle^{\otimes 2}$. The state of these auxiliary qubits is unchanged by the circuit, i.e., it acts as a catalyst.

Lemma 1 (Circuit for the bit-transfer unitary Transf_ℓ^j). Let $\ell \in \mathbb{N}$ and $j \in \{0, \ldots, \ell-1\}$ be arbitrary. There is a circuit V_ℓ^j acting on $(\mathbb{C}^{2^\ell} \otimes \mathbb{C}^2) \otimes (\mathbb{C}^2)^{\otimes 2}$ composed of fewer than $12j - 4$ maps belonging to the set $\mathcal{G}(\ell)$ of basic bit-manipulation operations such that

$$\begin{aligned} V_\ell^j[(|x\rangle \otimes |b\rangle) \otimes |0\rangle^{\otimes 2}] &= [|x - 2^j(b \oplus x_j)\rangle \otimes |b \oplus x_j\rangle] \otimes |0\rangle^{\otimes 2} \\ &= \left[\mathsf{Transf}_\ell^j(|x\rangle \otimes |b\rangle)\right] \otimes |0\rangle^{\otimes 2} \end{aligned}$$

for all $x = [x_{\ell-1}, \ldots, x_0] \in \mathbb{Z}_{2^\ell}$ and $b \in \{0,1\}$. In other words, the circuit V_ℓ^j realizes the unitary Transf_ℓ^j on the subspace

$$(\mathbb{C}^{2^\ell} \otimes \mathbb{C}^2) \otimes (\mathbb{C}|0\rangle^{\otimes 2}) \subset (\mathbb{C}^{2^\ell} \otimes \mathbb{C}^2) \otimes (\mathbb{C}^2)^{\otimes 2},$$

where the two auxiliary qubits are in the state $|0\rangle$.

Proof. For $j \in \{1, \ldots, \ell\}$ consider the right-shift isomorphism

$$\begin{array}{rccccc} R_j : & \mathbb{C}^{2^j} & \to & \mathbb{C}^{2^{j-1}} & \otimes & \mathbb{C}^2, \\ & |[x_{j-1}, \ldots, x_0]\rangle & \mapsto & |[x_{j-1}, \ldots, x_1]\rangle & \otimes & |x_0\rangle, \end{array} \tag{19}$$

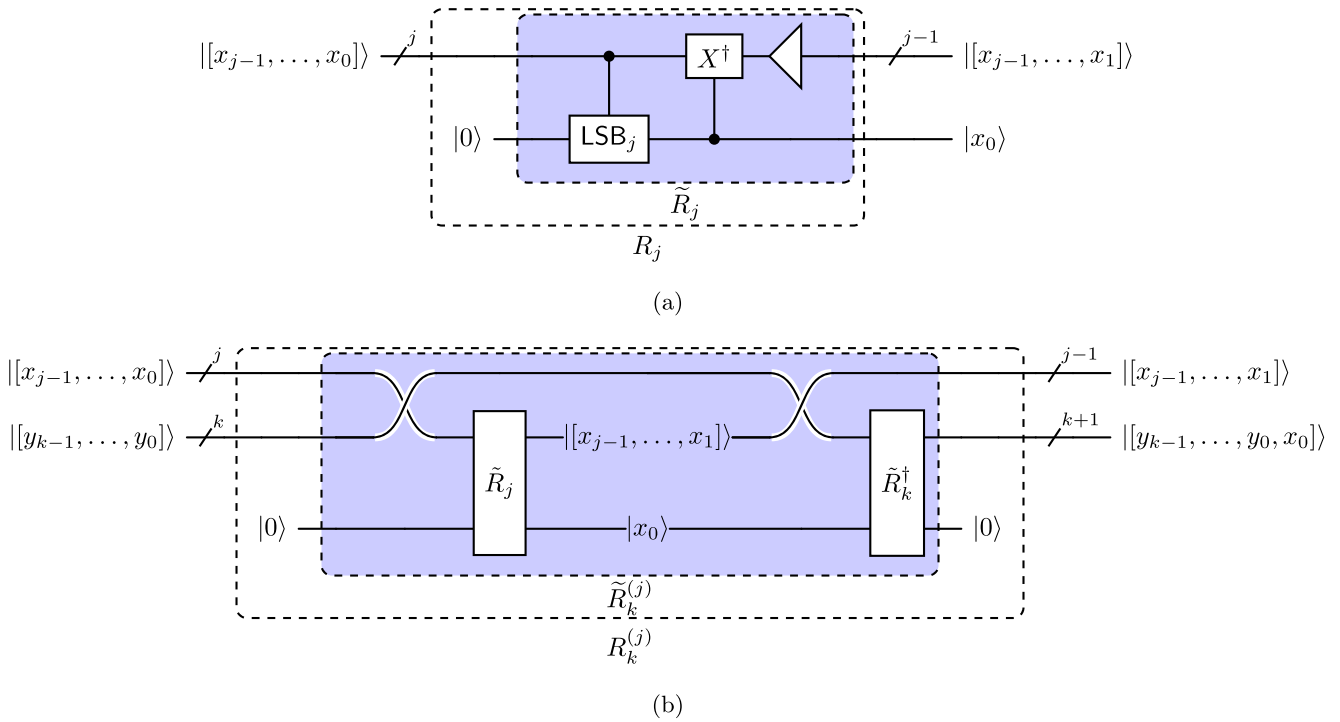


FIG. 1. Circuit realizing the isometry $R_k^{(j)}$ based on the isometry R_j and the adjoint of R_k. All blue subcircuits only involve basic bit-manipulation maps from the set $\mathcal{G}(\ell)$. (a) Circuit realizing the isometry R_j defined in Eq. (19). It is based on a subcircuit $\widetilde{R}_j$ (shaded in blue) acting on a bipartite system with the auxiliary (qubit) system in the state $|0\rangle$. The adjoint of the isometry Embed_{j-1} is represented by an inverted triangle. (b) Circuit realizing the isometry $R_k^{(j)}$ [see Eq. (20)]. It is based on a subcircuit $\widetilde{R}_k^{(j)}$ (blue) acting on an auxiliary (qubit) system in the state $|0\rangle$.

and for $j \in \{1,\ldots,\ell\}$ and $k \in \{1,\ldots,\ell\}$ the right-shift isomorphism

$$\begin{array}{rccccccc} R_k^{(j)}: & \mathbb{C}^{2^j} & \otimes & \mathbb{C}^{2^k} & \to & \mathbb{C}^{2^{j-1}} & \otimes & \mathbb{C}^{2^{k+1}}, \\ & |[x_{j-1},\ldots,x_0]\rangle & \otimes & |[y_{k-1},\ldots,y_0]\rangle & \mapsto & |[x_{j-1},\ldots,x_1]\rangle & \otimes & |[y_{k-1},\ldots,y_0,x_0]\rangle. \end{array} \tag{20}$$

A circuit realization of these isometries using basic bit-manipulation operations [i.e., operations belonging to the set $\mathcal{G}(\ell)$] is given in Fig. 1.

For $j \in \{0,\ldots,\ell-1\}$, define the unitary map

$$W_j^\ell = R_{j-1}^{[\ell-(j-1)]}\cdots R_1^{(\ell-1)}R_\ell : \mathbb{C}^{2^\ell} \to \mathbb{C}^{2^{\ell-j}} \otimes \mathbb{C}^{2^j}.$$

A circuit realizing the map W_j^ℓ is given in Fig. 2. It acts on computational basis states (in the binary representation) as

$$W_j^\ell |[x_{\ell-1},\ldots,x_0]\rangle = |[x_{\ell-1},\ldots,x_j]\rangle \otimes |[x_0,\ldots,x_{j-1}]\rangle \quad \text{for} \quad [x_{\ell-1},\ldots,x_0] \in \{0,\ldots,2^\ell-1\}, \tag{21}$$

and thus

$$\left(W_j^\ell\right)^\dagger (|[x_{\ell-1},\ldots,x_j]\rangle \otimes |[x_0,\ldots,x_{j-1}]\rangle) = |2^j[x_{\ell-1},\ldots,x_j] + [x_{j-1},\ldots,x_0]\rangle. \tag{22}$$

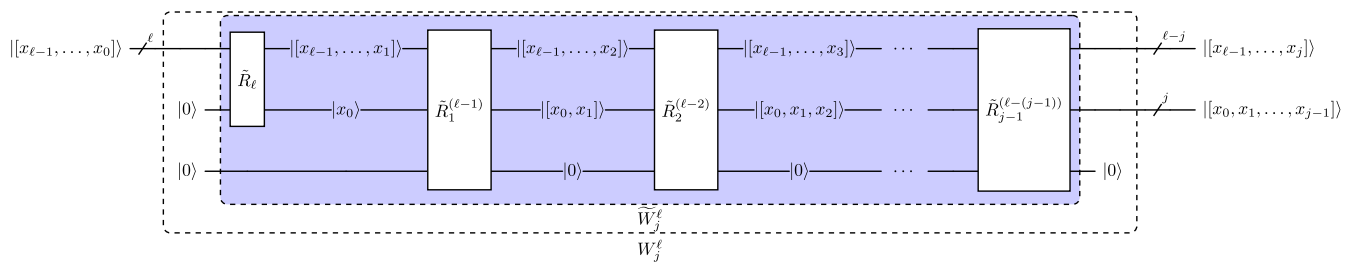


FIG. 2. Circuit for the isometry W_j^ℓ. It uses two auxiliary (qubit) systems each in the state $|0\rangle$. The construction is presented in terms of the unitaries $\widetilde{R}_j$ and $\widetilde{R}_k^{(j)}$ introduced in Figs. 1(a) and 1(b), respectively. We denote the blue subcircuit by $\widetilde{W}_j^\ell$.

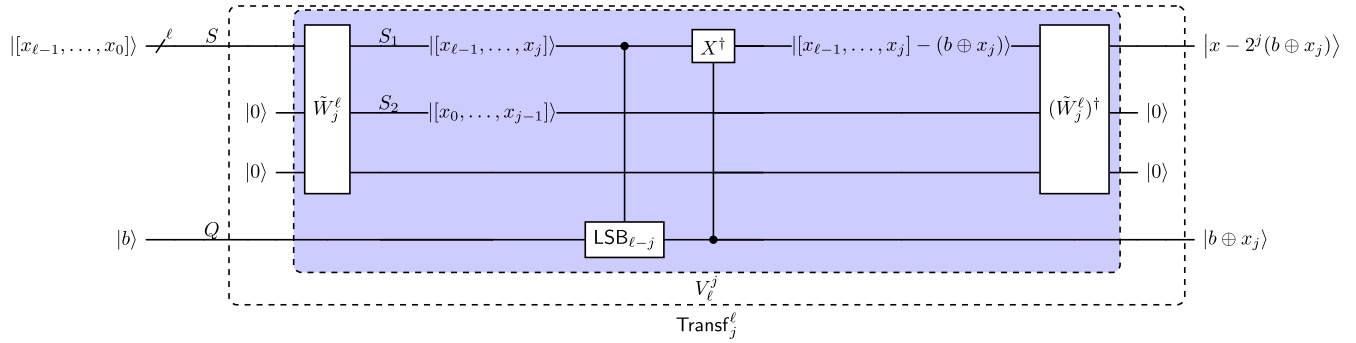


FIG. 3. Circuit for the bit-transfer unitary Transf_ℓ^j where $x=[x_{\ell-1},\ldots,x_0]$. Here $[x_{\ell-1},\ldots,x_j]-(b\oplus x_j)$ is understood modulo $2^{\ell-j}$. The unitary $\widetilde{W}_j^\ell$ is the blue subcircuit in Fig. 2. The blue subcircuit in this figure is the circuit V_ℓ^j which realizes Transf_ℓ^j.

It follows from Eqs. (21) and (22) that the map

$$\big(W_j^\ell\big)^\dagger_{S_1S_2\to S}(\mathsf{CX}_\ell)_{Q\to S_1}(\mathsf{LSB}^{\ell-j})_{S_1\to Q}\big(W_j^\ell\big)_{S\to S_1S_2}$$

realizes the desired functionality, where Q denotes the qubit system, whereas S, S_1, and S_2 are of dimensions 2^ℓ, $2^{\ell-j}$, and 2^j, respectively (see Fig. 3).

The circuits $\widetilde{R}_j$ and $\widetilde{R}_k^{(j)}$ require 3 and 6 maps from $\mathcal{G}(\ell)$, respectively. The circuit $\widetilde{W}_j^\ell$ in Fig. 2 consists of 1 circuit of the type $\widetilde{R}_j$ and $j-1$ of the type $\widetilde{R}_k^{(j)}$, amounting to a total of $6(j-1)+3=6j-3$ maps from $\mathcal{G}(\ell)$. Then the implementation of Transf_ℓ^j in Fig. 3 which includes two isometries of the type W_j^ℓ and two additional operations from $\mathcal{G}(\ell)$ uses up to $2(6j-3)+2=12j-4$ operations in $\mathcal{G}(\ell)$. ■

C. Bit-transfer-based implementation of multiqubit gates

In this section we consider a multiqubit unitary $U:(\mathbb{C}^2)^{\otimes\ell}\to(\mathbb{C}^2)^{\otimes\ell}$ and show how it can be realized on a system of the form $\mathbb{C}^{2^\ell}$ using bit transfer. (We consider that U acts nontrivially on two qubits and, by notational abuse, call U a two-qubit unitary.) Since the qudit dimension is of the form $d=2^\ell$ with $\ell\in\mathbb{N}$, we may also think of the qudit as consisting of ℓ qubits with a suitable isomorphism $\mathbb{C}^{2^\ell}\cong(\mathbb{C}^2)^{\otimes\ell}$. Combined with property (17), this allows us to outsource the computation of any two-qubit unitary to two auxiliary qubit systems which are initialized in the computational basis state $|0\rangle$.

In more detail, our construction uses a system of the form $\mathbb{C}^{2^\ell}\otimes(\mathbb{C}^2)^{\otimes4}$, i.e., four additional auxiliary qubits in addition to the system $\mathbb{C}^{2^\ell}$ which encodes the logical information (i.e., ℓ logical qubits). To address the latter, we use the isomorphism $J_\ell:(\mathbb{C}^2)^{\otimes\ell}\to\mathbb{C}^{2^\ell}$ defined by

$$J_\ell(|x_{\ell-1}\rangle\otimes\cdots\otimes|x_0\rangle)=|[x_{\ell-1},\ldots,x_0]\rangle\quad\text{for}\quad(x_0,\ldots,x_{\ell-1})\in\{0,1\}^\ell.$$

For better readability we sometimes write $|a\rangle|b\rangle$ for the tensor product $|a\rangle\otimes|b\rangle$.

Lemma 2 (Bit-transfer-based implementation of two-qubit gates). Let $\ell\geqslant2$ be an integer. Consider a "logical" ℓ-qubit system denoted

$$A_0\cdots A_{\ell-1}=(\mathbb{C}^2)^{\otimes\ell}.$$

Let $j<k$ and $j,k\in\{0,\ldots,\ell-1\}$ be arbitrary. Let $U_{A_jA_k}:(\mathbb{C}^2)^{\otimes\ell}\to(\mathbb{C}^2)^{\otimes\ell}$ be a unitary acting nontrivially only on the two qubits A_jA_k. Let

$$SQ_1Q_2Q_3Q_4=\mathbb{C}^{2^\ell}\otimes(\mathbb{C}^2)^{\otimes4}$$

be a system consisting of a 2^ℓ-dimensional ("code") space S and four auxiliary qubits $Q_1\cdots Q_4$. Then there is a circuit $V_{U_{A_jA_k}}$ on $SQ_1\cdots Q_4$ consisting of fewer than $48\ell-16$ operations belonging to the set $\mathcal{G}(\ell)$ and a single two-qubit operation which satisfies

$$\begin{aligned}&\big(J_\ell^{-1}\otimes I\big)V_{U_{A_jA_k}}(J_\ell\otimes I)((|x_{\ell-1}\rangle\cdots|x_0\rangle)\otimes|0\rangle^{\otimes4})\\&\quad=\big(U_{A_jA_k}(|x_{\ell-1}\rangle\cdots|x_0\rangle)\big)\otimes|0\rangle^{\otimes4}\end{aligned}\tag{23}$$

for all $x=(x_0,\ldots,x_{\ell-1})\in\{0,1\}^\ell$. In other words, the circuit $V_{U_{A_jA_k}}$ realizes the unitary $U_{A_jA_k}$ on the 2^ℓ-dimensional subspace

$$\mathbb{C}^{2^\ell}\otimes(\mathbb{C}|0\rangle^{\otimes4})\subset SQ_1\cdots Q_4=\mathbb{C}^{2^\ell}\otimes(\mathbb{C}^2)^{\otimes4},$$

where the four auxiliary qubits are in the state $|0\rangle$.

In Appendix E we give a generalization (see Lemma E1) of this result to the case where 2ℓ qubits are encoded into a bipartite Hilbert space of the form $(\mathbb{C}^{2^\ell})^{\otimes2}$.

Proof. The expressions (17) and (18) imply

$$\begin{aligned}&\big(\mathsf{Transf}_\ell^j\big)^\dagger_{SQ_1}\big(\mathsf{Transf}_\ell^k\big)^\dagger_{SQ_2}\big(I\otimes U_{Q_1Q_2}\big)\\&\quad\times\big(\mathsf{Transf}_\ell^k\big)_{SQ_2}\big(\mathsf{Transf}_\ell^j\big)_{SQ_1}(|x\rangle\otimes|0\rangle^{\otimes2})\\&\quad=\big(J_\ell U_{A_jA_k}J_\ell^{-1}|x\rangle\big)\otimes|0\rangle^{\otimes2}\end{aligned}\tag{24}$$

for all $j,k\in\{0,\ldots,\ell-1\}$ and for all $x\in\{0,\ldots,2^\ell-1\}$, where S, Q_1, and Q_2 are systems of dimension 2^ℓ, 2, and 2, respectively, see Fig. 4.

The construction of $V_{U_{A_jA_k}}$ relies on the identity (24), which directly implies that the unitary

$$\begin{aligned}\widetilde{V}_{U_{A_jA_k}}=&\big(\mathsf{Transf}_\ell^j\big)^\dagger_{SQ_1}\big(\mathsf{Transf}_\ell^k\big)^\dagger_{SQ_2}\big(I\otimes U_{Q_1Q_2}\big)\\&\times\big(\mathsf{Transf}_\ell^k\big)_{SQ_2}\big(\mathsf{Transf}_\ell^j\big)_{SQ_1}\end{aligned}$$

satisfies

$$\widetilde{V}_{U_{A_jA_k}}(|x\rangle\otimes|0\rangle^{\otimes2})$$

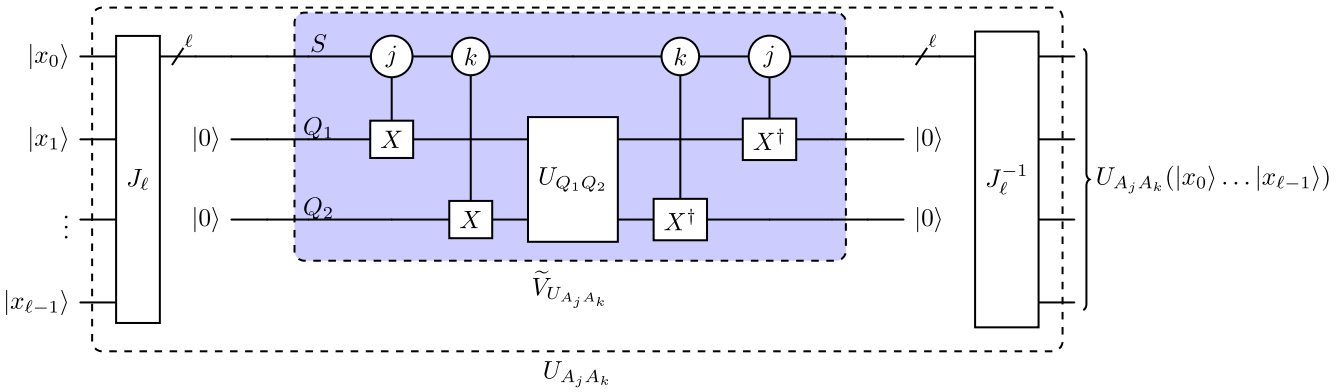


FIG. 4. Circuit implementing the two-qubit unitary $U_{A_jA_k}$. It uses the bit-transfer unitaries Transf_ℓ^j and Transf_ℓ^k, two additional qubit systems and the two-qubit unitary $U_{Q_1Q_2}$ acting on the auxiliary two-qubit system Q_1Q_2. The circuit in blue is $\widetilde{V}_{U_{A_jA_k}}$. Utilizing the implementation given in Lemma 1 for the bit-transfer unitaries results in the use of two additional qubits (which are reused across all four implementations of a bit-transfer unitary, since the two auxiliary qubits are coherently reset to the state $|0\rangle$, see Fig. 3). It follows that in this case, the total number of additional auxiliary qubits is four.

$$= \left(J_\ell U_{A_jA_k} J_\ell^{-1}|x\rangle\right) \otimes |0\rangle^{\otimes 2} \quad \text{for all } x \in \{0,\dots,2^\ell-1\}. \tag{25}$$

Then, substituting each bit-transfer unitary Transf_ℓ^j and its adjoint in $\widetilde{V}_{U_{A_jA_k}}$ by the circuit given in Lemma 1 results in a circuit $V_{U_{A_jA_k}}$ which uses two additional auxiliary qubits Q_3 and Q_4. Each transfer gate uses both auxiliary qubits Q_3 and Q_4, which is possible because the latter are reset to the state $|0\rangle$ after each utilization (cf. Fig. 4). Hence, four additional auxiliary qubits are used in total.

By Eq. (25), the circuit $V_{U_{A_jA_k}}$ satisfies property (23). Furthermore, $V_{U_{A_jA_k}}$ uses a two-qubit gate and four bit transfer unitaries, each of which requires $12\ell-4$ basic bit-manipulation maps in $\mathcal{G}(\ell)$, totalling $48\ell-16$ operations in $\mathcal{G}(\ell)$. This concludes the proof. ■

III. IMPLEMENTATION OF LOGICAL UNITARIES IN IDEAL GKP CODES

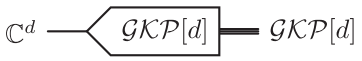
In this section, we show how logical unitaries can be implemented in ideal GKP codes. Although the code spaces are spanned by unphysical (unnormalizable) states, the action of certain Gaussian unitaries and phase-space displacements can conveniently be analyzed using position eigenstates. We use this framework to derive exact circuit identities for ideal GKP codes in order to motivate the construction of circuits we expect to exhibit similar behavior when acting on approximate (finitely squeezed) GKP codes.

In Sec. III A, we show how elementary operations in the hybrid qubit-oscillator model can be used to implement the basic bit-manipulation maps introduced in Sec. II A and the bit-transfer unitaries introduced in Sec. II B in ideal GKP codes. In Sec. III B, we extend this result by showing how two-qubit unitaries can be implemented in ideal GKP code spaces.

In the following, we use the encoding map

$$\mathsf{gkpEnc}[d]: \begin{array}{ccc} \mathbb{C}^d & \to & \mathcal{GKP}[d] \\ |j\rangle & \mapsto & |\mathsf{GKP}(j)_d\rangle \end{array} \text{ for } j \in \mathbb{Z}_d$$

between $\mathbb{C}^d$ and the ideal GKP code $\mathcal{GKP}[d]$ for an integer $d \geqslant 2$ [see Eq. (3) for the definition of the code states)]. In a circuit, we represent this map by the diagram



Similarly, the corresponding decoding map is defined by its action on basis states

$$\mathsf{gkpDec}[d] = (\mathsf{gkpEnc}[d])^{-1}: \begin{array}{ccc} \mathcal{GKP}[d] & \to & \mathbb{C}^d \\ |\mathsf{GKP}(j)_d\rangle & \to & |j\rangle \end{array} \text{ for } j \in \mathbb{Z}_d,$$

linearly extended to all of $\mathcal{GKP}[d]$. We represent it as

A. Basic bit-manipulation and bit-transfer maps

Here, we give physical realizations of basic bit-manipulation operations belonging to the set $\mathcal{G}(\ell)$ in appropriate codes. Recall that the considered basic bit-manipulation operations are defined in terms of a set $\mathcal{G}_\ell$ of maps, which consists of [see Eq. (15)]

(1) the qubit-controlled modular shift CX_ℓ,
(2) the least significant bit-gate LSB_ℓ,

both of which are unitaries on $\mathbb{C}^{2^\ell} \otimes \mathbb{C}^2$, as well as

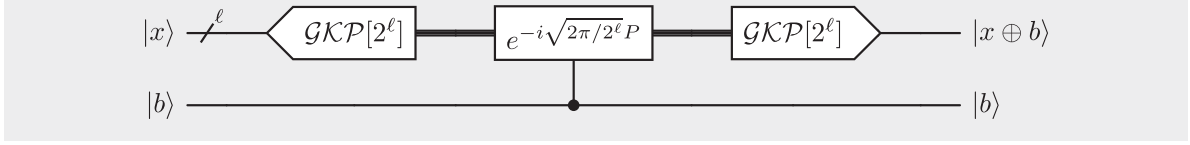
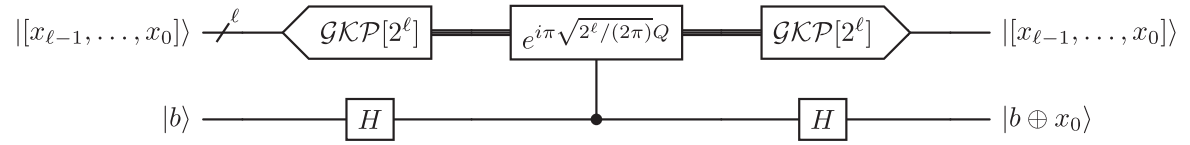
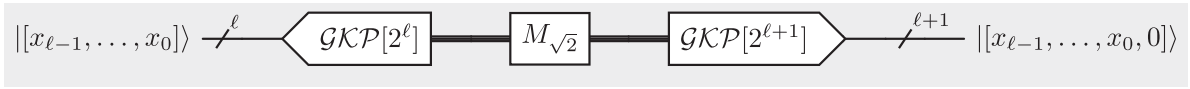
(3) the embedding isometry $\mathsf{Embed}_\ell : \mathbb{C}^{2^\ell} \to \mathbb{C}^{2^{\ell+1}}$.

The set $\mathcal{G}(\ell)$ of all bit-manipulation maps consists of all such maps and their adjoints, see Eq. (16).

Here we consider logical implementations of these maps when $\mathbb{C}^{2^\ell} \otimes \mathbb{C}^2$ is replaced by the subspace $\mathcal{GKP}[2^\ell] \otimes \mathbb{C}^2 \subset L^2(\mathbb{R}) \otimes \mathbb{C}^2$ of a qubit-oscillator system. In other words, the code is a product of an ideal, i.e., infinitely squeezed GKP code $\mathcal{GKP}[2^\ell]$ (encoding a 2^ℓ-dimensional qudit) and a "bare" (i.e., unencoded) physical qubit.

Recall that $\mathcal{U}_{\mathsf{elem}}^{n,m}$ denotes the set of elementary unitary operations (ii)–(v) acting on the system $L^2(\mathbb{R})^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes m}$ (see Sec. I).

TABLE III. Exact implementations of the unitary (logical) operations CX_ℓ and LSB_ℓ in the code $\mathcal{GKP}[2^\ell]\otimes\mathbb{C}^2$, and of the isometry Embed_ℓ embedding the code $\mathcal{GKP}[2^\ell]$ into the code $\mathcal{GKP}[2^{\ell+1}]$. Here we write $M_{\sqrt{2}} = e^{-i(\ln 2)(QP+PQ)/4}$ for brevity. All these implementations are defined by unitary maps on $L^2(\mathbb{R})\otimes\mathbb{C}^2$, except for Embed_ℓ which is defined on $L^2(\mathbb{R})$. The associated logical operations constitute the set $\mathcal{G}_\ell$. To obtain circuits composed of elementary gates, the controlled phase shift in the implementation of LSB_ℓ needs to be decomposed, see the proof of Lemma 3.

logical operation	implementation
CX_ℓ	
LSB_ℓ	
Embed_ℓ	

Lemma 3 (Implementation of bit-manipulation maps in ideal GKP codes). Let $\ell\in\mathbb{N}$ be an integer. Then the following holds:

(i) The (logical) unitaries CX_ℓ and LSB_ℓ for the code $\mathcal{GKP}[2^\ell]\otimes\mathbb{C}^2$ can be implemented exactly by a unitary circuit on $L^2(\mathbb{R})\otimes\mathbb{C}^2$ using one, respectively at most $\ell+6$ elementary operations in $\mathcal{U}^{1,1}_{\mathsf{elem}}$.

(ii) The (logical) isometry $\mathsf{Embed}_\ell : \mathcal{GKP}[2^\ell]\to\mathcal{GKP}[2^{\ell+1}]$ has an exact implementation by a unitary circuit on $L^2(\mathbb{R})$ using one elementary operation in $\mathcal{U}^{1,0}_{\mathsf{elem}}$.

See Table III for the corresponding circuits.

Proof. The circuit identities in Table III are straightforward to verify (see, e.g., Ref. [23], Sec. 1.1.4 for the case of the LSB_ℓ-gate). The circuits for CX_ℓ and Embed_ℓ consist of a single constant-strength unitary, thus giving the desired implementation.

The circuit for LSB_ℓ involves a qubit-controlled phase-space displacement where the displacement is of the form

$$W((0,\alpha)) = e^{i\alpha Q}$$

[see Eq. (1)] with $\alpha=\pi[2^\ell/(2\pi)]^{1/2}$. This is not of constant strength for general $\ell\in\mathbb{N}$ but can be decomposed into constant-strength operations in the set $\mathcal{U}^{1,0}_{\mathsf{elem}}$ as follows. Define the squeezing operator $M_\alpha = e^{-i(\ln\alpha)(QP+PQ)/2}$. We have

$$e^{i\alpha Q} = (M^\dagger_{\alpha'})^n e^{iQ}(M_{\alpha'})^n \quad \text{where } \alpha' = e^{\frac{\ln\alpha}{\lceil|\ln\alpha|\rceil}} \text{ and } n=\lceil|\ln\alpha|\rceil. \tag{26}$$

Equation (26) gives a decomposition of the unitary $e^{i\alpha Q}$ into $2n+1$ constant-strength unitaries. Notice that the circuit for LSB_ℓ uses two additional Hadamard gates, giving a total of $2n+3$ elementary operations in $\mathcal{U}^{1,1}_{\mathsf{elem}}$. The claim follows from the fact that

$$\begin{aligned} &I\otimes|0\rangle\langle 0| + e^{i\alpha Q}\otimes|1\rangle\langle 1| \\ &\quad = (M^\dagger_{\alpha'}\otimes I)^n(I\otimes|0\rangle\langle 0| + e^{iQ}\otimes|1\rangle\langle 1|)(M_{\alpha'}\otimes I)^n, \end{aligned}$$

where each factor on the right-hand side (r.h.s.). is a constant-strength unitary and $n=\lceil|\ln(\pi[2^\ell/(2\pi)]^{1/2})|\rceil\leqslant(\ell-1)/2+2$ which gives $2n+3\leqslant\ell+6$. ∎

An immediate consequence of Lemma 3 is a circuit implementation of the bit-transfer unitary Transf^j_ℓ introduced in Sec. II B. Namely, we have the following:

Lemma 4 (Implementation of bit-transfer in ideal GKP codes). Let $\ell\in\mathbb{N}$ be an integer and $j\in\{0,\ldots,\ell-1\}$. Then there is a unitary circuit $W_{\mathsf{Transf}^j_\ell}$ on $L^2(\mathbb{R})^{\otimes 2}\otimes(\mathbb{C}^2)^{\otimes 2}$ such that the following holds:

(i) The unitary $W_{\mathsf{Transf}^j_\ell}$ consists of fewer than $85\ell^2$ operations in $\mathcal{U}^{2,2}_{\mathsf{elem}}$ and satisfies

$$\begin{aligned} &W_{\mathsf{Transf}^j_\ell}[|\mathsf{GKP}(x)_{2^\ell}\rangle\otimes|\mathsf{GKP}(0)_2\rangle\otimes|0\rangle\otimes|b\rangle] \\ &\quad = |\mathsf{GKP}(x-2^j(b\oplus x_j))_{2^\ell}\rangle\otimes[|\mathsf{GKP}(0)_2\rangle\otimes|0\rangle] \\ &\qquad \otimes|b\oplus x_j\rangle \end{aligned}$$

for all $x=[x_{\ell-1},\ldots,x_0]\in\mathbb{Z}_{2^\ell}$ and $b\in\{0,1\}$. That is, the unitary $W_{\mathsf{Transf}^j_\ell}$ implements the bit-transfer unitary $\mathsf{Transf}^j_\ell : \mathbb{C}^{2^\ell}\otimes\mathbb{C}^2\to\mathbb{C}^{2^\ell}\otimes\mathbb{C}^2$ exactly on the space

$$\mathcal{GKP}[2^\ell]\otimes\mathbb{C}(|\mathsf{GKP}(0)_2\rangle\otimes|0\rangle)\otimes\mathbb{C}^2.$$

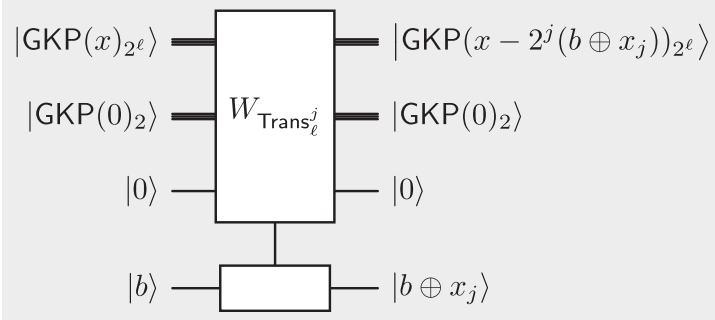
(ii) Define $\zeta=\sqrt{\pi}2^{(\ell-1)/2}$. There exists a family of $L\leqslant 36\ell$ unitaries $\{W^{(t)}\}_{t=1}^L$ with

$$W^{(t)}\in\mathcal{U}^{2,2}_{\mathsf{SD}}(2,\zeta) \quad \text{for all} \quad t\in\{1,\ldots,L\}$$

such that $W_{\mathsf{Transf}^j_\ell} = W^{(L)}\cdots W^{(1)}$. [Recall from Eq. (10) that $\mathcal{U}^{2,2}_{\mathsf{SD}}(\alpha,\zeta)$ is the set of unitary operations on $L^2(\mathbb{R})^{\otimes 2}\otimes(\mathbb{C}^2)^{\otimes 2}$ with squeezing and displacements bounded by $\alpha>1$ and $\zeta>0$, respectively.]

Proof. The physical realization $W_{\mathsf{Transf}^j_\ell}$ of the bit-transfer Transf^j_ℓ is obtained by applying Lemma 1 (cf. Fig. 3) which gives a circuit for the latter, and replacing each bit-manipulation map $M\in\mathcal{G}(\ell)$ by the implementation provided

TABLE IV. Diagram of the implementation $W_{\mathsf{Transf}_\ell^j}$ given in Lemma 4 of the bit-transfer unitary Transf_ℓ^j. It moves the jth bit of the encoded ℓ-bit string x into a physical qubit.

diagram	gate	type
	$W_{\mathsf{Transf}_\ell^j}$	unitary

in Lemma 3 (cf. Table III). We note here that since $\mathcal{GKP}[2^k]$ is a code defined on a single oscillator for any $k \in \mathbb{N}$, qudits of different dimensions occurring, i.e., in the course of applying the circuit for the bit-transfer unitary Transf_ℓ^j (see Lemma 1) can be embedded into a single oscillator. In other words, these substitutions give an implementation of Transf_ℓ^j whereby the circuit V_ℓ^j introduced in Lemma 1 is turned into the circuit $W_{\mathsf{Transf}_\ell^j}$ acting on $L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 := SBQQ'$ (i.e., two oscillators S, B and two qubits Q, Q'). This means that while Transf_ℓ^j acts on two systems, the implementation $W_{\mathsf{Transf}_\ell^j}$ acts on four systems. Namely, the implementation $W_{\mathsf{Transf}_\ell^j}$ acts by bit-transfer inside the subspace

$$\mathcal{GKP}[2^\ell] \otimes \mathbb{C}(|\mathsf{GKP}(0)_2\rangle \otimes |0\rangle) \otimes \mathbb{C}^2 \subset L^2(\mathbb{R})^{\otimes 2} \otimes (\mathbb{C}^2)^{\otimes 2}.$$

To bound the total number of elementary operations we note that the circuit V_ℓ^j is composed of at most $12\ell - 4$ basic bit-manipulation operations belonging to $\mathcal{G}(\ell)$ (see Lemma 1), and each such operation can be implemented in the code of interest using at most $\ell + 6$ elementary physical operations (see Lemma 3). Therefore, we need at most $(12\ell - 4)(\ell + 6) < 85\ell^2$ elementary operations. This shows Claim (i).

Claim (ii) follows immediately from Lemma 3 by observing that the physical implementation of each bit manipulation map can be written as a circuit of at most three elements contained in the set $\mathcal{G}(\ell)$ in Eq. (16). Therefore we can bound $L \leqslant 3(12\ell - 4) \leqslant 36\ell$. ■

We use the diagram given in Table IV to represent $W_{\mathsf{Transf}_\ell^j}$.

B. Two-qubit gates

The implementations of bit-transfer unitaries obtained in Sec. III A can be used to implement logical unitaries following the procedure given in Sec. II C. Namely, we have the following results:

Lemma 5 (Implementation of two-qubit unitaries in ideal GKP codes). Let $\ell \geqslant 2$ be an integer. Consider a logical ℓ-qubit system denoted

$$A_0 \cdots A_{\ell-1} = (\mathbb{C}^2)^{\otimes \ell}.$$

Let $j < k$ and $j, k \in \{0, \ldots, \ell - 1\}$ be arbitrary. Let $U := U_{A_jA_k} : (\mathbb{C}^2)^{\otimes \ell} \to (\mathbb{C}^2)^{\otimes \ell}$ be a unitary acting nontrivially only on the two qubits A_jA_k. Let

$$SBQQ_1Q_2 = L^2(\mathbb{R}) \otimes [L^2(\mathbb{R}) \otimes (\mathbb{C}^2)^{\otimes 3}]$$

be a system consisting of a 2^ℓ-dimensional code space S, one auxiliary oscillator B, and three auxiliary qubits Q, Q_1, Q_2. Then there is a unitary circuit W_U on $SBQQ_1Q_2$ consisting of at most $340\ell^2$ elementary operations in $\mathcal{U}_{\mathsf{elem}}^{2,3}$ such that

$$W_U(|\mathsf{GKP}(x)_{2^\ell}\rangle \otimes [|\mathsf{GKP}(0)_2\rangle \otimes |0\rangle^{\otimes 3}]) = \mathsf{gkpEnc}[2^\ell](J_\ell U J_\ell^{-1}|x\rangle) \otimes [|\mathsf{GKP}(0)_2\rangle \otimes |0\rangle^{\otimes 3}]$$

for all $x \in \{0, \ldots, 2^\ell - 1\}$. In other words, the circuit W_U implements the gate U exactly on the subspace

$$\mathcal{GKP}[2^\ell] \otimes \mathbb{C}(|\Psi\rangle) \subset L^2(\mathbb{R}) \otimes [L^2(\mathbb{R}) \otimes (\mathbb{C}^2)^{\otimes 3}],$$

where $|\Psi\rangle = |\mathsf{GKP}(0)_2\rangle \otimes |0\rangle^{\otimes 3} \in L^2(\mathbb{R}) \otimes (\mathbb{C}^2)^{\otimes 3}$, i.e., the auxiliary oscillator is in the ideal GKP state $|\mathsf{GKP}(0)_2\rangle$ and the three auxiliary qubits are in the state $|0\rangle^{\otimes 3}$.

We show an illustration of the implementation of a two-qubit unitary in Fig. 5. We give a generalization of Lemma 5 in Appendix E (see Lemma E2) where 2ℓ qubits are encoded into two ideal GKP codes $(\mathcal{GKP}[2^\ell])^{\otimes 2}$.

Proof. The circuit W_U is obtained from the circuit V_U introduced in Lemma 2 by replacing each bit-transfer map Transf_ℓ^j by its implementation $W_{\mathsf{Transf}_\ell^j}$ given in Lemma 4 (and similarly for the adjoint of the bit-transfer map). (We note the two-qubit gate U in the circuit V_U introduced in Lemma 2 acts as a two-qubit gate in the circuit W_U). Concretely, the circuit W_U is (leaving out identities on the remaining systems)

$$W_U = \left(W_{\mathsf{Transf}_\ell^j}\right)^\dagger_{SBQQ_1} \left(W_{\mathsf{Transf}_\ell^k}\right)^\dagger_{SBQQ_2} \times U_{Q_1Q_2} \left(W_{\mathsf{Transf}_\ell^k}\right)_{SBQQ_2} \left(W_{\mathsf{Transf}_\ell^j}\right)_{SBQQ_1}. \tag{27}$$

Each transfer gate uses the auxiliary oscillator B and the auxiliary qubit Q_3 as per Lemma 4. By reusing the auxiliary oscillator B for each circuit realizing different bit-transfers (and their adjoints), we obtain a circuit W_U on $L^2(\mathbb{R}) \otimes [L^2(\mathbb{R}) \otimes (\mathbb{C}^2)^{\otimes 3}]$ with the desired action.

The circuit W_U uses a single two-qubit gate, which is an elementary operation in our setup. It further uses four

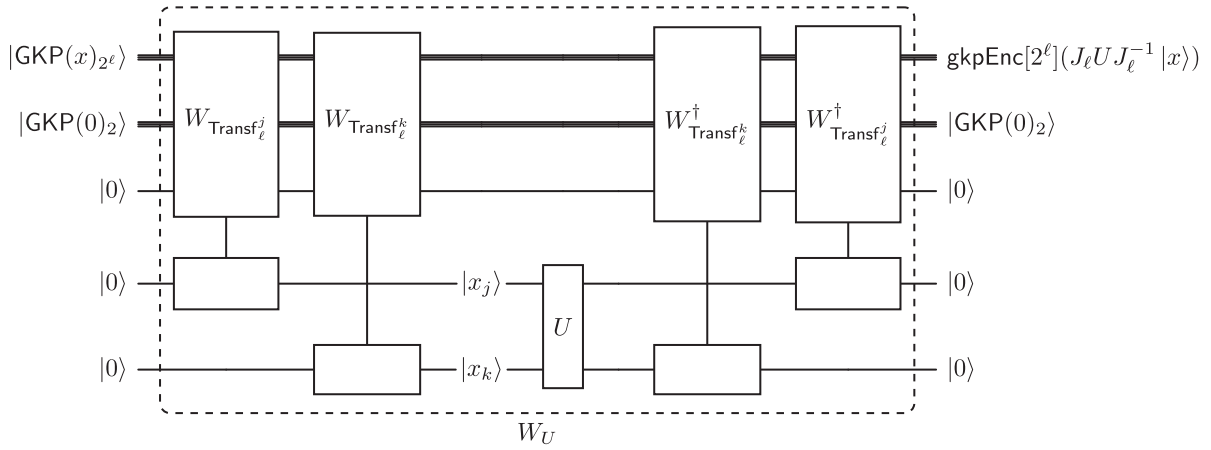


FIG. 5. Illustration of the implementation W_U in Lemma 5 [cf. Eq. (27)] of a logical two-qubit unitary $U = U_{A_jA_k}$ acting on the jth and kth qubits for $j, k \in \{0, \ldots, \ell - 1\}$.

implementations of bit-transfer unitaries and their adjoints, each of which can be implemented using fewer than $85\ell^2$ elementary operations by Lemma 4. Therefore, the total number of elementary operations necessary to implement W_U is less than $4 \times 85\ell^2 + 1 = 340\ell^2 + 1$, i.e., it is less than or equal to $340\ell^2$. ■

IV. LOGICAL GATE ERRORS OF IMPLEMENTATIONS IN APPROXIMATE GKP CODES

In this section we show that the circuit identities derived in Sec. III B for ideal GKP codes are approximately satisfied when working with finitely squeezed GKP states. More concretely, we quantify the error arising from the use of approximate GKP states $|\mathsf{GKP}^\varepsilon_{\kappa,\Delta}(j)_{2^\ell}\rangle$, $j \in \{0, \ldots, 2^\ell - 1\}$ defined in Eq. (4) by bounding the logical gate error of several implementations of logical unitaries in the approximate GKP codes $\mathcal{GKP}^\varepsilon_{\kappa,\Delta}[2^\ell] = \mathsf{span}\{\mathsf{GKP}^\varepsilon_{\kappa,\Delta}(j)^\ell_2\}_{j\in\{0,\ldots,2^\ell-1\}}$. Recall the definition of the logical gate error in Appendix A.

In Sec. IV A we derive bounds on the logical gate error of implementations of basic bit-manipulation maps as well as bit-transfer unitaries in symmetrically squeezed GKP codes. In Sec. IV B we use these results together with subadditivity to establish bounds on the logical gate error of implementations of two- and multiqubit unitaries, proving Theorem 1. In Sec. IV C we apply these bounds to implementations of Cliffords.

A. Basic bit-manipulation and bit-transfer maps

In the following, we upper bound the logical gate error for the implementations of the bit manipulation maps given in Table III. Recall that we denote by $\mathcal{U}^{n,m}_{\mathsf{elem}}$ the set of elementary unitary operations in (ii)–(v) acting on the system $L^2(\mathbb{R})^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes m}$.

Lemma 6 (Implementation of CX_ℓ, LSB_ℓ and Embed_ℓ). Let $\ell \in \mathbb{N}$. Let $\kappa, \Delta \in (0, 1/4)$ and $\varepsilon \in (0, 2^{-(\ell+1)}]$. Consider the space $\mathcal{GKP}^\varepsilon_{\kappa,\Delta}[2^\ell] \otimes \mathbb{C}^2 \simeq \mathbb{C}^{2^\ell} \otimes \mathbb{C}^2$. Then we have the following:

(i) The physical implementation of the unitary CX_ℓ on $\mathbb{C}^{2^\ell} \otimes \mathbb{C}^2$ given by

$$W_{\mathsf{CX}_\ell} = \mathsf{ctrl}e^{-i\sqrt{2\pi/2^\ell}P}$$

can be realized by a single elementary operation in $\mathcal{U}^{1,1}_{\mathsf{elem}}$. It satisfies

$$\max\left\{\mathsf{err}_{\mathcal{GKP}^\varepsilon_{\kappa,\Delta}[2^\ell]\otimes\mathbb{C}^2}\left(W_{\mathsf{CX}_\ell}, \mathsf{CX}_\ell\right), \mathsf{err}_{\mathcal{GKP}^\varepsilon_{\kappa,\Delta}[2^\ell]\otimes\mathbb{C}^2}\left(W^\dagger_{\mathsf{CX}_\ell}, \mathsf{CX}^\dagger_\ell\right)\right\} \leqslant 3\kappa.$$

(ii) The physical implementation of the unitary LSB_ℓ on $\mathbb{C}^{2^\ell} \otimes \mathbb{C}^2$ given by

$$W_{\mathsf{LSB}_\ell} = (I \otimes H)(M^\dagger_{\alpha'} \otimes I)^n \mathsf{ctrl}e^{iQ}(M_{\alpha'} \otimes I)^n(I \otimes H),$$

where

$$\alpha' = e^{\frac{\ln\alpha}{\lceil|\ln\alpha|\rceil}} \text{ and } n = \lceil|\ln\alpha|\rceil \text{ with } \alpha = \sqrt{\pi}2^{(\ell-1)/2}$$

defines a circuit $W_{\mathsf{LSB}_\ell} = W_T \cdots W_1$ composed of $T \leqslant \ell + 6 \times 2\,(6 \times 2)$ elementary gates in $\mathcal{U}^{1,1}_{\mathsf{elem}}$. It satisfies

$$\max\left\{\mathsf{err}_{\mathcal{GKP}^\varepsilon_{\kappa,\Delta}[2^\ell]\otimes\mathbb{C}^2}\left(W_{\mathsf{LSB}_\ell}, \mathsf{LSB}_\ell\right), \mathsf{err}_{\mathcal{GKP}^\varepsilon_{\kappa,\Delta}[2^\ell]\otimes\mathbb{C}^2}\left(W^\dagger_{\mathsf{LSB}_\ell}, \mathsf{LSB}^\dagger_\ell\right)\right\} \leqslant 6 \times 2^\ell\Delta + 12(\Delta/\varepsilon)^2.$$

Consider the spaces $\mathcal{GKP}^\varepsilon_{\kappa,\Delta}[2^\ell] \simeq \mathbb{C}^{2^\ell}$ and $\mathcal{GKP}^\varepsilon_{\kappa,\Delta}[2^{\ell+1}] \simeq \mathbb{C}^{2^{\ell+1}}$. Then we have the following:

(iii) The isometry $\mathsf{Embed}_\ell : \mathbb{C}^{2^\ell} \to \mathbb{C}^{2^{\ell+1}}$ can be implemented by the unitary $W_{\mathsf{Embed}_\ell} = M_{\sqrt{2}}$. It uses a single elementary operation in $\mathcal{U}^{1,0}_{\mathsf{elem}}$ and satisfies

$$\max\left\{\mathsf{err}_{\mathcal{GKP}^\varepsilon_{\kappa,\Delta}[2^\ell],\mathcal{GKP}^\varepsilon_{\kappa,\Delta}[2^{\ell+1}]}\left(W_{\mathsf{Embed}_\ell}, \mathsf{Embed}_\ell\right), \mathsf{err}_{\mathcal{GKP}^\varepsilon_{\kappa,\Delta}[2^{\ell+1}],\mathcal{GKP}^\varepsilon_{\kappa,\Delta}[2^\ell]}\left(W^\dagger_{\mathsf{Embed}_\ell}, \mathsf{Embed}^\dagger_\ell\right)\right\} = 0.$$

We prove Lemma 6 in Appendix C.

An immediate consequence of the bounds on the logical gate error of the basic bit-manipulation maps in Lemma 6

is the following result on the physical implementation of bit-transfer unitaries Transf_ℓ^j.

Lemma 7 (Implementation of bit-transfer unitaries). Let $\ell \in \mathbb{N}$ be an integer and $j \in \{0, \ldots, \ell-1\}$. Let $\kappa \in (0, 1/4)$. Then there is a unitary circuit $W_{\mathsf{Transf}_\ell^j}$ on $L^2(\mathbb{R}) \otimes \mathbb{C}^2 \otimes L^2(\mathbb{R}) \otimes \mathbb{C}^2$ which consists of fewer than $85\ell^2$ operations in $\mathcal{U}^{2,2}_{\mathsf{elem}}$. It satisfies

$$\max\left\{\mathsf{err}_{\mathcal{L}}\left(W_{\mathsf{Transf}_\ell^j}, \mathsf{Transf}_\ell^j\right), \mathsf{err}_{\mathcal{L}}\left(\left(W_{\mathsf{Transf}_\ell^j}\right)^\dagger, \left(\mathsf{Transf}_\ell^j\right)^\dagger\right)\right\} \leqslant 36\ell\kappa,$$

where the code space is given by $\mathcal{L} = \mathcal{GKP}^\star_\kappa[2^\ell] \otimes \mathbb{C}^2 \otimes \mathbb{C}(|\mathsf{GKP}_\kappa^{2^{-(\ell+1)}}(0)_2\rangle \otimes |0\rangle)$.

Proof. Let $W_{\mathsf{Transf}_\ell^j}$ be the circuit implementation of Transf_ℓ^j introduced in Lemma 5. By the proof of Lemma 4 (cf. Lemma 1), the implementation $W_{\mathsf{Transf}_\ell^j}$ is constructed by replacing at most

$$12\ell - 4 \leqslant 12\ell \tag{28}$$

basic bit-manipulation maps from the set $\mathcal{G}(\ell) = \cup_{k=1}^{\ell}(\mathcal{G}_k \cup \mathcal{G}_k^\dagger)$ [see Eq. (15)] by suitable implementations. By Lemma 6, there is a physical implementation W_V for every basic bit-manipulation map $V \in \mathcal{G}_k \cup \mathcal{G}_k^\dagger$ such that

$$\mathsf{err}_{\mathcal{L}_{\mathrm{in}},\mathcal{L}_{\mathrm{out}}}(W_V, V) \leqslant \max\{3\kappa, 6 \times 2^k\Delta + 12(\Delta/\varepsilon)^2\}, \tag{29}$$

where if $V = \mathsf{Embed}_k$ the logical spaces are $\mathcal{L}_{\mathrm{in}} = \mathcal{GKP}^\varepsilon_{\kappa,\Delta}[2^k]$ and $\mathcal{L}_{\mathrm{out}} = \mathcal{GKP}^\varepsilon_{\kappa,\Delta}[2^{k+1}]$; otherwise, we have $\mathcal{L}_{\mathrm{in}} = \mathcal{L}_{\mathrm{out}} = \mathcal{GKP}^\varepsilon_{\kappa,\Delta}[2^k] \otimes \mathbb{C}^2$.

We have $k \in \{0, \ldots, \ell-1\}$ and, by definition of $\mathcal{GKP}^\star_\kappa[2^k]$, $\Delta = \kappa/(2\pi\, 2^\ell)$ and $\varepsilon = 2^{-(\ell+1)}$ (see Sec. I). Therefore, we may upper bound Eq. (29) as

$$\begin{aligned}
&\mathsf{err}_{\mathcal{L}_{\mathrm{in}},\mathcal{L}_{\mathrm{out}}}(W_V, V) \\
&\leqslant \max_{k\in\{1,\ldots,\ell\}} \max\{3\kappa, 6 \times 2^k\kappa/(2\pi\, 2^\ell) + 12(\kappa/\pi)^2\} \\
&= \max\{3\kappa, 6 \times 2^\ell\kappa/(2\pi\, 2^\ell) + 12(\kappa/\pi)^2\} \\
&\leqslant 3\kappa,
\end{aligned} \tag{30}$$

for $\kappa \in (0, 1/4)$.

Combining Eqs. (28) and (30) with the subadditivity of the logical gate error under composition, see Lemma A1, gives

$$\begin{aligned}
&\max\left\{\mathsf{err}_{\mathcal{L}}\left(W_{\mathsf{Transf}_\ell^j}, \mathsf{Transf}_\ell^j\right)\right\} \\
&\leqslant 12\ell \max_{V\in\mathcal{G}(\ell)} \mathsf{err}_{\mathcal{L}_{\mathrm{in}},\mathcal{L}_{\mathrm{out}}}(W_V, V) \\
&\leqslant 12\ell \times 3\kappa = 36\ell\kappa.
\end{aligned}$$

The bound for the adjoint follows by the same arguments, giving the claim. ∎

B. Multiqubit gates

Here we prove the main result stated as Theorem 1. We start by analyzing implementations of two-qubit unitaries in symmetrically squeezed GKP codes. We then generalize the result to implementations of (multiqubit) circuits composed of T two-qubit gates.

Theorem 8 (Implementation of a two-qubit gate in symmetrically squeezed GKP codes). Let $\ell \geqslant 2$ be an integer. Let $\kappa \in (0, 1/4)$. Consider a logical ℓ-qubit system denoted

$$A_0 \cdots A_{\ell-1} = (\mathbb{C}^2)^{\otimes\ell}.$$

Let $j < k$ and $j, k \in \{0, \ldots, \ell-1\}$ be arbitrary. Let $U := U_{A_jA_k} : (\mathbb{C}^2)^{\otimes\ell} \to (\mathbb{C}^2)^{\otimes\ell}$ be a unitary acting nontrivially only on the two qubits A_jA_k. Then, there exists a physical unitary implementation W_U acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \otimes (\mathbb{C}^2)^{\otimes 3}$ of the logical unitary U using at most $340\ell^2$ elementary operations in $\mathcal{U}^{2,3}_{\mathsf{elem}}$. It satisfies

$$\mathsf{err}_{\mathcal{L}}\left(W_U, J_\ell U J_\ell^{-1}\right) \leqslant 150\ell\kappa,$$

with code space $\mathcal{L} = \mathcal{GKP}^\star_\kappa[2^\ell] \otimes \mathbb{C}(|\mathsf{GKP}_\kappa^{2^{-(\ell+1)}}(0)_2\rangle \otimes |0\rangle^{\otimes 3})$.

We give a generalization of Theorem 8 in Appendix E (see Theorem E3) where 2ℓ qubits are encoded into two symmetrically squeezed GKP codes $(\mathcal{GKP}^\star_\kappa[2^\ell])^{\otimes 2}$.

Proof. Let W_U be the unitary implementation of U introduced in Lemma 5. It can be realized using at most $340\ell^2$ elementary operations. Moreover, it consists of four physical implementations of a bit-transfer operations of the form $W_{\mathsf{Transf}_\ell^j}$, $(W_{\mathsf{Transf}_\ell^j})^\dagger$ (together with a single two-qubit unitary acting on two-physical qubits which does not contribute to the logical gate error). Recall Lemma 7, which states

$$\max\left\{\mathsf{err}_{\mathcal{L}}\left(W_{\mathsf{Transf}_\ell^j}, \mathsf{Transf}_\ell^j\right), \mathsf{err}_{\mathcal{L}}\left(\left(W_{\mathsf{Transf}_\ell^j}\right)^\dagger, \left(\mathsf{Transf}_\ell^j\right)^\dagger\right)\right\} \leqslant 36\ell\kappa.$$

This together together with the subadditivity of the logical gate error under composition, see Lemma A1, gives the claim

$$\mathsf{err}_{\mathcal{L}}\left(W_U, J_\ell U J_\ell^{-1}\right) \leqslant 4 \times 36\ell\kappa \leqslant 150\ell\kappa.$$

∎

With this preparation we can give the proof of the main result of this work, see Theorem 1.

Proof of Theorem 1. Theorem 8 bounds the logical gate error of the implementation of a single two-qubit gate. The claim follows using the subadditivity of the logical gate error under composition (see Lemma A1), that is, using the triangle inequality of the diamond norm T times. ∎

C. Qudit Clifford gates

In this section we demonstrate how Theorem 1 (and its extension to the case of 2ℓ qubits encoded in two modes, see Theorem E3) can be leveraged to find implementations of Cliffords.

For completeness, we first introduce qudit Paulis and Cliffords. Let $d \geqslant 2$ be an integer. We define the Pauli operators X, Z for a qudit as

$$\begin{aligned}
X|x\rangle &= |x \oplus 1\rangle \\
Z|x\rangle &= \omega_d^x|x\rangle \quad \text{for all} \quad x \in \mathbb{Z}_d,
\end{aligned}$$

where $\oplus$ denotes addition modulo d and $\omega_d = e^{i2\pi/d}$ is a dth root of unity. The Pauli operators X, Z satisfy the commutation relation

$$ZX = \omega_d XZ$$

TABLE V. Action of single- and two-qudit operations on an orthonormal basis $\{|x\rangle\}_{x\in\mathbb{Z}_d}$ of $\mathbb{C}^d$. Here $c_d = d \mod 2$.

diagram	gate	type of unitary
$\|x\rangle$ — X — $\|x\oplus 1\rangle$	modular shift X	Pauli
$\|x\rangle$ — Z — $\omega_b^x \|x\rangle$	modular phase shift Z	Pauli
$\|x\rangle$ — P — $\omega_d^{x(x+c_d)/2} \|x\rangle$	phase gate P	Clifford
$\|x\rangle$ — F — $\frac{1}{\sqrt{b}}\sum_{y\in\mathbb{Z}_d}\omega_d^{xy}\|y\rangle$	Fourier transform F	Clifford
$\|x\rangle$, $\|y\rangle$ — Z — $\omega_d^{xy}\|x\rangle\|y\rangle$	controlled modular phase CZ	Clifford
$\|x\rangle$ — $Z(e^{i\theta})$ — $e^{i\theta x}\|x\rangle$	phase shift $Z(e^{i\theta})$	Clifford for $\theta\in(\pi/d)\mathbb{Z}$

and generate the single-qudit Pauli group. The n-qudit Pauli group acting on the tensor product space $(\mathbb{C}^d)^{\otimes n}$ is the group $\langle X_j, Z_j \mid j = 1,\dots,n\rangle$, i.e., it is the group generated by single-qudit Pauli-operators on each qudit $j \in \{1,\dots,n\}$. The n-qudit Clifford group is the normalizer of the n-qudit Pauli group in the unitary group. It is given by

$$\langle\{\mathsf{P}_j, \mathsf{F}_j \mid j = 1,\dots,n\} \cup \{\mathsf{CZ}_{j,k} \mid j \neq k,\, j,k \in \{1,\dots,n\}\}\rangle,$$

where P_j is the phase gate in Eq. (7) acting on the jth qubit, F_j is the Fourier transform in Eq. (11) acting on the jth qubit and $\mathsf{CZ}_{j,k}$ is the two-qudit controlled phase gate in Eq. (12) with the jth and the kth qubits the control and target, respectively.

To generate a universal gate set, we need additional gates. We use the phase gate $Z(e^{i\theta})$ defined for $\theta \in \mathbb{R}$ by its action

$$Z(e^{i\theta})|x\rangle = e^{i\theta x}|x\rangle \quad \text{for all} \quad x \in \mathbb{Z}_d$$

on computational basis states. We note that $Z(e^{i\theta})$ is a power of the Pauli-Z operator whenever θ is an integer multiple of $2\pi/d$.

The generators of the qudit Clifford group, as well as the phase gate $Z(e^{i\theta})$ are illustrated in Table V. In the special case $d = 2^\ell$, we give implementations of a complete set of generators of the Clifford group in terms of qubit-oscillator operations. More precisely, we have the following.

Corollary 1 (Implementations of Cliffords in symmetrically squeezed GKP codes). Let $\ell \in \mathbb{N}$ and $\kappa \in (0, 1/4)$. Then there is a universal constant $C > 0$ (independent of ℓ) such that following holds:

(i) Let $U \in \{\mathsf{F}, \mathsf{P}, Z(e^{i\theta})\}$ be a unitary acting on $\mathbb{C}^{2^\ell}$ as in Table V. Then there exists a physical unitary implementation W_U acting on the Hilbert space $L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \otimes (\mathbb{C}^2)^{\otimes 3}$ which uses $O(\ell^4)$ elementary operations in $\mathcal{U}^{2,3}_{\mathsf{elem}}$. It satisfies

$$\mathsf{err}_{\mathcal{L}}\left(W_U, J_\ell U J_\ell^{-1}\right) \leqslant C\ell^3\kappa,$$

with code space $\mathcal{L} = \mathcal{GKP}^\star_\kappa[2^\ell] \otimes \mathbb{C}(|\mathsf{GKP}_\kappa^{2^{-(\ell+1)}}(0)_2\rangle \otimes |0\rangle^{\otimes 3})$.

(ii) Consider the unitary CZ acting on $\mathbb{C}^{2^\ell} \otimes \mathbb{C}^{2^\ell}$ as in Table V. Then there exists a physical unitary implementation W_{CZ} acting on the Hilbert space $L^2(\mathbb{R})^{\otimes 2} \otimes L^2(\mathbb{R}) \otimes (\mathbb{C}^2)^{\otimes 3}$ which uses $O(\ell^4)$ elementary operations in $\mathcal{U}^{3,3}_{\mathsf{elem}}$. It satisfies

$$\mathsf{err}_{\mathcal{L}}\left(W_{\mathsf{CZ}}, J_\ell^{\otimes 2}\mathsf{CZ}\left(J_\ell^{\otimes 2}\right)^{-1}\right) \leqslant C\ell^3\kappa,$$

with code space $\mathcal{L} = (\mathcal{GKP}^\star_\kappa[2^\ell])^{\otimes 2} \otimes \mathbb{C}(|\mathsf{GKP}_\kappa^{2^{-(\ell+1)}}(0)_2\rangle \otimes |0\rangle^{\otimes 3})$.

Proof. We show that for $U \in \{\mathsf{F}, \mathsf{P}, Z(e^{i\theta})\}$ a unitary on $\mathbb{C}^{2^\ell}$, the ℓ-qubit unitary $\widetilde{U} = J_\ell^{-1} U J_\ell$ on $(\mathbb{C}^2)^{\otimes\ell}$ has the decomposition

$$\widetilde{U} = \prod_{t=1}^{T} U^{(t)}_{A_jA_k},$$

where $T = O(\ell^2)$, A_j is a two-dimensional system for all $j \in \{0,\dots,\ell-1\}$ and $U^{(t)}_{A_jA_k}$ is a (two-qubit unitary) acting only on the system A_jA_k for all $t \in \{1,\dots,T\}$.

For $U = \mathsf{F}$ the decomposition is the standard realization of the quantum Fourier-transform for $d = 2^\ell$ on ℓ qubits using only (single-qubit) Hadamard gates, SWAP-gates and qubit-controlled rotations of the form

$$|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes R(\omega_{2^k}) \quad \text{for} \quad k \in \{0,\dots,\ell-1\}.$$

where we define the single-qubit rotation with angle $\alpha \in \mathbb{R}$ as

$$R(e^{i\alpha})|b\rangle = e^{i\alpha b}|b\rangle \quad \text{for} \quad b \in \{0,1\}. \tag{31}$$

We refer to Ref. [24] for a detailed discussion. For completeness, we give explicit decompositions for the remaining unitaries.

Let $U = \mathsf{P}$. As $d = 2^\ell$ is even, we have $c_d = 0$. Then we have $\mathsf{P}|x\rangle = \omega_{2^\ell}^{x^2}|x\rangle$, which can be rewritten as

$$\mathsf{P}|x\rangle = \left(\prod_{j,k=0}^{\ell-1} \omega_{2^\ell}^{2^{j+k}x_jx_k}\right)|x\rangle \quad \text{for} \quad x = [x_{\ell-1},\dots,x_0].$$

This implies that

$$\widetilde{U} = \prod_{j,k=0}^{\ell-1} U\left(\omega_{2^\ell}^{2^{j+k}}\right)_{A_jA_k},$$

where we introduced the (two-qubit) unitary $U(e^{i\theta})_{A_jA_k}$ acting as

$$\begin{aligned} &U(e^{i\theta})_{A_jA_k}: \\ &\quad (\mathbb{C}^2)^{\otimes\ell} \quad \to \quad (\mathbb{C}^2)^{\otimes\ell} \\ &\quad |x_{\ell-1}\rangle \otimes \cdots \otimes |x_0\rangle \quad \mapsto \quad e^{i\theta x_j x_k}|x_{\ell-1}\rangle \otimes \cdots \otimes |x_0\rangle. \end{aligned}$$

Note that if $j = k$ we can regard $U(e^{i\theta})_{A_jA_j}$ as a two-qubit unitary.

Finally, let $U = Z(e^{i\theta})$. Then we have

$$Z(e^{i\theta})|x\rangle = \left(\prod_{j=0}^{\ell-1} e^{i\theta x_j 2^j}\right)|x\rangle \text{ for } x = [x_{\ell-1}, \ldots, x_0].$$

This implies that

$$\widetilde{U} = \prod_{j=0}^{\ell-1} R(e^{i\theta 2^j})_{A_j},$$

where the subscript A_j indicates that the (single-qubit) rotation $R(e^{i\theta 2^j})$ acts on the jth qubit, see Eq. (31).

Similarly, the 2ℓ-qubit unitary $\widetilde{U} = (J_\ell^{\otimes 2})^{-1}\mathsf{CZ}J_\ell^{\otimes 2}$ on $(\mathbb{C}^2)^{\otimes\ell} \otimes (\mathbb{C}^2)^{\otimes\ell}$ has a decomposition

$$\widetilde{U} = \prod_{t=1}^{T} U^{(t)}_{A_jB_k},$$

where $T = O(\ell^2)$, A_j and B_k are (distinct) two-dimensional systems for all $j, k \in \{0, \ldots, \ell-1\}$ and $U^{(t)}_{A_jB_k}$ is a two-qubit unitary acting only the system A_jB_k for $t \in \{1, \ldots, T\}$. A concrete decomposition can be obtained as follows: We have $\mathsf{CZ}(|x\rangle \otimes |y\rangle) = \omega_{2^\ell}^{xy}(|x\rangle \otimes |y\rangle)$ or

$$\begin{aligned} &\mathsf{CZ}(|x\rangle \otimes |y\rangle) \\ &\quad = \left(\prod_{j,k=0}^{\ell-1} \omega_{2^\ell}^{ix_jy_k2^{j+k}}\right)(|x\rangle \otimes |y\rangle) \text{ for all } x, y \in \mathbb{Z}_{2^\ell}. \end{aligned}$$

This implies that

$$\widetilde{U} = \prod_{j,k=0}^{\ell-1} U\left(\omega_{2^\ell}^{2^{j+k}}\right)_{A_jB_k},$$

where we introduced the (two-qubit) unitary $U(e^{i\theta})_{A_jB_k}: (\mathbb{C}^2)^{\otimes\ell} \otimes (\mathbb{C}^2)^{\otimes\ell} \to (\mathbb{C}^2)^{\otimes\ell} \otimes (\mathbb{C}^2)^{\otimes\ell}$ acting as

$$\begin{aligned} &|x_{\ell-1}\rangle \otimes \cdots \otimes |x_0\rangle \otimes |y_{\ell-1}\rangle \otimes \cdots \otimes |y_0\rangle \\ &\quad \mapsto e^{ix_jy_k\theta}(|x_{\ell-1}\rangle \otimes \cdots \otimes |x_0\rangle \otimes |y_{\ell-1}\rangle \otimes \cdots \otimes |y_0\rangle). \end{aligned}$$

Claim (i) follows from Theorem 1 with $T = O(\ell^2)$. Claim (ii) follows from a straightforward generalization of Theorem 1 for 2ℓ qubits encoded into two approximate codes $(\mathcal{GKP}^\star_\kappa[2^\ell])^{\otimes 2}$ which is obtained by combining Lemma E2, Theorem E3, and Lemma A1. ■

By combining Corollary 1 with Ref. [1, Theorem 6.1], we find that there exist physical implementations of a complete generating set of the qudit Clifford group on $\mathbb{C}^d \otimes \mathbb{C}^d$ (for $d = 2^\ell$) on a code space defined by symmetrically squeezed GKP codes tensored with a fixed auxiliary state, i.e.,

$$(\mathcal{GKP}^\star_\kappa[2^\ell])^{\otimes 2} \otimes \left(\left|\mathsf{GKP}_\kappa^{2^{-(\ell+1)}}(0)_2\right\rangle \otimes |0\rangle^{\otimes 3}\right).$$

That is, the auxiliary system consists of one additional oscillator and three physical qubits. Each of these gate implementations requires $O(\ell^4)$ elementary operations and achieves a logical gate error of order $O(\ell^3\kappa)$.

V. DISCUSSION

In this work, we presented novel constructions for implementing a universal set of logical unitaries on arbitrarily many encoded qubits using oscillators and only three qubits. Importantly, our construction is modular and can be used beyond the case of a constant number of physical information carriers (i.e., oscillators): multiple logical qubits can be encoded across several modes while keeping the number of auxiliary physical qubits fixed, thereby making the architecture inherently scalable. Furthermore, we proved that in the limit of large squeezing these implementations become exact. This remedies a limitation of existing unitary Gaussian-only schemes [1,20] without relying on adaptive circuits.

We emphasize that our analysis focuses on a particular finite-dimensional subspace of the CV Hilbert space associated with an approximate GKP code. In this setting, leakage out of the subspace provides a natural mechanism for a nonvanishing lower bound on the logical error in certain implementations, as was analyzed in Ref. [1]. By contrast, the unitary approach based on the hybrid qubit-oscillator model presented in this work prevents leakage for sufficiently squeezed approximate GKP codes by exploiting qubit-controlled displacement operations as an elementary resource. It will be important to compare this unitary approach with more traditional GKP error-correction schemes for the suppression and correction of leakage. A rigorous analysis of the performance of error-correction gadgets in handling leakage outside the approximate GKP code space is an open direction for future work.

Our construction is based on a non-Gaussian gadget which coherently transfers bit-based information stored in a data mode—more precisely, a GKP encoded qudit—to a constant number of physical qubits. The latter act as the data-processing unit. While our analysis does not include fault-tolerance considerations, our construction can be combined with existing fault-tolerance schemes for GKP codes on the oscillator level and standard qubit-based fault-tolerant schemes on the qubit level. Since our scheme only requires a constant number of auxiliary qubits, the overhead for implementing logical gates should remain moderate. However, it remains a challenging open question how qubit-controlled displacements can be implemented in a fault-tolerant manner. We hope this work stimulates further research in this direction.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: THE LOGICAL GATE ERROR AND ITS PROPERTIES

In this Appendix we briefly recall the definition and several properties of the logical gate error defined in [1]. Let $d \geqslant 2$ be an integer and let $U : \mathbb{C}^d \to \mathbb{C}^d$ be a unitary. Correspondingly, we define the CPTP map

$$\mathcal{U} : \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{C}^d)$$
$$\rho \mapsto \mathcal{U}(\rho) = U\rho U^\dagger.$$

We are interested in implementing the logical CPTP map $\mathcal{U}$ on a larger "physical" space by a suitable implementation when we consider $\mathbb{C}^d$ as the codespace of some code.

More precisely, let $\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}}$ be Hilbert spaces, and let $\mathcal{L}_{\text{in}} \subseteq \mathcal{H}_{\text{in}}$, $\mathcal{L}_{\text{out}} \subseteq \mathcal{H}_{\text{out}}$ be (logical) subspaces isomorphic to $\mathbb{C}^d$. We define an (isometric) encoding map $\mathsf{Enc}_{\mathcal{L}_{\text{in}}} : \mathbb{C}^d \to \mathcal{L}_{\text{in}}$, and a corresponding decoding map $\mathsf{Dec}_{\mathcal{L}_{\text{out}}} : \mathcal{L}_{\text{out}} \to \mathbb{C}^d$. We define the map

$$\mathbf{U} = \mathsf{Dec}_{\mathcal{L}_{\text{out}}} U \mathsf{Enc}_{\mathcal{L}_{\text{in}}} : \mathcal{L}_{\text{in}} \to \mathcal{L}_{\text{out}}.$$

The map $\mathbf{U}$ induces the CPTP map

$$\boldsymbol{\mathcal{U}} : \mathcal{B}(\mathcal{L}_{\text{in}}) \to \mathcal{B}(\mathcal{L}_{\text{out}})$$
$$\rho \mapsto \mathbf{U}\rho\mathbf{U}^\dagger. \tag{A1}$$

In the following we denote by $\pi_{\mathcal{L}}$ the orthogonal projection onto the subspace $\mathcal{L}$ of a Hilbert space $\mathcal{H}$, and we define the corresponding map $\Pi_{\mathcal{L}}$ as $\Pi_{\mathcal{L}}(\rho) = \pi_{\mathcal{L}}\rho\pi_{\mathcal{L}}$.

The logical gate error of an implementation $\mathcal{W} : \mathcal{B}(\mathcal{H}_{\text{in}}) \to \mathcal{B}(\mathcal{H}_{\text{out}})$ of a logical unitary $U : \mathbb{C}^d \to \mathbb{C}^d$ on input and output code spaces $\mathcal{L}_{\text{in}}$ and $\mathcal{L}_{\text{out}}$, respectively, is defined as

$$\mathsf{err}_{\mathcal{L}_{\text{in}},\mathcal{L}_{\text{out}}}(\mathcal{W}, U) = \|(\mathcal{W} - \boldsymbol{\mathcal{U}}) \circ \Pi_{\mathcal{L}}\|_\diamond.$$

We frequently consider the case where $\mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}} = \mathcal{H}$, $\mathcal{L}_{\text{in}} = \mathcal{L}_{\text{out}}$ and an implementation of the form $\mathcal{W}(\rho) = W\rho W^\dagger$ where W is a unitary on $\mathcal{H}$. In this case, we write

$$\mathsf{err}_{\mathcal{L}}(W, U) = \mathsf{err}_{\mathcal{L}}(\mathcal{W}, U)$$

by a slight abuse of notation.

In the following we collect several results about the logical gate error.

Lemma A1 (Subadditivity of the logical gate error [1]). Let $\mathcal{H}_{\text{in}}^{(1)}$, $\mathcal{H}_{\text{out}}^{(1)}$, $\mathcal{H}_{\text{out}}^{(2)}$, $\mathcal{K}$ be Hilbert spaces. Set $\mathcal{H}_{\text{in}}^{(2)} := \mathcal{H}_{\text{out}}^{(1)}$. Let $U_1, U_2 : \mathcal{K} \to \mathcal{K}$ be unitaries and let $\mathcal{W}_1 : \mathcal{B}(\mathcal{H}_{\text{in}}^{(1)}) \to \mathcal{B}(\mathcal{H}_{\text{out}}^{(1)})$ and $\mathcal{W}_2 : \mathcal{B}(\mathcal{H}_{\text{in}}^{(2)}) \to \mathcal{B}(\mathcal{H}_{\text{out}}^{(2)})$ be CPTP maps. Let $\mathcal{L}_{\text{in}}^{(1)} \subseteq \mathcal{H}_{\text{in}}^{(1)}$, $\mathcal{L}_{\text{out}}^{(1)} \subseteq \mathcal{H}_{\text{out}}^{(1)}$, $\mathcal{L}_{\text{in}}^{(2)} \subseteq \mathcal{H}_{\text{in}}^{(2)}$, and $\mathcal{L}_{\text{out}}^{(2)} \subseteq \mathcal{H}_{\text{out}}^{(2)}$ be code spaces with $\mathcal{L}_{\text{in}}^{(2)} := \mathcal{L}_{\text{out}}^{(1)}$, and $\mathcal{L}_{\text{in}}{}^{(1)} \simeq \mathcal{L}_{\text{out}}{}^{(1)} \simeq \mathcal{L}_{\text{out}}{}^{(2)} \simeq \mathcal{K}$. We have

$$\mathsf{err}_{\mathcal{L}_{\text{in}}^{(1)},\mathcal{L}_{\text{out}}^{(2)}}(\mathcal{W}_2 \circ \mathcal{W}_1, U_2U_1)$$
$$\leqslant \mathsf{err}_{\mathcal{L}_{\text{in}}^{(1)},\mathcal{L}_{\text{out}}^{(1)}}(\mathcal{W}_1, U_1) + \mathsf{err}_{\mathcal{L}_{\text{in}}^{(2)},\mathcal{L}_{\text{out}}^{(2)}}(\mathcal{W}_2, U_2).$$

The logical gate error is invariant under tensoring additional systems. More precisely, we have the following:

Lemma A2 (see Ref. [1]). Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. Let $\mathcal{L}_j \subset \mathcal{H}_j$ be a subspace for $j = 1, 2$ and let $\mathcal{W} : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_1)$ be a CPTP map. Then

$$\mathsf{err}_{\mathcal{L}_1\otimes\mathcal{L}_2}\big(\mathcal{W} \otimes \mathsf{id}_{\mathcal{B}(\mathcal{H}_2)}, U \otimes I_{\mathbb{C}^d}\big) = \mathsf{err}_{\mathcal{L}_1}(\mathcal{W}, U).$$

We bound the logical gate error of a (unitary) implementation in terms of the matrix elements with respect to a choice of basis of the logical subspaces. More precisely, let $\{|j\rangle\}_{j=0}^{d-1}$ be an orthonormal basis of $\mathbb{C}^d$, $\{|\mathbf{j}\rangle_{\mathcal{L}_{\text{in}}}\}_{j=0}^{d-1}$ an orthonormal basis of $\mathcal{L}_{\text{in}}$, and $\{|\mathbf{j}\rangle_{\mathcal{L}_{\text{out}}}\}_{j=0}^{d-1}$ an orthonormal basis of $\mathcal{L}_{\text{out}}$. Then we can fix a concrete encoding of $\mathbb{C}^d$ into $\mathcal{L}_{\text{in}} \subseteq \mathcal{H}_{\text{in}}$ by mapping basis states as

$$\mathsf{Enc}_{\mathcal{L}_{\text{in}}} : \begin{array}{ccc} \mathbb{C}^d & \to & \mathcal{H}_{\text{in}} \\ |j\rangle & \mapsto & |\mathbf{j}\rangle_{\mathcal{L}_{\text{in}}} \end{array} \quad \text{for } j \in \{0, \ldots, d-1\},$$

and linearly extending to all of $\mathbb{C}^d$. We denote the inverse ("decoding") map by $\mathsf{Dec}_{\mathcal{L}_{\text{in}}} = (\mathsf{Enc}_{\mathcal{L}_{\text{in}}})^{-1}$. For the subspace $\mathcal{L}_{\text{out}} \subseteq \mathcal{H}_{\text{out}}$, we define a pair of maps $(\mathsf{Enc}_{\mathcal{L}_{\text{out}}}, \mathsf{Dec}_{\mathcal{L}_{\text{out}}})$ in an analogous manner. The following definition will be useful:

Definition A1 (Definition 3.6 in Ref. [1]). We define the operator

$$B = B^U_{\mathcal{L}_{\text{in}},\mathcal{L}_{\text{out}}}(W, U) = \mathbf{U}^\dagger \pi_{\mathcal{L}_{\text{out}}} W \pi_{\mathcal{L}_{\text{in}}} : \mathcal{L}_{\text{in}} \to \mathcal{L}_{\text{in}}.$$

It is characterized by the matrix elements with respect to the basis $\{|\mathbf{k}\rangle_{\mathcal{L}_{\text{in}}}\}_{k=0}^{d-1}$ given by

$$B_{j,k} = \langle \mathbf{j}|_{\mathcal{L}_{\text{in}}} \mathbf{U}^\dagger \pi_{\mathcal{L}_{\text{out}}} W \pi_{\mathcal{L}_{\text{in}}} |\mathbf{k}\rangle_{\mathcal{L}_{\text{in}}}$$

for $j, k \in \{0, \ldots, d-1\}$.

It is straightforward to check that

$$B_{j,k} := \sum_{m=0}^{d-1} \overline{U_{m,j}} \langle \mathbf{m}|_{\mathcal{L}_{\text{out}}} W |\mathbf{k}\rangle_{\mathcal{L}_{\text{in}}} \quad \text{for} \quad j, k \in \{0, \ldots, d-1\}.$$

We call a matrix $B \in \mathsf{Mat}_{d\times d}(\mathbb{C})$ s-sparse if the number of nonzero entries in each row and column is at most s. We need the following bound on the logical gate error:

Corollary A1 (Gate error of unitary implementations in terms of matrix elements [1]). Let $W : \mathcal{H}_{\text{in}} \to \mathcal{H}_{\text{out}}$ be unitary and $\mathcal{W}(\rho) = W\rho W^\dagger$. Let $U : \mathbb{C}^d \to \mathbb{C}^d$ be unitary, and let $B = B^U_{\mathcal{L}_{\text{in}},\mathcal{L}_{\text{out}}}(W, U)$ be the operator introduced in Definition A1. Suppose that B is s-sparse and has positive entries $\{B_{j,j}\}_{j=0}^{d-1}$ on the diagonal. Then

$$\max\left\{\mathsf{err}_{\mathcal{L}_{\text{in}},\mathcal{L}_{\text{out}}}(W, U), \mathsf{err}_{\mathcal{L}_{\text{out}},\mathcal{L}_{\text{in}}}(W^\dagger, U^\dagger)\right\}$$
$$\leqslant 3\Big(\big(1 - \min_j B_{j,j}\big) + (s-1) \max_{j,\ell:j\neq\ell} |B_{j,\ell}|\Big)^{1/2}.$$

APPENDIX B: MATRIX ELEMENTS OF IMPLEMENTATIONS OF BASIC BIT-MANIPULATION MAPS

In this Appendix we bound the matrix elements of physical implementations introduced in Sec. III A of the CX_ℓ-gate, the LSB_ℓ-gate and the embedding isometry.

Let $d \geqslant 2$ be an integer, $\kappa, \Delta > 0$ and $\varepsilon \leqslant 1/2$. We introduce the integer-centered (normalized) GKP states $|\mathsf{GKP}^\varepsilon_{\kappa,\Delta}\rangle$ in position space as

$$\mathsf{GKP}^\varepsilon_{\kappa,\Delta}(x) \propto \sum_{s\in\mathbb{Z}} e^{-\kappa^2 s^2/2}\, e^{-x^2/(2\Delta^2)}$$

if $\mathsf{dist}(x,\mathbb{Z}) \leqslant \varepsilon$ and $\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(x) = 0$ otherwise. We note that we can relate the GKP code states $|\mathsf{GKP}_{\kappa,\Delta}(j)_{2^\ell}\rangle$ [cf. Eq. (3)] to the state $|\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}\rangle$ as

$$|\mathsf{GKP}_{\kappa,\Delta}(j)_{2^\ell}\rangle = e^{-i\sqrt{2\pi/2^\ell}jP} M_{\sqrt{2^{\ell+1}\pi}} \big|\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}\big\rangle \quad \text{for } j \in \{0,\dots,2^\ell-1\}, \tag{B1}$$

where we use the shorthand notation $M_\alpha = e^{-i\ln\alpha(PQ+QP)/2}$ for $\alpha > 0$. We give matrix elements of implementations with respect to the basis consisting of the states

$$|\mathsf{GKP}_{\kappa,\Delta}(j)_{2^\ell}\rangle \otimes |b\rangle \in L^2(\mathbb{R}) \otimes \mathbb{C}^2 \text{ for } j \in \mathbb{Z}_{2^\ell},\ b \in \{0,1\}.$$

1. The $\mathsf{C}X_\ell$ gate

We bound the matrix elements of the implementation of the $\mathsf{C}X_\ell$ gate given in Table III.

Lemma B1 (Matrix elements of the $\mathsf{C}X_\ell$ gate). Let $\ell \in \mathbb{N}$. Let $\kappa, \Delta \in (0, 1/4)$ and $\varepsilon \in (0, 2^{-(\ell+1)}]$. Define the unitary $W_{\mathsf{C}X_\ell}$ acting on $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ as

$$W_{\mathsf{C}X_\ell} = I \otimes |0\rangle\langle 0| + e^{-i\sqrt{2\pi/2^\ell}P} \otimes |1\rangle\langle 1|.$$

For $j,k \in \mathbb{Z}_{2^\ell}$ and $b,b' \in \{0,1\}$, define the matrix element

$$M_{(j,b),(k,b')} = \big(\big\langle \mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(j)_{2^\ell}\big| \otimes \langle b|\big) W_{\mathsf{C}X_\ell} \big(\big|\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(k)_{2^\ell}\big\rangle \otimes |b'\rangle\big)$$

of the operator $W_{\mathsf{C}X_\ell}$ with respect to the basis

$$\big\{\big|\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(j)_{2^\ell}\big\rangle \otimes |b\rangle\big\}_{(j,b)\in\mathbb{Z}_{2^\ell}\times\{0,1\}} \quad \text{of} \quad \mathcal{GKP}^{\varepsilon}_{\kappa,\Delta}[2^\ell] \otimes \mathbb{C}^2.$$

Then we have

$$(1-\kappa^2)\delta_{b,b'}\delta_{j,k\oplus b} \leqslant M_{(j,b),(k,b')} \leqslant \delta_{b,b'}\delta_{j,k\oplus b} \quad \text{for all} \quad j,k \in \mathbb{Z}_d \quad \text{and} \quad b,b' \in \{0,1\}.$$

In particular, the matrix $M = (M_{(j,b),(k,b')})$ is 1-sparse.

Proof. A straightforward computation shows that

$$M_{(j,b),(k,b')} = \delta_{b,b'}\big((1-b)\delta_{j,k} + b\big\langle \mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(j)_{2^\ell}, \times e^{-i\sqrt{2\pi/2^\ell}P}\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(k)_{2^\ell}\big\rangle\big).$$

The claim follows from Ref. [1, Lemma C.1], which states

$$(1-\kappa^2)\delta_{j,k\oplus 1} \leqslant \big\langle \mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(j)_{2^\ell}, e^{-i\sqrt{2\pi/2^\ell}P}\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(k)_{2^\ell}\big\rangle \leqslant \delta_{j,k\oplus 1}.$$

■

2. The LSB_ℓ gate

We bound the matrix elements of the implementation of the LSB_ℓ gate given in Table III.

Lemma B2 (Matrix elements of the LSB gate). Let $\ell \in \mathbb{N}$. Let $\kappa, \Delta > 0$ and $\varepsilon \in (0, 2^{-(\ell+1)}]$. Define the unitary W_{LSB_ℓ} acting on $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ as

$$W_{\mathsf{LSB}_\ell} = (I\otimes H)(I \otimes |0\rangle\langle 0| + e^{i\pi\sqrt{2^\ell/(2\pi)}Q} \otimes |1\rangle\langle 1|)(I\otimes H). \tag{B2}$$

For $j,k \in \mathbb{Z}_{2^\ell}$ and $b,b' \in \{0,1\}$, define the matrix element

$$M_{(j,b),(k,b')} = \big(\big\langle \mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(j)_{2^\ell} \otimes \langle b|\big) \times W_{\mathsf{LSB}_\ell}\big(\big|\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(k)_{2^\ell}\big\rangle \otimes |b'\rangle\big)$$

of the operator W_{LSB_ℓ} with respect to the basis

$$\big\{\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(j) \otimes |b\rangle\big\}_{(j,b)\in\mathbb{Z}_{2^\ell}\times\{0,1\}} \text{ of } \mathcal{GKP}^{\varepsilon}_{\kappa,\Delta}[2^\ell] \otimes \mathbb{C}^2.$$

Then,

$$\begin{aligned} M_{(j,b),(j,b\oplus j_0)} &\geqslant 1 - 2^{2\ell+1}\Delta^2 - 8(\Delta/\varepsilon)^4, \\ \big|M_{(j,b),(j,b\oplus j_0\oplus 1)}\big| &\leqslant 2^{2\ell+1}\Delta^2 + 8(\Delta/\varepsilon)^4, \\ M_{(j,b),(k,b')} &= 0 \text{ otherwise} \end{aligned} \tag{B3}$$

for all $j,k \in \mathbb{Z}_{2^\ell}$ and $b,b' \in \{0,1\}$, where $j_0 = j \mod 2$ is the least significant bit in the binary representation of j. In particular, the matrix $M = (M_{(j,b),(k,b')})$ is 2-sparse.

Proof. By Eq. (B1), we have

$$\begin{aligned} &\big(\big\langle \mathsf{GKP}^{\varepsilon}_{\Delta,\kappa}(j)_{2^\ell}\big| \otimes \langle b|\big) W_{\mathsf{LSB}_\ell}\big(\big|\mathsf{GKP}_{\kappa,\Delta}(k)^{\varepsilon}_{2^\ell}\big\rangle \otimes |b'\rangle\big) \\ &\quad = \big(\big\langle \mathsf{GKP}^{\varepsilon}_{\Delta,\kappa}\big| \otimes \langle b|\big) U^{(\ell)}_{j,k}\big(\big|\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}\big\rangle \otimes |b'\rangle\big), \end{aligned}$$

where

$$U^{(\ell)}_{j,k} = \big((M_{\sqrt{2\pi\times 2^\ell}})^\dagger\big(e^{-i\sqrt{2\pi/2^\ell}jP}\big)^\dagger \otimes I\big) \times W_{\mathsf{LSB}_\ell}\big(e^{-i\sqrt{2\pi/2^\ell}kP} M_{\sqrt{2\pi\times 2^\ell}} \otimes I\big). \tag{B4}$$

Plugging Eq. (B2) into Eq. (B4) and using the identities

$$\begin{aligned} (M_\alpha)^\dagger P M_\alpha &= P/\alpha, \\ (M_\alpha)^\dagger Q M_\alpha &= \alpha Q, \\ (e^{-i\beta P})^\dagger Q e^{-i\beta P} &= Q + \beta I \quad \text{for} \quad \alpha > 0\ \beta \in \mathbb{R} \end{aligned}$$

to derive the transformation of the quadrature operators, we obtain

$$\begin{aligned} U^{(\ell)}_{j,k} &= (I\otimes H)\big(I\otimes|0\rangle\langle 0| + e^{i\pi\sqrt{2^\ell/(2\pi)}(\sqrt{2\pi}dQ+\sqrt{2\pi/2^\ell}j)} \otimes |1\rangle\langle 1|\big)(I\otimes H)\big(e^{-i(k-j)P/2^\ell}\otimes I\big) \\ &= (I\otimes H)\big(I\otimes|0\rangle\langle 0| + e^{i\pi j}e^{i\pi\times 2^\ell Q} \otimes |1\rangle\langle 1|\big)(I\otimes H)\big(e^{-i(k-j)P/2^\ell}\otimes I\big). \end{aligned}$$

Using $H|b\rangle = (|0\rangle + (-1)^b|1\rangle)/\sqrt{2}$ gives

$$\begin{aligned} M_{(j,b),(k,b')} &= \tfrac{1}{2}\big(\delta_{j,k} + (-1)^b(-1)^{b'}e^{i\pi j}\big\langle \mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}, e^{i\pi\times 2^\ell Q}e^{-i(k-j)P/2^\ell}\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}\big\rangle\big) \\ &= \tfrac{1}{2}\delta_{j,k}\big(1 + (-1)^{b+b'+j_0}\big\langle \mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}, e^{i\pi\times 2^\ell Q}\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}\big\rangle\big), \end{aligned} \tag{B5}$$

where we used Ref. [23, Lemma 7.6], which implies that

$$\big\langle \mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}, e^{-i(k-j)P/2^\ell}\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}\big\rangle \neq 0 \quad \text{if and only if } j = k \quad \text{for} \quad \varepsilon \leqslant 2^{-(\ell+1)},$$

together with the fact that the unitary $e^{i\pi dQ}$ does not change the support of the state $\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}$. Moreover, we used that

$$(-1)^b(-1)^{b'}e^{i\pi j} = e^{i\pi(b+b'+j)} = (-1)^{b+b'+j_0},$$

where $j_0 = j \mod 2$ is the least significant bit of j in its binary representation, i.e., $j_0 = 0$ if j is even and $j_0 = 1$ if j is odd. By Ref. [1, Lemma B.15] with $z = 2^{\ell-1}$ we have

$$\left\langle \mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}, e^{i\pi\times 2^{\ell}Q}\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}\right\rangle \geqslant 1 - 10\times 2^{2\ell-2}\Delta^2 - 16(\Delta/\varepsilon)^4. \tag{B6}$$

Inserting Eq. (B6) into Eq. (B5) gives the claim (B3).

Finally, Eq. (B5) implies that for fixed $j \in \mathbb{Z}_{2^\ell}$ only the matrix elements $M_{(j,0),(j,0)}$, $M_{(j,0),(j,1)}$, $M_{(j,1),(j,0)}$, and $M_{(j,1),(j,1)}$ are nonzero, which implies that the matrix M is 2-sparse. ■

3. The embedding isometry

In the following lemma, we establish that the squeezing operation $M_{\sqrt{2}}$ implements the embedding map Embed_ℓ defined in Eq. (14) perfectly.

Lemma B3. Let $\ell \in \mathbb{N}$. Let $\kappa, \Delta > 0$ and $\varepsilon \in (0, 2^{-(\ell+1)}]$. Then we have

$$M_{\sqrt{2}}\left|\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(j)_{2^\ell}\right\rangle = \left|\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(2j)_{2^{\ell+1}}\right\rangle \quad \text{for all} \quad j \in \mathbb{Z}_d.$$

Proof. By definition, we have

$$\begin{aligned} M_{\sqrt{2}}\left|\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(j)_{2^\ell}\right\rangle &= M_{\sqrt{2}}e^{-i\sqrt{2\pi/2^\ell}jP}M_{\sqrt{2\pi\times 2^\ell}}\left|\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}\right\rangle \\ &= e^{-i\sqrt{4\pi/2^\ell}jP}M_{\sqrt{2}}M_{\sqrt{2\pi\times 2^\ell}}\left|\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}\right\rangle \\ &= e^{-i\sqrt{2\pi/(2^{\ell+1})}2jP}M_{\sqrt{2\pi\times 2^{\ell+1}}}\left|\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}\right\rangle \\ &= \left|\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(2j)_{2^{\ell+1}}\right\rangle, \end{aligned}$$

where we used that $M_\alpha P M_\alpha^\dagger = \alpha P$ for all $\alpha > 0$. ■

APPENDIX C: LOGICAL GATE ERRORS OF IMPLEMENTATIONS OF BASIC BIT-MANIPULATION MAPS

Here we prove Lemma 6, which bounds the logical gate error of the implementations given in Table III of the basic bit-manipulation maps. The proof uses Corollary A1 and relies on the bounds for the matrix elements of the corresponding implementations derived in Appendix B.

Proof Lemma 6. The claim about the number of elementary operations that compose the respective physical implementation W_U of $U \in \{\mathsf{CX}_\ell, \mathsf{LSB}_\ell, \mathsf{Embed}_\ell\}$ follows directly from Lemma 3. Next, we derive the bounds on the logical gate error.

Define

$$|\mathbf{k}, \mathbf{b}\rangle = \left|\mathsf{GKP}^{\varepsilon}_{\kappa,\Delta}(k)_{2^\ell}\right\rangle \otimes |b\rangle \quad \text{for} \quad k \in \mathbb{Z}_{2^\ell}, \quad b \in \{0,1\}.$$

We first focus on the logical CX_ℓ-gate defined by its matrix elements $(\mathsf{CX}_\ell)_{(j,b),(k,c)} = \delta_{b,c}\delta_{j,k\oplus b}$ and its (approximate) implementation W_{CX_ℓ}. The matrix elements of the corresponding matrix B in Definition A1 are

$$\begin{aligned} B_{(r,a),(s,b)} &= \sum_{c=0}^{1}\sum_{j=0}^{d-1}\overline{(\mathsf{CX}_\ell)_{(j,c),(r,a)}}\langle \mathbf{j}, \mathbf{c}|W_{\mathsf{CX}_\ell}|\mathbf{s}, \mathbf{b}\rangle \\ &= \langle \mathbf{r}\oplus\mathbf{a}, \mathbf{a}|W_{\mathsf{CX}_\ell}|\mathbf{s}, \mathbf{b}\rangle \quad \text{for} \quad r, s \in \mathbb{Z}_{2^\ell} \quad \text{and} \\ &\quad a, b \in \{0,1\}. \end{aligned}$$

In particular, the diagonal elements are

$$B_{(r,a),(r,a)} = \langle \mathbf{r}\oplus\mathbf{a}, \mathbf{a}|W_{\mathsf{CX}_\ell}|\mathbf{r}, \mathbf{a}\rangle.$$

According to Lemma B1, the matrix B is 1-sparse ($s = 1$) and

$$B_{(r,a),(r,a)} \geqslant 1 - \kappa^2$$

for every $r \in \mathbb{Z}_{2^\ell}$ and $a \in \{0,1\}$. Using Corollary A1 with $s = 1$ the logical gate error is bounded by

$$\begin{aligned} \mathsf{err}_{\mathcal{GKP}^{\varepsilon}_{\kappa,\Delta}[2^\ell]\otimes\mathbb{C}^2}\left(W_{\mathsf{CX}_\ell}, \mathsf{CX}_\ell\right) &\leqslant 3\left(1 - \min_{(r,a)} B_{(r,a),(r,a)}\right)^{1/2} \\ &\leqslant 3\kappa. \end{aligned}$$

Similarly, we bound the logical gate error of the LSB_ℓ gate. Recall that the logical LSB_ℓ gate is defined by its matrix elements $(\mathsf{LSB}_\ell)_{(j,b),(k,c)} = \delta_{b,c\oplus j_0}\delta_{j,k}$ where $j_0 = j \mod 2$. The matrix elements of the corresponding matrix B in Definition A1 are given by

$$\begin{aligned} B_{(r,a),(s,b)} &= \sum_{c=0}^{1}\sum_{j=0}^{d-1}\overline{(\mathsf{LSB}_\ell)_{(j,c),(r,a)}}\langle \mathbf{j}, \mathbf{c}|W_{\mathsf{LSB}_\ell}|\mathbf{s}, \mathbf{b}\rangle \\ &= \langle \mathbf{r}, \mathbf{a}\oplus\mathbf{r_0}|W_{\mathsf{LSB}_\ell}|\mathbf{s}, \mathbf{b}\rangle \quad \text{for} \quad r, s \in \mathbb{Z}_{2^\ell} \quad \text{and} \\ &\quad a, b \in \{0,1\}. \end{aligned}$$

Therefore,

$$B_{(r,a),(r,a)} = \langle \mathbf{r}, \mathbf{a}\oplus\mathbf{r_0}|W_{\mathsf{LSB}_\ell}|\mathbf{r}, \mathbf{a}\rangle,$$

where $r_0 = r \mod 2$. According to Lemma B2, the matrix B is 2-sparse, and the only nonvanishing matrix elements are bounded by

$$\begin{aligned} B_{(r,a),(r,a)} &\geqslant 1 - 2^{2\ell+1}\Delta^2 - 8(\Delta/\varepsilon)^4 \\ |B_{(r,a),(r,a\oplus 1)}| &\leqslant 2^{2\ell+1}\Delta^2 + 8(\Delta/\varepsilon)^4 \text{ for every } r \in \mathbb{Z}_{2^\ell}, \\ &\quad a \in \{0,1\}. \end{aligned}$$

Using Corollary A1 (with $s = 2$), the logical gate error is bounded by

$$\begin{aligned} &\mathsf{err}_{\mathcal{GKP}^{\varepsilon}_{\kappa,\Delta}[2^\ell]\otimes\mathbb{C}^2}\left(W_{\mathsf{LSB}_\ell}, \mathsf{LSB}_\ell\right) \\ &\leqslant 3\left[(1 - \min_{(r,a)} B_{(r,a),(r,a)}) + (s-1)\max_{r,a,s,b:(r,a)\neq(s,b)}|B_{(r,a),(s,b)}|\right]^{1/2} \\ &\leqslant 3[2^{2\ell+2}\Delta^2 + 16(\Delta/\varepsilon)^4]^{1/2} \\ &\leqslant 6\times 2^\ell\Delta + 12(\Delta/\varepsilon)^2, \end{aligned}$$

where we used that $\sqrt{x+y} \leqslant \sqrt{x} + \sqrt{y}$ for $x, y \geqslant 0$.

The claim about the implementation of the isometry Embed_ℓ follows from Lemma B3.

Finally, the fact that the bounds on the logical gate error of the implementations were derived using Corollary A1 implies that the same bounds hold for the implementations of the adjoint operations. ■

APPENDIX D: LOGICAL GATE ERROR FOR RECTANGULAR-ENVELOPE GKP CODES

In this Appendix, we argue that similar conclusions apply to (truncated) rectangular-envelope GKP states (or comb states) and the corresponding approximate codes. In Appendix D 1 we define the corresponding states and codes. In Appendix D 2, we compute matrix elements of implementations of basic bit-manipulations. In Appendix D 3, we then establish bounds on the corresponding logical gate errors.

1. Definition of rectangular-envelope GKP codes

Here we define so-called comb states. Instead of a Gaussian envelope parametrized by a squeezing parameter $\kappa > 0$, these are defined in terms of a rectangular envelope whose width is determined by an integer L. We still use a parameter $\Delta > 0$ for the width of individual peaks (local maxima), which are (truncated) Gaussians. Recall that we use the Gaussian

$$\Psi_\Delta(x) = \frac{1}{(\pi\Delta^2)^{1/4}} e^{-x^2/(2\Delta^2)}$$

and its truncated variant

$$\Psi_\Delta^\varepsilon = \frac{\Pi_{[-\varepsilon,\varepsilon]}\Psi_\Delta}{\|\Pi_{[-\varepsilon,\varepsilon]}\Psi_\Delta\|},$$

where $\|\cdot\|$ is the Euclidean norm, and where $\varepsilon \in (0, 1/2)$. Defining the translated versions $\chi_\Delta^\varepsilon(z)(\cdot) = \Psi_\Delta^\varepsilon(\cdot - z)$, the integer-spaced comb state (or "rectangular-envelope GKP state") is defined as

$$\left|\mathrm{III}_{L,\Delta}^\varepsilon\right\rangle = \frac{1}{\sqrt{L}} \sum_{z=-L/2}^{L/2-1} \left|\chi_\Delta^\varepsilon(z)\right\rangle. \tag{D1}$$

Support properties are especially important for our considerations. The following is an immediate consequence of the definitions: We have

$$\mathsf{supp}\left(e^{-i\delta P}\mathrm{III}_{L,\Delta}^\varepsilon\right) \cap \mathsf{supp}\left(\mathrm{III}_{L,\Delta}^\varepsilon\right) = \emptyset \quad \text{if} \quad \varepsilon < |\delta| < 1 - 2\varepsilon.$$

We can also show that the state $\mathrm{III}_{L,\Delta}^\varepsilon$ is fixed by momentum-translations which are small integer multiples of 2π.

Lemma D1 (Momentum-translated rectangular-envelope GKP states). Let $z \in \mathbb{Z}$. Let $L \in 2\mathbb{N}$, $\Delta > 0$, and $\varepsilon \leqslant 1/(2d)$. Then

$$\left\langle \mathrm{III}_{L,\Delta}^\varepsilon, e^{2\pi i z Q}\mathrm{III}_{L,\Delta}^\varepsilon\right\rangle \geqslant 1 - 10z^2\Delta^2 - 16(\Delta/\varepsilon)^4.$$

Proof. We have

$$\left\langle \mathrm{III}_{L,\Delta}^\varepsilon, e^{2\pi i z Q}\mathrm{III}_{L,\Delta}^\varepsilon\right\rangle = \frac{1}{L}\sum_{y=-L/2}^{L/2-1} \left\langle \chi_\Delta^\varepsilon(y), e^{2\pi i z Q}\chi_\Delta^\varepsilon(y)\right\rangle$$

since the operator $e^{2\pi i z Q}$ does not change the support, and the functions $\{\chi_\Delta^\varepsilon(y)\}_{y\in\mathbb{Z}}$ have pairwise disjoint support by definition. A straightforward calculation (see the proof of Lemma B.14 in Ref. [1]) gives

$$\left\langle \chi_\Delta^\varepsilon(y), e^{2\pi i z Q}\chi_\Delta^\varepsilon(y)\right\rangle \geqslant 1 - 10z^2\Delta^2 - 16(\Delta/\varepsilon)^4 \quad \text{for any} \quad y \in \mathbb{Z}.$$

The claim follows from this. ■

As for GKP codes with a Gaussian envelope, we use the state defined by Eq. (D1) as a starting point to define approximate (rectangular-envelope) GKP codes: For any integer $d \geqslant 2$,we define the d approximate GKP states

$$\left|\mathrm{III}_{L,\Delta}^\varepsilon(j)_d\right\rangle = e^{-i\sqrt{2\pi/d}jP} M_{\sqrt{2\pi d}} \left|\mathrm{III}_{L,\Delta}^\varepsilon\right\rangle \text{ for } j \in \{0,\ldots,d-1\}.$$

In this definition, we have used the single-mode squeezing operator M_α for $\alpha > 0$, a Gaussian unitary which acts on $\Psi \in L^2(\mathbb{R})$ as $(M_\alpha\Psi)(x) = \frac{1}{\sqrt{\alpha}}\Psi(x/\alpha)$ for $x \in \mathbb{R}$. A straightforward computation gives

$$M_\alpha\left|\chi_\Delta^\varepsilon(z)\right\rangle = \chi_{\Delta\alpha}^{\varepsilon\alpha}(z\alpha)$$

and thus

$$e^{-i\sqrt{2\pi/d}jP} M_{\sqrt{2\pi d}}\left|\chi_\Delta^\varepsilon(z)\right\rangle = \chi_{\Delta\sqrt{2\pi d}}^{\varepsilon\sqrt{2\pi d}}\left(z\sqrt{2\pi d} + \sqrt{\frac{2\pi}{d}}j\right).$$

It follows that

$$\left|\mathrm{III}_{L,\Delta}^\varepsilon(j)_d\right\rangle = \frac{1}{\sqrt{L}}\sum_{z=-L/2}^{L/2-1} \chi_{\Delta\sqrt{2\pi d}}^{\varepsilon\sqrt{2\pi d}}\left(z\sqrt{2\pi d} + \sqrt{\frac{2\pi}{d}}j\right) \tag{D2}$$

for any $j \in \{0,\ldots,d-1\}$. In particular, the states $\{\mathrm{III}_{L,\Delta}^\varepsilon(j)_d\}_{j=\{0,\ldots,d-1\}}$ have essentially pairwise disjoint support for $\varepsilon \leqslant 1/(2d) =: \varepsilon_d$ and thus they are orthogonal.

For an integer $d \geqslant 2$, $L \in 2\mathbb{N}$, $\Delta > 0$, and $\varepsilon \leqslant 1/(2d)$, we define the (rectangular-envelope truncated) GKP code with parameters L, Δ, ε by

$$\mathrm{III}\mathcal{GKP}_{L,\Delta}^\varepsilon[d] := \mathsf{span}\left\{\mathrm{III}_{L,\Delta}^\varepsilon(j)_d\right\}_{j=\{0,\ldots,d-1\}}.$$

Corresponding isometric encoding and decoding maps are defined as before using the orthonormal basis $\{\mathrm{III}_{L,\Delta}^\varepsilon(j)_d\}_{j=\{0,\ldots,d-1\}}$.

As before, we write

$$\mathrm{III}\mathcal{GKP}_{L,\Delta}^\star[d] := \mathrm{III}\mathcal{GKP}_{L,\Delta}^{\varepsilon_d}[d]$$

for the GKP code with optimal truncation parameter $\varepsilon_d = 1/(2d)$. We typically choose the integer $L \in \mathbb{N}$ as a certain function of $L \in \mathbb{N}$, i.e., we set

$$L_{\Delta,d} = 2^{2(\lceil \log_2 1/\Delta\rceil - \lfloor \log_2 d\rfloor)}, \tag{D3}$$

where we assume that $\Delta < 1/d$. With these choices, we end up with a one-parameter family of approximate GKP codes depending only on the parameter $\Delta > 0$. We write

$$\mathrm{III}\mathcal{GKP}_\Delta^\star[d] := \mathrm{III}\mathcal{GKP}_{L_{\Delta,d},\Delta}^{\varepsilon_d}[d]$$

for the code associated with $\Delta > 0$, and call this the approximate (rectangular-envelope truncated) GKP code with parameter Δ.

For later use, we observe that (for sufficiently large L), the operator $e^{-i\sqrt{2\pi/d}P}$ approximately acts as a logical-Pauli operator, as expressed by the following matrix elements:

Lemma D2. Let $d \geqslant 2$ be an integer, $\varepsilon \in (0, 1/(2d))$, $\Delta > 0$ and $L \in 2\mathbb{N}$. For any $j, k \in \{0, \dots, d-1\}$ we then have

$$\left\langle \mathrm{III}^{\varepsilon}_{L,\Delta}(j)_d, e^{-i\sqrt{2\pi/d}P}\mathrm{III}^{\varepsilon}_{L,\Delta}(k)_d \right\rangle = \begin{cases} 1 - 2/L & \text{if } j = k \oplus 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\oplus$ denotes addition modulo d.

Proof. This follows immediately from Eq. (D2) using the fact that the states $\chi^{\varepsilon\sqrt{2\pi d}}_{\Delta\sqrt{2\pi d}}(z\sqrt{2\pi d} + \sqrt{\frac{2\pi}{d}}j)$ and $\chi^{\varepsilon\sqrt{2\pi d}}_{\Delta\sqrt{2\pi d}}(z'\sqrt{2\pi d} + \sqrt{\frac{2\pi}{d}}j')$ have disjoint support unless $(z, j) = (z', j')$. ∎

2. Matrix elements of hybrid qubit-oscillator implementations

Lemma D3 (Matrix elements of the CX_ℓ gate). Let $\ell \in \mathbb{N}$, $\Delta > 0$, $L \in 2\mathbb{N}$, and $\varepsilon \in (0, 2^{-(\ell+1)}]$. Define the unitary W_{CX_ℓ} acting on $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ as

$$W_{\mathsf{CX}_\ell} = I \otimes |0\rangle\langle 0| + e^{-i\sqrt{2\pi/2^\ell}P} \otimes |1\rangle\langle 1|.$$

For $j, k \in \mathbb{Z}_{2^\ell}$ and $b, b' \in \{0, 1\}$, define the matrix element

$$M_{(j,b),(k,b')} = \left(\left\langle \mathrm{III}^{\varepsilon}_{L,\Delta}(j)_{2^\ell}\right| \otimes \langle b|\right) W_{\mathsf{CX}_\ell} \left(\left|\mathrm{III}^{\varepsilon}_{L,\Delta}(k)_{2^\ell}\right\rangle \otimes |b'\rangle\right)$$

of the operator W_{CX_ℓ} with respect to the basis

$$\left\{\left|\mathrm{III}^{\varepsilon}_{L,\Delta}(j)_{2^\ell}\right\rangle \otimes |b\rangle\right\}_{(j,b)\in\mathbb{Z}_{2^\ell}\times\{0,1\}} \text{ of } \mathrm{III}\mathcal{GKP}^{\varepsilon}_{L,\Delta}[2^\ell] \otimes \mathbb{C}^2.$$

Then we have

$$(1 - 2/L)\,\delta_{b,b'}\,\delta_{j,k\oplus b} \leqslant M_{(j,b),(k,b')} \leqslant \delta_{b,b'}\,\delta_{j,k\oplus b} \text{ for all } j, k \in \mathbb{Z}_d \text{ and } b, b' \in \{0, 1\}.$$

In particular, the matrix $M = (M_{(j,b),(k,b')})$ is 1-sparse.

Proof. This follows from the same arguments as Lemma B1 together with Lemma D2. ∎

Lemma D4 (Matrix elements of the LSB gate). Let $\ell \in \mathbb{N}$, $\Delta > 0$, $L \in 2\mathbb{N}$, and $\varepsilon \in (0, 2^{-(\ell+1)}]$. Define the unitary W_{LSB_ℓ} acting on $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ as

$$W_{\mathsf{LSB}_\ell} = (I \otimes H)\left(I \otimes |0\rangle\langle 0| + e^{i\pi\sqrt{2^\ell/(2\pi)}Q} \otimes |1\rangle\langle 1|\right) \times (I \otimes H).$$

For $j, k \in \mathbb{Z}_{2^\ell}$ and $b, b' \in \{0, 1\}$, define the matrix element

$$M_{(j,b),(k,b')} = \left(\left\langle \mathrm{III}^{\varepsilon}_{L,\Delta}(j)_{2^\ell} \right| \otimes \langle b|\right) W_{\mathsf{LSB}_\ell} \left(\left|\mathrm{III}^{\varepsilon}_{L,\Delta}(k)_{2^\ell}\right\rangle \otimes |b'\rangle\right)$$

for matrix element of the operator W_{LSB_ℓ} with respect to the basis

$$\left\{\mathrm{III}^{\varepsilon}_{\kappa,\Delta}(j) \otimes |b\rangle\right\}_{(j,b)\in\mathbb{Z}_{2^\ell}\times\{0,1\}} \quad \text{of} \quad \mathrm{III}\mathcal{GKP}^{\varepsilon}_{L,\Delta}[2^\ell] \otimes \mathbb{C}^2.$$

Then

$$M_{(j,b),(j,b\oplus j_0)} \geqslant 1 - 2^{2\ell+1}\Delta^2 - 8(\Delta/\varepsilon)^4,$$
$$\left|M_{(j,b),(j,b\oplus j_0\oplus 1)}\right| \leqslant 2^{2\ell+1}\Delta^2 + 8(\Delta/\varepsilon)^4,$$
$$M_{(j,b),(k,b')} = 0 \text{ otherwise}$$

for all $j, k \in \mathbb{Z}_{2^\ell}$ and $b, b' \in \{0, 1\}$, where $j_0 = j \mod 2$ is the least significant bit in the binary representation of j. In particular, the matrix $M = (M_{(j,b),(k,b')})$ is 2-sparse.

Proof. This follows from the same arguments as Lemma B2 together with Lemma D1. ∎

3. Logical gate errors for rectangular-envelope GKP codes

We find the following bounds on the logical gate error of implementations of basic bit-manipulation maps in the code $\mathrm{III}\mathcal{GKP}^{\varepsilon}_{L,\Delta}[2^\ell]$:

Lemma D5 (Implementation of CX_ℓ, LSB_ℓ, and Embed_ℓ in rectangular-envelope GKP codes). Let $\ell \in \mathbb{N}$, $\Delta > 0$, $L \in 2\mathbb{N}$, and $\varepsilon \in (0, 2^{-(\ell+1)}]$. Then we have the following:

(i) The physical implementation of the unitary CX_ℓ on $\mathbb{C}^{2^\ell} \otimes \mathbb{C}^2$ given by

$$W_{\mathsf{CX}_\ell} = \mathsf{ctrl}e^{-i\sqrt{2\pi/2^\ell}P}$$

can be realized by a single operation in $\mathcal{U}^{1,1}_{\mathsf{elem}}$ and satisfies

$$\mathsf{err}_{\mathrm{III}\mathcal{GKP}^{\varepsilon}_{L,\Delta}[2^\ell]\otimes\mathbb{C}^2}\left(W_{\mathsf{CX}_\ell}, \mathsf{CX}_\ell\right) \leqslant 5L^{-1/2}.$$

(ii) The physical implementation of the unitary LSB_ℓ on $\mathbb{C}^{2^\ell} \otimes \mathbb{C}^2$ given by

$$W_{\mathsf{LSB}_\ell} = (I \otimes H)(M^\dagger_{\alpha'} \otimes I)^n \mathsf{ctrl}e^{iQ}(M_{\alpha'} \otimes I)^n(I \otimes H)$$

where

$$\alpha' = e^{\frac{\ln\alpha}{\lceil|\ln\alpha|\rceil}} \text{ and } n = \lceil|\ln\alpha|\rceil \text{ with } \alpha = \sqrt{\pi}2^{(\ell-1)/2}$$

defines a circuit $W_{\mathsf{LSB}_\ell} = W_T \cdots W_1$ composed of $T \leqslant \ell + 6$ elementary gates in $\mathcal{U}^{1,1}_{\mathsf{elem}}$ such that

$$\mathsf{err}_{\mathrm{III}\mathcal{GKP}^{\varepsilon}_{L,\Delta}[2^\ell]\otimes\mathbb{C}^2}\left(W_{\mathsf{LSB}_\ell}, \mathsf{LSB}_\ell\right) \leqslant 6 \times 2^\ell\Delta + 12(\Delta/\varepsilon)^2.$$

(iii) The isometry $\mathsf{Embed}_\ell : \mathbb{C}^{2^\ell} \to \mathbb{C}^{2^{\ell+1}}$ can be implemented by a unitary $W_{\mathsf{Embed}_\ell} = M_{\sqrt{2}}$ acting on $L^2(\mathbb{R})$ using one elementary operation in $\mathcal{U}^{1,0}_{\mathsf{elem}}$ such that

$$\mathsf{err}_{\mathrm{III}\mathcal{GKP}^{\varepsilon}_{L,\Delta}[2^\ell],\mathrm{III}\mathcal{GKP}^{\varepsilon}_{L,\Delta}[2^{\ell+1}]}\left(W_{\mathsf{Embed}_\ell}, \mathsf{Embed}_\ell\right) = 0.$$

Proof. Comparing Lemmas B1 and B2 with Lemmas D3 and D4, respectively, we observe that the proof follows similarly to that of Lemma 6, where we replace the parameter κ with $\sqrt{2/L}$. ∎

With the necessary modifications, the proofs from Sec. IV B can be applied. This leads to the following result concerning the implementation of multiqubit circuits using rectangular-envelope GKP codes. It is the counterpart of Theorem 1 for (regular) symmetrically squeezed GKP codes.

Corollary D1 (Implementations of multiqubit circuits in rectangular-envelope GKP codes). Let $\ell \in \mathbb{N}$ and $\Delta \in (0, 2^{-\ell})$. Define $L_{\Delta,2^\ell} = 2^{2(\lceil\log_2 1/\Delta\rceil - \ell)}$. Let $U = U_T \cdots U_1$ be a unitary on $(\mathbb{C}^2)^{\otimes\ell}$ where each unitary U_t acts nontrivially only on (any) two qubits for $t \in \{1, \dots, T\}$. There exists a physical unitary implementation W_U using at most $340\ell^2 T$ elementary operations in $\mathcal{U}^{2,3}_{\mathsf{elem}}$ and a universal constant $C > 0$ (independent of ℓ) such that

$$\mathsf{err}_{\mathcal{L}}\left(W_U, J_\ell U\left(J_\ell^{-1}\right)\right) \leqslant 2600 \times 2^{2\ell}\, T\,\Delta,$$

where the code space is given by $\mathcal{L} = \mathrm{III}\mathcal{GKP}^{\star}_{\Delta}[2^\ell] \otimes \mathbb{C}(|\mathrm{III}^{2^{-(\ell+1)}}_{L_{\Delta,2^\ell},\Delta}(0)_2\rangle \otimes |0\rangle^{\otimes 3})$.

Proof. Consider a two-qubit unitary U on $(\mathbb{C}^2)^{\otimes\ell}$. Following the same arguments as in the proof of Theorem 2 (treating a single two-qubit unitary) and Theorem 1 (treating the case of a circuit), we can construct an implementation

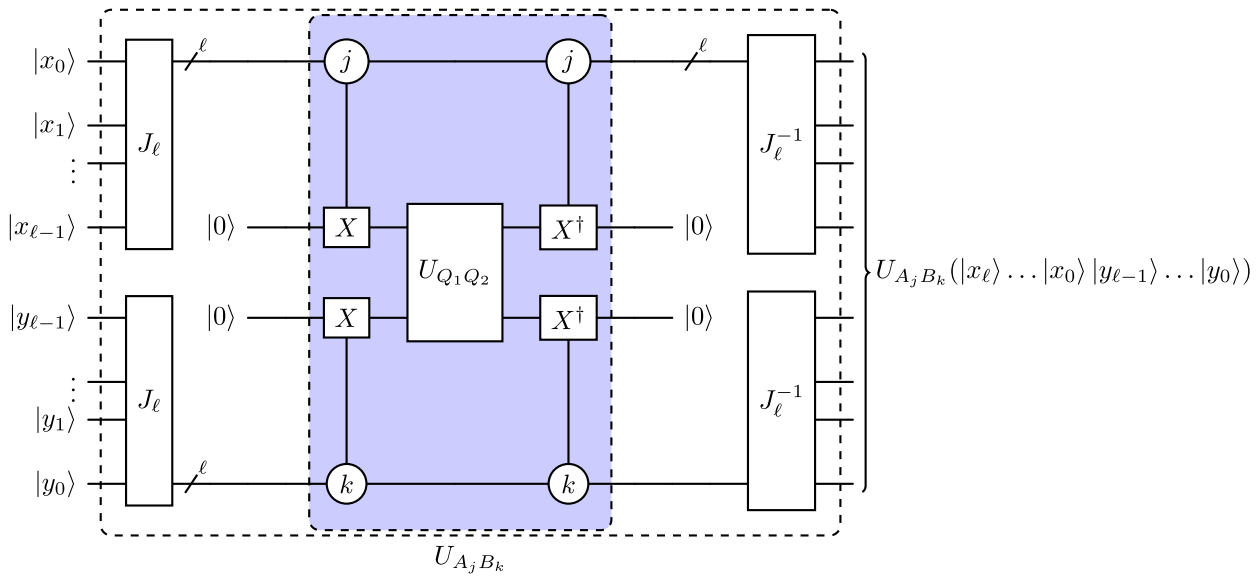

FIG. 6. Circuit implementing the two-qubit unitary $U_{A_jB_k}$ using the bit-transfer unitary Transf_ℓ^j and Transf_ℓ^k and two additional qubit systems.

W_U using at most $340\ell^2 T$ elementary operations. To obtain an upper bound on the logical gate error, we use Lemma D5 (for rectangular-envelope GKP codes) and find by the same arguments as in the proofs of Lemma 7 and Theorem 8:

$$\begin{aligned}
&\mathsf{err}_{\mathcal{L}}\big(W_U, J_\ell U J_\ell^{-1}\big)\\
&\quad\leqslant 4\times 12\ell\,\max\left\{5L_{\Delta,2^\ell}^{-1/2}, 6\times 2^\ell\Delta + 12\{\Delta/[1/(2^{\ell+1})]\}^2\right\}\\
&\quad\leqslant 48\ell\,\max\{5\times 2^\ell\Delta, 6\times 2^\ell\Delta + 48(2^\ell\Delta)^2\}\\
&\quad\leqslant 2600\ell\times 2^\ell\, T\Delta,
\end{aligned}$$

where we used the definition (D3) of $L_{\Delta,2^\ell}$ which implies that $L_{\Delta,2^\ell}^{-1/2}\leqslant 2^\ell\Delta$. The claim follows from this because $\ell\, 2^\ell\leqslant 2^{2\ell}$ for all $\ell\in\mathbb{N}$.

APPENDIX E: GENERALIZATION TO TWO-MODE ENCODINGS OF 2ℓ LOGICAL QUBITS

The following generalizes Lemma 2 to a scenario where we have an even number 2ℓ of qubits encoded in a bipartite Hilbert space of the form $(\mathbb{C}^2)^{\otimes\ell}\otimes(\mathbb{C}^2)^{\otimes\ell}$. The construction again uses four auxiliary qubits. For brevity, we write

$$|\vec{x}\rangle = |x_{\ell-1}\rangle\cdots|x_0\rangle \quad \text{for} \quad \vec{x}=(x_0,\dots,x_{\ell-1})\in\{0,1\}^\ell.$$

Lemma E1 (Bit-transfer-based implementation of multiqubit gates—bipartite case). Let $\ell\geqslant 2$ be an integer. Consider a system of 2ℓ logical qubits denoted

$$(A_0\cdots A_{\ell-1})(B_0\cdots B_{\ell-1}) = (\mathbb{C}^2)^{\otimes\ell}\otimes(\mathbb{C}^2)^{\otimes\ell}.$$

Let $j,k\in\{0,\dots,\ell-1\}$ be arbitrary. Let $U_{A_jB_k}:(\mathbb{C}^2)^{\otimes\ell}\otimes(\mathbb{C}^2)^{\otimes\ell}\to(\mathbb{C}^2)^{\otimes\ell}\otimes(\mathbb{C}^2)^{\otimes\ell}$ be a unitary acting nontrivially only on the two qubits A_jB_k. Let

$$S_1S_2Q_1Q_2Q_3Q_4 = \mathbb{C}^{2^\ell}\otimes\mathbb{C}^{2^\ell}\otimes(\mathbb{C}^2)^{\otimes 4}$$

be a system consisting of a bipartite $2^{2\ell}$-dimensional (code) space $S_1S_2\cong(\mathbb{C}^{2^\ell})^{\otimes 2}$ and four auxiliary qubits $Q_1\cdots Q_4$. Then there is a circuit $V_{U_{A_jB_k}}$ on $S_1S_2Q_1\cdots Q_4$ consisting of fewer than $48\ell-16$ operations belonging to the set $\mathcal{G}(\ell)$ and a single two-qubit operation such that

$$\begin{aligned}
&\big[\big(J_\ell^{-1}\big)^{\otimes 2}\otimes I\big]V_{U_{A_jB_k}}\big(J_\ell^{\otimes 2}\otimes I\big)[(|\vec{x}\rangle|\vec{y}\rangle)\otimes|0\rangle^{\otimes 4}]\\
&\qquad=\big[U_{A_jB_k}(|\vec{x}\rangle|\vec{y}\rangle)\big]\otimes|0\rangle^{\otimes 4}
\end{aligned}$$

for all $\vec{x}=(x_0,\dots,x_{\ell-1})$, $\vec{y}=(y_0,\dots,y_{\ell-1})\in\{0,1\}^\ell$. That is, the circuit $V_{U_{A_jB_k}}$ realizes the unitary $U_{A_jB_k}$ on the 2^ℓ-dimensional subspace

$$\begin{aligned}
&\mathbb{C}^{2^\ell}\otimes\mathbb{C}^{2^\ell}\otimes(\mathbb{C}|0\rangle^{\otimes 4})\subset S_1S_2Q_1\cdots Q_4\\
&\qquad=\mathbb{C}^{2^\ell}\otimes\mathbb{C}^{2^\ell}\otimes(\mathbb{C}^2)^{\otimes 4},
\end{aligned}$$

where the four auxiliary qubits are in the state $|0\rangle$.

Proof. The construction of $V_{U_{A_jB_k}}$ follows using the same reasoning as used in the proof of Lemma 2, but relies on the circuit identity shown in Fig. 6 instead of the circuit given in Fig. 4

The following two lemmas generalize Lemma 5 and Theorem 8, respectively. Lemma E2 considers 2ℓ qubits encoded into two ideal GKP codes $(\mathcal{GKP}[2^\ell])^{\otimes 2}$. Theorem E3 considers symmetrically squeezed GKP codes $(\mathcal{GKP}^\star_\kappa[2^\ell])^{\otimes 2}$ instead of ideal ones. In both cases, the constructions use one auxiliary oscillator and three auxiliary qubits.

Lemma E2 (Implementation of multiqubit unitaries in ideal GKP codes—bipartite case). Let $\ell\geqslant 2$ be an integer. Consider a system of 2ℓ logical qubits denoted

$$(A_0\cdots A_{\ell-1})(B_0\cdots B_{\ell-1}) = (\mathbb{C}^2)^{\otimes\ell}\otimes(\mathbb{C}^2)^{\otimes\ell}.$$

Let $j,k\in\{0,\dots,\ell-1\}$ be arbitrary. Let $U:=U_{A_jB_k}:(\mathbb{C}^2)^{\otimes\ell}\otimes(\mathbb{C}^2)^{\otimes\ell}\to(\mathbb{C}^2)^{\otimes\ell}\otimes(\mathbb{C}^2)^{\otimes\ell}$ be a unitary acting nontrivially only on the two qubits A_jB_k. Let

$$S_1S_2BQ_1Q_2Q_3 = L^2(\mathbb{R})\otimes L^2(\mathbb{R})\otimes[L^2(\mathbb{R})\otimes(\mathbb{C}^2)^{\otimes 3}]$$

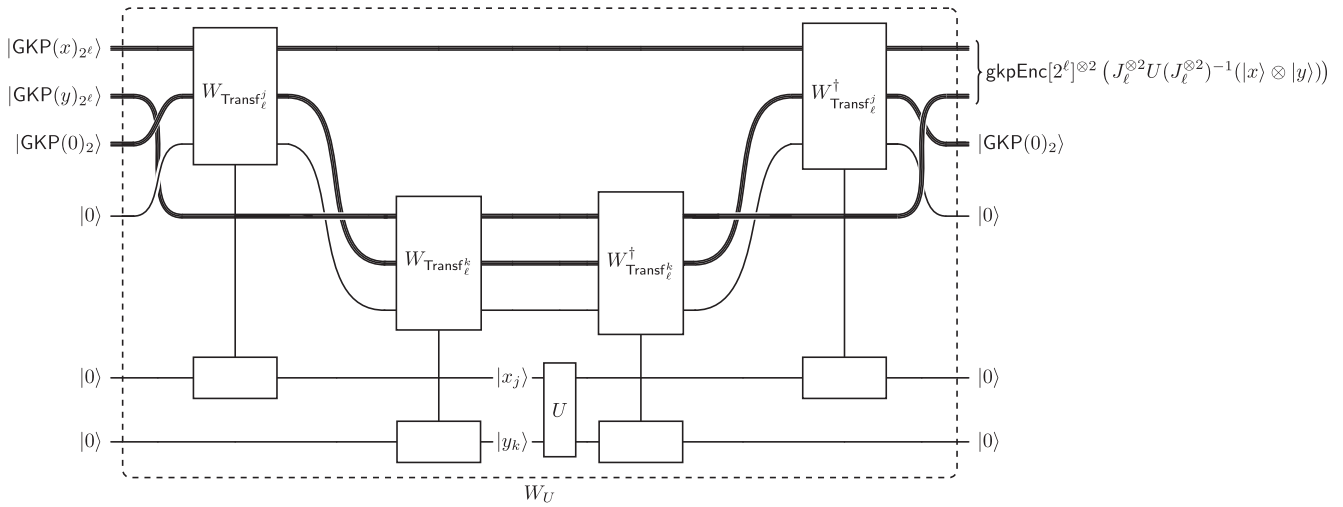

FIG. 7. The implementation W_U of a logical two-qubit unitary $U = U_{A_jB_k}$, acting on the jth qubit A_j and kth qubit B_k for $j, k \in \{0, \ldots, \ell - 1\}$, see Lemma E2. Here the first ℓ qubits $A_0 \cdots A_{\ell-1}$ are encoded in a GKP code on the first mode, whereas the remaining qubits $B_0 \cdots B_{\ell-1}$ are encoded in a second mode. In contrast with the construction given in Fig. 5 where the entire logical information is encoded in a single mode, bit-transfer gates (and their adjoints) are used, which act on the respective modes.

be a system consisting of a bipartite $2^{2\ell}$-dimensional (code) space $S_1S_2 \cong [L^2(\mathbb{R})]^{\otimes 2}$, one auxiliary oscillator B and three auxiliary qubits $Q_1Q_2Q_3$. Then there is a unitary circuit W_U on $L^2(\mathbb{R})^{\otimes 2} \otimes [L^2(\mathbb{R}) \otimes (\mathbb{C}^2)^{\otimes 3}]$ consisting of at most $340\ell^2$ elementary operations in $\mathcal{U}^{3,3}_{\mathsf{elem}}$ such that

$$\begin{aligned} W_U[|\mathsf{GKP}(x)_{2^\ell} \otimes \mathsf{GKP}(y)_{2^\ell}\rangle \otimes (|\mathsf{GKP}(0)_2\rangle \otimes |0\rangle^{\otimes 3})] \\ = \mathsf{gkpEnc}[2^\ell]^{\otimes 2}\left[J_\ell^{\otimes 2}U\left(J_\ell^{\otimes 2}\right)^{-1}(|x\rangle \otimes |y\rangle)\right] \\ \otimes [|\mathsf{GKP}(0)_2\rangle \otimes |0\rangle^{\otimes 3}] \end{aligned}$$

for all $x, y \in \{0, \ldots, 2^\ell - 1\}$. In other words, the circuit W_U implements the gate U exactly on the subspace

$$\mathcal{GKP}[2^\ell]^{\otimes 2} \otimes \mathbb{C}(|\Psi\rangle) \subset L^2(\mathbb{R})^{\otimes 2} \otimes [L^2(\mathbb{R}) \otimes (\mathbb{C}^2)^{\otimes 3}],$$

where $|\Psi\rangle = |\mathsf{GKP}(0)_2\rangle \otimes |0\rangle^{\otimes 3} \in L^2(\mathbb{R}) \otimes (\mathbb{C}^2)^{\otimes 3}$, i.e., the auxiliary oscillator is in the ideal GKP state $|\mathsf{GKP}(0)_2\rangle$ and the three auxiliary qubits are in the state $|0\rangle^{\otimes 3}$.

Proof. The construction of $U_{A_jB_k}$ is analogous to the construction of $U_{A_jA_k}$ in the proof of Lemma 5. It is illustrated in Fig. 7. ■

Combining the constructions illustrated in Figs. 5 and 7, we obtain the following:

Theorem E3 (Implementation of multiqubit gates in approximate GKP codes—bipartite case). Let $\ell \geqslant 2$ be an integer. Consider a system of 2ℓ logical qubits denoted $Q_1 \cdots Q_{2\ell} \cong (\mathbb{C}^2)^{\otimes \ell}$. Let $\alpha, \beta \in \{1, \ldots, 2\ell\}$, $\alpha < \beta$ be arbitrary. Let $U := U_{Q_\alpha Q_\beta} : (\mathbb{C}^2)^{\otimes 2\ell} \to (\mathbb{C}^2)^{\otimes 2\ell}$ be a unitary acting nontrivially only on the pair of qubits $Q_\alpha Q_\beta$. Then there exists a physical unitary implementation W_U using at most $340\ell^2$ elementary operations in $\mathcal{U}^{3,3}_{\mathsf{elem}}$ acting on the Hilbert space $L^2(\mathbb{R})^{\otimes 2} \otimes L^2(\mathbb{R}) \otimes (\mathbb{C}^2)^{\otimes 3}$ such that the following holds:

(i) Let $\kappa \in (0, 1/4)$. For the code space

$$\begin{aligned} \mathcal{L} = (\mathcal{GKP}^\star_\kappa[2^\ell])^{\otimes 2} \otimes \mathbb{C}\left(\left|\mathsf{GKP}^{2^{-(\ell+1)}}_\kappa(0)_2\right\rangle \otimes |0\rangle^{\otimes 3}\right) \\ \subset L^2(\mathbb{R})^{\otimes 2} \otimes L^2(\mathbb{R}) \otimes (\mathbb{C}^2)^{\otimes 3} \end{aligned}$$

defined in terms of the symmetrically squeezed GKP-code $\mathcal{GKP}^\star_\kappa[2^\ell]$, we have

$$\mathsf{err}_{\mathcal{L}}\left(W_U, J_\ell^{\otimes 2}U\left(J_\ell^{\otimes 2}\right)^{-1}\right) \leqslant 150\ell\kappa. \tag{E1}$$

(ii) Let $\ell \in \mathbb{N}$ and $\Delta \in (0, 2^{-\ell})$. For the code space

$$\begin{aligned} \mathcal{L} = (\mathrm{III}\mathcal{GKP}^\star_\Delta[2^\ell])^{\otimes 2} \otimes \mathbb{C}\left(\left|\mathrm{III}^{2^{-(\ell+1)}}_{L_{\Delta,2^\ell},\Delta}(0)_2\right\rangle \otimes |0\rangle^{\otimes 3}\right) \\ \times \subset L^2(\mathbb{R})^{\otimes 2} \otimes L^2(\mathbb{R}) \otimes (\mathbb{C}^2)^{\otimes 3} \end{aligned}$$

defined in terms of the rectangular-envelope GKP-code $\mathrm{III}\mathcal{GKP}^\star_\Delta[2^\ell]$ and $L_{\Delta,2^\ell} = 2^{2(\lceil \log_2 1/\Delta \rceil - \ell)}$, we have

$$\mathsf{err}_{\mathcal{L}}\left(W_U, J_\ell^{\otimes 2}U\left(J_\ell^{\otimes 2}\right)^{-1}\right) \leqslant 2600 \times 2^{2\ell}\Delta. \tag{E2}$$

At first sight, the bounds on the logical gate error from Theorem E3 suggest a different scaling for symmetrically squeezed Gaussian-envelope and rectangular-envelope GKP codes. This is not the case. We note that the bound for the logical gate error of an implementation using symmetrically squeezed Gaussian-envelope GKP codes in Eq. (E1) requires $\kappa = O(1/\mathsf{poly}(\ell))$ in order to be nontrivial. When expressed in terms of the corresponding parameter $\Delta = \kappa/(2\pi \times 2^\ell)$, this implies that $\Delta = 1/\exp[\Omega(\ell)]$. This is in accordance with the scaling suggested in Eq. (E2) for rectangular-envelope GKP codes.

We note that—again by the composability of the logical gate error, Theorem E3 implies an analogous statement for an (tensor product) encoding of $m\ell$ qubits into a code space of the form

$$\begin{aligned} \mathcal{L} = (\mathcal{GKP}^\star_\kappa[2^\ell])^{\otimes m} \otimes \mathbb{C}\left(\left|\mathsf{GKP}^{2^{-(\ell+1)}}_\kappa(0)_2\right\rangle \otimes |0\rangle^{\otimes 3}\right) \\ \subset L^2(\mathbb{R})^{\otimes m} \otimes L^2(\mathbb{R}) \otimes (\mathbb{C}^2)^{\otimes 3}, \end{aligned}$$

respectively,

$$\begin{aligned} \mathcal{L} = (\mathrm{III}\mathcal{GKP}^\star_\Delta[2^\ell])^{\otimes m} \otimes \mathbb{C}\left(\left|\mathrm{III}^{2^{-(\ell+1)}}_{L_{\Delta,2^\ell},\Delta}(0)_2\right\rangle \otimes |0\rangle^{\otimes 3}\right) \\ \subset L^2(\mathbb{R})^{\otimes m} \otimes L^2(\mathbb{R}) \otimes (\mathbb{C}^2)^{\otimes 3}. \end{aligned}$$

That is, any two-qubit unitary U acting on any pair of logical qubits can be implemented with $340\ell^2$ elementary operations and a logical gate error bounded as stated in Theorem E3.

Proof. The proof of statement (ii) is analogous to the proof of (i) and relies on Corollary D1. We thus focus on the proof of Claim (i).

Let us denote the three bosonic modes by B_1, B_2, and $\mathsf{B}_{\mathsf{aux}}$, respectively. We note that $\mathcal{GKP}^{\star}_{\kappa}[2^{\ell}]^{\otimes 2} \cong (\mathbb{C}^2)^{\otimes 2\ell}$, hence the code space $\mathcal{L}$ indeed encodes 2ℓ qubits. Let us write

$$\begin{aligned} Q_1 \cdots Q_{2\ell} &:= (A_0 \cdots A_{\ell-1})(B_0 \cdots B_{\ell-1}) \\ &= (\mathbb{C}^2)^{\otimes \ell} \otimes (\mathbb{C}^2)^{\otimes \ell}, \end{aligned}$$

where the qubits $A_0 \cdots A_{\ell-1}$ are encoded in the code space $\mathcal{GKP}^{\star}_{\kappa}[2^{\ell}])$ of the first mode B_1, whereas the qubits $B_0 \cdots B_{\ell-1}$ are encoded analogously in the second mode B_2.

Consider a pair $Q_\alpha Q_\beta$ of (logical) qubits, where $\alpha < \beta$. We distinguish between the following three cases:

(i) $Q_\alpha Q_\beta = A_j A_k$ for some $j, k \in \{0, \ldots, \ell - 1\}$, $j < k$: Here the (logical) gate U acts exclusively on qubits encoded in the first mode. We can thus use Theorem 8, i.e., the implementation W_U illustrated in Fig. 5. We apply this implementation to the first mode B_1, the auxiliary bosonic mode $\mathsf{B}_{\mathsf{aux}}$ and the three qubits (while leaving the second mode B_2 untouched). By Theorem 8, this implementation uses at most $340\ell^2$ elementary operations belonging to $\mathcal{U}^{3,3}_{\mathsf{elem}}$, and has a gate error upper bounded by $400\ell\kappa$. The claim follows from this because the logical gate error is composable, i.e., stable under tensoring in an additional system (the bosonic mode B_2).

(ii) $Q_\alpha Q_\beta = B_j B_k$ for some $j, k \in \{0, \ldots, \ell - 1\}$, $j < k$: Here we can proceed analogously, applying the construction of Theorem 8 (see Fig. 5) to the second mode B_2 instead, while leaving the mode B_1 untouched.

(iii) $Q_\alpha Q_\beta = A_j B_k$ for some $j, k \in \{0, \ldots, \ell - 1\}$, $j < k$: Here we have to implement a two-qubit unitary U where the first qubit is encoded in the mode B_1, whereas the second qubit is encoded in the mode B_2. We can use the construction from Lemma E2, see Fig. 7. The proof of the corresponding upper bound on the gate error proceeds analogously as the proof of Theorem 8 and is omitted here. ■

[1] L. Brenner, B. Dias, and R. Koenig, Composable logical gate error in approximate quantum error correction: Reexamining gate implementations in Gottesman-Kitaev-Preskill codes, arXiv:2509.14658.

[2] K. Banaszek and K. Wódkiewicz, Testing quantum nonlocality in phase space, Phys. Rev. Lett. **82**, 2009 (1999).

[3] J. Wenger, M. Hafezi, F. Grosshans, R. Tualle-Brouri, and P. Grangier, Maximal violation of Bell inequalities using continuous-variable measurements, Phys. Rev. A **67**, 012105 (2003).

[4] J. Etesse, R. Blandino, B. Kanseri, and R. Tualle-Brouri, Proposal for a loophole-free violation of Bell's inequalities with a set of single photons and homodyne measurements, New J. Phys. **16**, 053001 (2014).

[5] N. C. Menicucci, P. van Loock, M. Gu, C. Weedbrook, T. C. Ralph, and M. A. Nielsen, Universal quantum computation with continuous-variable cluster states, Phys. Rev. Lett. **97**, 110501 (2006).

[6] M. Gu, C. Weedbrook, N. C. Menicucci, T. C. Ralph, and P. van Loock, Quantum computing with continuous-variable clusters, Phys. Rev. A **79**, 062318 (2009).

[7] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Gaussian quantum information, Rev. Mod. Phys. **84**, 621 (2012).

[8] J. Niset, J. Fiurášek, and N. J. Cerf, No-go theorem for Gaussian quantum error correction, Phys. Rev. Lett. **102**, 120501 (2009).

[9] C. Vuillot, H. Asasi, Y. Wang, L. P. Pryadko, and B. M. Terhal, Quantum error correction with the toric Gottesman-Kitaev-Preskill code, Phys. Rev. A **99**, 032344 (2019).

[10] J. Eisert, S. Scheel, and M. B. Plenio, Distilling Gaussian states with Gaussian operations is impossible, Phys. Rev. Lett. **89**, 137903 (2002).

[11] J. Fiurášek, Gaussian transformations and distillation of entangled Gaussian states, Phys. Rev. Lett. **89**, 137904 (2002).

[12] G. Giedke and J. I. Cirac, Characterization of Gaussian operations and distillation of Gaussian states, Phys. Rev. A **66**, 032316 (2002).

[13] D. Gottesman, A. Kitaev, and J. Preskill, Encoding a qubit in an oscillator, Phys. Rev. A **64**, 012310 (2001).

[14] T. Matsuura, N. C. Menicucci, and H. Yamasaki, Continuous-variable fault-tolerant quantum computation under general noise, Nat. Commun. **17**, 1709 (2026) .

[15] I. Tzitrin, J. E. Bourassa, N. C. Menicucci, and K. K. Sabapathy, Progress towards practical qubit computation using approximate Gottesman-Kitaev-Preskill codes, Phys. Rev. A **101**, 032315 (2020).

[16] M. H. Shaw, A. C. Doherty, and A. L. Grimsmo, Logical gates and read-out of superconducting Gottesman-Kitaev-Preskill qubits, arXiv:2403.02396.

[17] M. H. Shaw, A. C. Doherty, and A. L. Grimsmo, Stabilizer subsystem decompositions for single- and multimode Gottesman-Kitaev-Preskill codes, PRX Quantum **5**, 010331 (2024).

[18] M. E. Shirokov, On the energy-constrained diamond norm and its application in quantum information theory, Probl. Inf. Transm. **54**, 20 (2018).

[19] G. Pantaleoni, B. Q. Baragiola, and N. C. Menicucci, Modular bosonic subsystem codes, Phys. Rev. Lett. **125**, 040501 (2020).

[20] I. Rojkov, P. M. Röggla, M. Wagener, M. Fontboté-Schmidt, S. Welte, J. Home, and F. Reiter, Two-qubit operations for finite-energy Gottesman-Kitaev-Preskill encodings, Phys. Rev. Lett. **133**, 100601 (2024).

[21] L. Brenner, L. Caha, X. Coiteux-Roy, and R. Koenig, Complexity of Gottesman-Kitaev-Preskill states, Phys. Rev. X **15**, 031073 (2025).

[22] Y. Liu, S. Singh, K. C. Smith, E. Crane, J. M. Martyn, A. Eickbusch, A. Schuckert, R. D. Li, J. Sinanan-Singh, M. B. Soley, T. Tsunoda, I. L. Chuang, N. Wiebe, and S. M. Girvin, Hybrid oscillator-qubit quantum processors: Instruction set architectures, abstract machine models, and applications PRX Quantum **7**, 010201 (2026).

[23] L. Brenner, L. Caha, X. Coiteux-Roy, and R. Koenig, Factoring an integer with three oscillators and a qubit, Nat. Commun. **17**, 227 (2026).

[24] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, Cambridge, 2010).