Trapping-potential dependence of the unitary Fermi gas at the BCS-BEC crossover

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Cold-atom experiments that measure Fermi-gas properties near unitarity confine fermionic atoms to a region of space using trapping potentials of various shapes. The presence of a trapping potential introduces a new characteristic physical scale in the superfluid effective-field theory (EFT) description of the unitary Fermi gas which, among other things, describes the acoustic branch of excitations in the far infrared well below the scale of the superfluid gap. In this EFT there is a clear hierarchy of scales, and corrections to the homogeneous system due to the trapping potential may be organized into three regions with distinct power counting that relies on both the EFT derivative expansion and the Wentzel–Kramers–Brillouin approximation, which is an expansion in gradients of the trapping potential. The energy spectrum of the superfluid system is obtained in each of the regions by explicit computation of the phonon-field fluctuations and by the modifications to the dynamic structure factor due to the corresponding density fluctuations. This work presents a systematic and quantitative method for treating the presence of a trapping potential, which is essential for interpreting experimental realizations of the unitary Fermi gas. It provides clear predictions for how the trapping potential modifies the dispersion relation's curvature, a key factor in characterizing the relaxation mechanisms of the superfluid. The most significant deviations from linear dispersion due to the trapping potential are found in the far-infrared region of the superfluid EFT.

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I. INTRODUCTION

A gas of spin-1/2 fermionic atoms, whose effective two-body interactions may be tuned using Feshbach resonances, at and near unitarity, provides a remarkable strongly interacting many-body system that can be studied experimentally to a high degree of accuracy using cold-atom techniques, while also being tractable theoretically due to the large degree of symmetry and clear scale separation [1]. From the perspective of the fundamental field theoretic formulation of the strongly interacting fermion system, the absence of a small expansion parameter implies the necessity of nonperturbative methods for a first-principles computation of observables. So, for instance, the energy density of the system, characterized by the so-called Bertsch parameter, is computed using Quantum Monte Carlo (QMC) simulations [2] and measured in cold-atom experiments [3,4]. However, at unitarity and indeed over the entire BCS-BEC crossover region, where the large scattering length approaches infinity (unitarity) from positive (BEC) and negative (BCS) values, the infrared degrees of freedom are superfluid and are therefore described by a single Goldstone boson (phonon) degree of freedom [5–7]. At unitarity, the effective-field theory (EFT) of the Goldstone mode is Schrödinger invariant and therefore is a nonrelativistic conformal field theory (NRCFT) which admits powerful techniques for computing observables such as the state-operator correspondence [8] and the large-charge expansion [9–18], where an EFT can be formulated as an expansion in powers of $1/Q$, where Q is the particle number. This is the approach that will be adopted in the following. The conformal data that are encoded in the low-energy constants of the EFT—including the Bertsch parameter—is external to the EFT and must be obtained via theoretical modeling, experimental measurement, and/or numerical simulation. Promising systematic methods for obtaining these data include the so-called ϵ expansion [19], and the large-N expansion [15], where N is the number of fermion flavors. At leading order in the $1/N$ expansion, this model reproduces well-known mean-field theory results, but has the advantage of providing arbitrary systematic improvements in a perturbative expansion in powers of $1/N$.

A main focus of this paper is on the phonon dispersion relation, which describes the acoustic excitation branch of the superfluid throughout the BCS-BEC crossover. In the absence of a confining potential, the dispersion relation may be written as [20]

$$q_0(q) = c_s q\left(1 + \frac{\gamma}{8c_s^2}q^2 + O(q^4 \log q)\right). \tag{1.1}$$

Here q_0 is the energy, q is the momentum transfer, c_s is the speed of sound, and γ is a parameter which governs the deviation of the spectrum from linearity. The form of the dispersion relation is a simple consequence of the spontaneously broken particle number in the superfluid phase [6]. Throughout we work in units where $\hbar = M = 1$, where M is the atomic mass. At unitarity, the EFT is a NRCFT, constrained by Schrödinger invariance, with the sole scale given by the chemical potential, μ, which is generated by spontaneous symmetry breaking. Hence, the expansion of the dispersion relation given in Eq. (1.1) is in q relative to the ultraviolet (UV) scale μ.

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Observables of special interest in the superfluid EFT at unitarity are correlation functions at fixed particle-number charge Q. In this paper, the focus is on the fluctuations of the Goldstone boson field relative to the scale μ of the fermion chemical potential, which is fixed by the number of particles that play the role of a $U(1)$ charge. Working at large charge directly translates to working at large μ.[1]

The sign of γ is essential for understanding the gas relaxation mechanism at low temperature. So, for instance, if $\gamma > 0$ (a convex branch) then the three-phonon Beliaev–Landau process is kinematically allowed, whereas if $\gamma < 0$ (a concave branch) then phonon damping can occur only at $T \neq 0$ via the as-yet unobserved four-phonon Landau–Khalatnikov process [21]. A recent experiment using Bragg spectroscopy [22] has found $\gamma_{\rm expt} = -0.085(8)$ (a concave branch). However, this result has been questioned in Refs. [20,23] based on potentially uncontrolled finite-temperature effects and a fitting-range bias that favors higher momentum transfers.

Atomic physics experiments that explore the BCS-BEC crossover suffer from nontrivial experimental influences, which must be carefully considered in extracting the dispersion-relation parameters. In particular, the fermionic atoms are contained in a trapping potential at a finite temperature. The impact of these environmental scales on the many-body system must be understood quantitatively in order to correctly interpret the implications of the experimental data for the zero-temperature homogeneous system. This paper will focus on developing a systematic EFT analysis of the effect of the trapping potential on the phonon fluctuations at zero temperature.

The presence of a trapping potential, here denoted V, breaks the translational symmetry of the spatially uniform system and introduces a new scale in the superfluid EFT, which can be taken as the size of the confined "droplet" or "cloud" of superfluid matter, defined as the distance scale at which the leading order of the density vanishes. The EFT hierarchy may then be set up to probe distance scales that are larger than the typical interparticle separation fixed by the density, but less than the size of the confined droplet. A standard tool for dealing with spatially dependent densities is the local density approximation (LDA) [24], which amounts to shifting the fermion chemical potential by V so that it becomes spatially dependent. Here, we use the superfluid EFT to investigate, systematically, corrections to the LDA that account for the effect of the trapping potential on the dispersion relation that describes the acoustic branch of excitations in the superfluid. In practice, this amounts to solving the phonon equation of motion in the presence of fluctuations on top of the homogeneous ground state. This turns out be achievable for an arbitrary (static) potential, V, using the Wentzel–Kramers–Brillouin (WKB) approximation together with the homogeneous EFT power counting.

As the potential V breaks spatial translations, the three-momentum components are no longer good quantum numbers in the trap. Strictly speaking, the dispersion relation, which relates energy to momentum, is not defined. A natural way to connect to atomic physics measurements is to instead consider the dynamic structure factor S, which is naturally defined as a matrix element of local operators in the superfluid EFT. In translation-invariant systems, S is a delta function with support on the curve of the dispersion relation. We can define an approximate dispersion relation as the curve in the energy-momentum plane along which the dynamic structure factor is peaked. The flatter the potential, the better the dispersion relation is approximated.

Unlike the dispersion relation, the energy spectrum of the fluctuation is well-defined because of time-translation invariance. Due to the compactness of the system, the spectrum is discrete, as will be proven below (see Sec. IV). In general, the spectrum and the dispersion relation carry distinct information. They have the same functional form only in translation-invariant systems.

For purposes of clarity, it is worth giving an overview here of some of the main points that should be kept in mind while reading this paper. Our goal in what follows is to describe the small density fluctuations $\delta\rho$ of the fermion cloud. First, consider the symmetries. The trap is time independent and spherically symmetric, so it only depends on the distance from the center. The system preserves time-translation invariance and rotational invariance. This implies that $\delta\rho$ can be decomposed into modes of the form

$$\delta\rho_{nlm} = e^{iq_0(n,\ell)t} f_{n\ell}(r) Y_{\ell m}(\theta,\phi), \quad \ell = 0, 1, \ldots, \quad m = -\ell, \ldots, \ell, \tag{1.2}$$

where $Y_{\ell m}$ are spherical harmonics. The function $q_0 = q_0(n,\ell)$ identifies the allowed values for the energy of the fluctuations: it is the spectrum. If the potential is confining, then the spectrum is discrete and the parameter n is an integer. The information contained in the radial function $f_{n\ell}(r)$ can be extracted by defining the response function

$$\chi(t-t';\mathbf{r},\mathbf{r}') = \langle \delta\rho(t,\mathbf{r})\delta\rho(t',\mathbf{r}')\rangle_T, \tag{1.3}$$

where $\langle \ldots \rangle_T$ represents time ordering.

In the special case of invariance under translations in space, the radial function is an exponential (or a sum of exponentials as needed to satisfy the boundary conditions), and the Fourier transform from $\mathbf{r}$ to $\mathbf{q}$ is proportional to a delta function that fixes a linear relation between q and n. For example, for a spherical box of radius R_{cl}, with zero potential inside the box and infinite potential outside

$$q(n) = \frac{\pi n}{R_{cl}}. \tag{1.4}$$

When the dynamic structure factor, which is related to the absorptive part of the response function, is a sum of delta functions, the spectrum equation $q_0 = q_0(n)$ has the same functional form as the dispersion relation $q_0 = q_0(q)$, and one is justified in thinking of n as a discrete momentum.

By contrast, with broken spatial translations, the spectrum and dispersion relation contain distinct information. However, if we compute the fluctuations in a WKB expansion, which is equivalent to a gradient expansion in the potential, at leading order, $f_{nl}(r)$ will take the same form as in the case of no potential, but with the substitution of the fermion chemical

[1]Operationally this corresponds to a change of ensemble from Canonical (particle number specified) to Grand Canonical (chemical potential specified).

potential, $\mu \to \mu - V(r)$. This is nothing other than the LDA. The Fourier integral will not give a delta function, but some appropriately peaked smooth function. And the relationship between spectrum and dispersion relation remains true only at leading order (LO).

Our main findings on the dependence of the spectrum and dispersion relation on the trapping potential are as follows:

(1) In first approximation, the behavior of the system depends on the details of the trapping potential close to the center of the trap.

(2) The spectrum is more dense for steep potentials [k smaller for $V(r) \sim r^{2k}$, $k \neq 0$] for fixed size of the cloud R_{cl}. The flatter the potential in the center (k larger), the more the special case of a flat spherical box potential is approximated and the states become equidistant. Asymptotically, for $k \to \infty$, we reproduce the spectrum of the spherical box $q_0 = \sqrt{2\mu/3}\pi n/R_{cl}$.

(3) The dynamic structure factor is more peaked for flatter potentials. In turn this means that the dispersion relation is more precise.

(4) At low energies [$q_0 \lessapprox O(\sqrt{\mu}/R_{cl})$], the dispersion relation receives corrections that can be concave. The leading effect depends on the details of the trap and is suppressed for flatter potentials.

(5) At high energies [$q_0 \gtrapprox O(\sqrt{\mu}/R_{cl})$], the dispersion relation receives a correction that depends on the low-energy parameters in the EFT and not on the details of the trap. Given the present knowledge of these parameters (that cannot be computed within the EFT framework), this correction is always convex.

This paper is organized as follows: Section II introduces the superfluid EFT and reviews what is currently known about the low-energy constants that enter up to next-to-leading order (NLO) in the homogeneous system. A scale hierarchy in the presence of the trapping potential is constructed in Sec. III. The fluctuation spectrum and the dynamic structure factor are investigated in the various tractable regions in Secs. IV and V, respectively. Conclusions are given in Sec. VI. Finally, in the Appendix we provide a derivation of the dynamic structure factor and response function in the (second-quantized) path-integral formalism.

II. SUPERFLUID EFFECTIVE-FIELD THEORY

The unitarity limit. Our starting point is the EFT that describes the low-energy dynamics of a superfluid at unitarity. As originally observed in Refs. [5–7], the infrared (IR) degree of freedom (DOF) is a Goldstone field θ controlled by the action[2]

$$\mathcal{L} = c_0 X^{5/2} + c_1 X^{-1/2}(\nabla X)^2 + c_2 X^{1/2}[(\Delta\theta)^2 - 3(\nabla \otimes \nabla\theta)^2] + \cdots, \tag{2.1}$$

where the dots are operators with more derivatives, and

$$X = \dot{\theta} - \tfrac{1}{2}(\nabla\theta)^2 - V, \tag{2.2}$$

with $\dot{\theta} \equiv \partial_t\theta$, is a Galilean-invariant combination that frequently appears as a building block in nonrelativistic theories in which the external potential V also appears. It is convenient to define the shifted field,

$$\theta(t, \mathbf{r}) = \mu t + \pi(t, \mathbf{r}), \tag{2.3}$$

where μt is the time-dependent vacuum expectation value (VEV) that appears in the classical ground-state and $\pi(t, \mathbf{r})$ is the (phonon) fluctuation. The particle-number density, ρ, is the variable conjugate to the Goldstone field and is given by

$$\rho(t, \mathbf{r}) = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{\partial\mathcal{L}}{\partial X}. \tag{2.4}$$

The c_i are dimensionless low-energy constants that are inputs to the EFT. Note that

$$c_0 = \frac{2^{5/2}}{15\pi^2\xi^{3/2}}, \tag{2.5}$$

with ξ the Bertsch parameter whose value is available from experiment, $\xi = 0.376(4)$ [$c_0 = 0.168(3)$)][3] and from Monte Carlo simulations $\xi = 0.372(5)$ [$c_0 = 0.166(3)$] [2]. As the static transverse response function depends solely on c_2, there is a strict negativity bound $c_2 < 0$ [6]. There is no direct knowledge of c_2 from experiment or simulation. By contrast, the parameter c_1 is directly related to the static longitudinal response function and the energy of Q fermions in a trap, which is proportional to the large-charge conformal dimension [6,11,17,25,26]

$$\Delta_Q(Q) = \frac{3^{4/3}}{4}\xi^{1/2}Q^{4/3} - 3^{2/3}\sqrt{2}\pi^2\xi\, c_1\, Q^{2/3} + O(Q^{5/9}) + \cdots + \frac{1}{3\sqrt{3}}\log Q, \tag{2.6}$$

where the universal Casimir correction, corresponding to the $Q^0\log(Q)$ term, has been included [26]. There have been studies of the energy of Q fermions in a trap using many methods [27–30], and one may fit both ξ and c_1 given sufficiently precise data. Figure 1 shows simulation data over a range $Q = 8$–80 from diffusion MC (DMC) and auxiliary-field MC (AFMC) [29]. Here, fitting to the DMC data of Ref. [29] over the range $Q = 10$–80 yields $c_1 = -0.011(1)$ and $\xi = 0.416(7)$ $c_0 = 0.143(3)$].

There are various mean-field-related determinations of the low-energy constants,[3] all of which are in agreement [7,15,31] and find $\xi = 0.5906$ ($c_0 = 0.0842$), $c_1 = -0.0153$, and $c_2 = -0.0035$. In the ϵ expansion [32], $c_1 = -0.0209$, $c_2 = O(\epsilon^2)$.[4] It is noteworthy that the mean-field value of c_1 is in excellent agreement with the Monte Carlo simulation data,

[2]Note that the form of the next-to-leading (NLO) corrections in Eq. (2.1) differs from that given in Ref. [6]. The correct—conformally invariant—operator structure is given in Ref. [7], which noted that Ref. [6] made an invalid use of the leading-order equations of motion to remove an operator that is necessary to maintain conformal invariance at NLO.

[3]Leading order in the large-N expansion reproduces mean-field theory.

[4]Note that ϵ-expansion conclusions regarding c_2, associated with the new operator in Eq. (2.1), are unchanged.

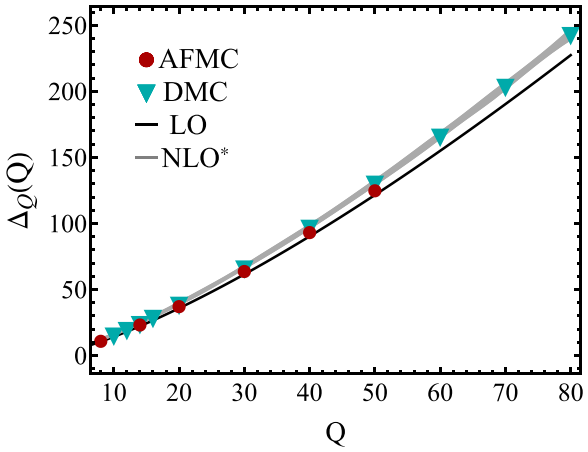

FIG. 1. Fit of DMC simulation data [29] over the range $Q = 8$–80, as described in the text. The solid black line is leading-order in the large-charge expansion, and the gray band is the NLO fit, including the Casimir correction. The simulation data are as described in the text. (Note that the raw data have been provided by the authors of Ref. [29].)

while the Bertsch parameter is significantly overestimated by the mean-field value.[5]

The dispersion-relation parameters in Eq. (1.1) are found to be [6,11]

$$c_s = \sqrt{\frac{2\mu}{3}} = \sqrt{\frac{\xi}{3}}k_F, \quad \gamma = \frac{64}{45}\frac{3c_2 - c_1}{c_0}, \tag{2.7}$$

where k_F is the Fermi momentum. It is important to emphasize that at unitarity, the speed of sound is determined by symmetry.

III. SCALE SEPARATION AND PERTURBATIVE EXPANSIONS

Strategy. In the homogeneous system, the EFT power counting is simple: all observable quantities are systematically computable for momentum transfers well below the scale of the superfluid gap, which, at unitarity, is set by the chemical potential μ (times a dimensionless number of order one) [6]. The presence of the trapping potential introduces a new scale—roughly the size of the fermion cloud—which in turn implies three distinct regions of interest with the homogeneous power counting enhanced by the WKB expansion.

The small parameter ϵ. To identify the controlling parameters, it is convenient to formulate dimensionless variables. A natural length scale is provided by the classical radius of the particle cloud, i.e., the distance from the origin at which the charge density vanishes in the LO EFT,

$$\mu - V(R_{cl}) = 0. \tag{3.1}$$

We can use R_{cl} to define the dimensionless variable

$$u = r/R_{cl}. \tag{3.2}$$

The energies are naturally measured in units of the chemical potential μ, so we introduce

$$V = \mu\bar{V}. \tag{3.3}$$

In terms of the new variables, the action at the saddle becomes

$$\mathcal{L} = \mu^{5/2}\left[c_0(1-\bar{V})^{5/2} + c_1\frac{1}{R_{cl}^2\mu}(\nabla_u\bar{V})^2\right], \tag{3.4}$$

which shows that in the large-charge expansion, which is equivalent to $\mu \gg (\nabla V)^2/V^2$, the subleading term is controlled by the parameter

$$\epsilon^2 = \frac{2}{R_{cl}^2\mu}, \tag{3.5}$$

as already observed in Ref. [15] (see also Ref. [16] for a careful analysis of the scales involved in the problem).

In this parametrization, the coefficient c_2 does not contribute to the action at the saddle and does not enter, e.g., in the expression for the conformal dimension of the lowest operator of fixed charge. Conversely, in the following, we show that it does contribute at the same order as c_1 to the dynamics of the fluctuations.

It is convenient to rewrite ϵ as the ratio of a low-energy scale and a high-energy scale, introducing ϖ such that

$$\varpi = \frac{\sqrt{2\mu}}{R_{cl}} \text{ and } \epsilon = \frac{\varpi}{\mu}. \tag{3.6}$$

This parameter generically depends both on the confining potential and the chemical potential. For a power-law potential of the form $V \propto r^{2k}$, we can write

$$V = \mu\left(\frac{r}{R_{cl}}\right)^{2k} = \frac{\varpi^{2k}}{2^k\mu^{k-1}}r^{2k}. \tag{3.7}$$

In the special case of a harmonic trap $V = \omega^2 r^2/2$, we have $\varpi = \omega$ and the dependence on μ of the low-energy scale drops out entirely. In what follows, potentials with $k > 1$ are referred to as "superharmonic."

The small parameter η. Expanding the action up to second order in the field, we obtain the action that encodes the small fluctuations which are the subject of this work:

$$\begin{aligned}\mathcal{L}^{(2)}[\pi] = &-\frac{5}{8}c_0\sqrt{\mu - V}[2(\mu - V)(\nabla\pi)^2 - 3\dot{\pi}^2] \\ &+ c_1\left[\frac{(\nabla\dot{\pi})^2}{\sqrt{\mu - V}} + \frac{\nabla V\cdot\nabla\dot{\pi}^2}{2(\mu - V)^{3/2}}\right. \\ &\left. + \frac{(\nabla V)^2[2(\mu - V)(\nabla\pi)^2 + 3\dot{\pi}^2]}{8(\mu - V)^{5/2}}\right] \\ &+ c_2\sqrt{\mu - V}[(\Delta\pi)^2 - 3(\nabla\otimes\nabla\pi)^2].\end{aligned} \tag{3.8}$$

One qualitative difference with respect to the analysis of the ground state is that, together with the low and high energy scales ϖ and μ, a third scale appears, namely, the typical energy of the fluctuations q_0, which we use to rescale the time variable:

$$\bar{t} = q_0 t. \tag{3.9}$$

[5]A similar observation is made in the relativistic analog of this problem, the large-charge effective theory for the $O(2)$ model. While the low-energy coefficient of the leading operator in the EFT computed at large N has an error of order 20% with respect to the lattice result, the large-N result for the NLO correction in the EFT lies within the error bars of the lattice computation [33–35].

Expressed in terms of dimensionless quantities, the action now takes the form

$$\mathcal{L}^{(2)}[\pi] = -\frac{5}{8}c_0\frac{\mu^{3/2}}{R_{cl}^2}\sqrt{1-\bar{V}}\left[2(1-\bar{V})(\nabla_u\pi)^2 - 3\frac{q_0^2R_{cl}^2}{\mu}\left(\frac{\partial\pi}{\partial\bar{t}}\right)^2\right] + c_2\frac{\mu^{1/2}}{R_{cl}^4}\sqrt{1-\bar{V}}[(\Delta_u\pi)^2 - 3(\nabla_u\otimes\nabla_u\pi)^2]$$
$$+ c_1\frac{\mu^{1/2}}{R_{cl}^4}\left[\frac{(\nabla_u\bar{V})^2}{4(1-\bar{V})^{3/2}}(\nabla_u\pi)^2 + \frac{q_0^2R_{cl}^2}{\mu}\left(\frac{\left(\nabla_u\frac{\partial\pi}{\partial\bar{t}}\right)^2}{\sqrt{1-\bar{V}}} + \frac{\nabla_u\bar{V}\cdot\nabla_u\left(\frac{\partial\pi}{\partial\bar{t}}\right)^2}{2(1-\bar{V})^{3/2}} + \frac{3\left(\frac{\partial\pi}{\partial\bar{t}}\nabla_u\bar{V}\right)^2}{8(1-\bar{V})^{5/2}}\right)\right]. \tag{3.10}$$

Comparing the different terms we see the following:

(1) the terms proportional to c_1 and c_2 are both by μR_{cl}^2 parametrically smaller than the corresponding terms proportional to c_0, as it was the case of the action at the saddle.

(2) The two terms proportional to c_0 scale in the same way if we take $q_0 = O(\sqrt{\mu}/R_{cl})$. The same is true for all the terms proportional to c_1 and c_2.

The conclusion of this analysis is that we have a consistent expansion if at leading order $q_0 \sim \sqrt{\mu}/R_{cl}$ and the controlling parameter is

$$\eta = \frac{q_0}{\mu}. \tag{3.11}$$

Then the leading term is controlled by the low-energy constant c_0 and the NLO correction is proportional to c_1 and c_2, which contribute both at the same order.

The small parameter δ. The above is, however, not the only expansion that we need to use: even if we limit ourselves to the leading order in the EFT, the equations of motion (EOM) cannot, in general, be solved exactly. They take the form

$$\frac{1}{3}\frac{2\mu}{q_0^2R_{cl}^2}\nabla_u[(1-\bar{V})^{3/2}\nabla_u\pi] = \sqrt{1-\bar{V}}\frac{\partial^2\pi}{\partial\bar{t}^2}. \tag{3.12}$$

Since the system is invariant under time translation we can always write the solution in the form

$$\pi(t,\mathbf{u}) = e^{iq_0t}\pi(\mathbf{u}). \tag{3.13}$$

The EOM then become

$$\frac{2\mu}{(q_0R_{cl})^2}\nabla_u[(1-\bar{V})^{3/2}\nabla_u\pi(\mathbf{u})] + 3\pi(\mathbf{u})\sqrt{1-\bar{V}} = 0. \tag{3.14}$$

We now have a differential equation in which the leading derivative is multiplied by a coefficient δ^2, where we have defined

$$\delta = \frac{\sqrt{2\mu}}{q_0R_{cl}} = \frac{\varpi}{q_0}. \tag{3.15}$$

In the limit in which the typical energy of the fluctuations is much larger than the low-energy scale, $\delta \ll 1$ and the EOM lends itself to a WKB expansion. We now have our second control parameter.

Regimes. The final result of the above considerations is that we have identified three scales: ϖ, q_0, and μ. We assume scale separation:

$$\varpi \ll q_0 \ll \mu. \tag{3.16}$$

From these scales we can construct two dimensionless parameters to realize a double expansion:

$$\eta = \frac{q_0}{\mu} \text{ controlling the EFT,} \tag{3.17}$$

$$\delta = \frac{\varpi}{q_0} \text{ controlling the WKB expansion.} \tag{3.18}$$

The product $\eta\delta = \varpi/\mu = \epsilon \ll 1$ is the parameter that controls the perturbative large-charge expansion of the ground state.

We can distinguish two regimes:

(1) If $\eta \ll \delta \ll 1$ (i.e., $\varpi \ll q_0 \ll \sqrt{\varpi\mu}$), we are in the low-energy regime in which the dynamics is described by the LO EFT and higher terms in the WKB expansion.

(2) If $\delta \ll \eta \ll 1$ (i.e., $\sqrt{\varpi\mu} \ll q_0 \ll \mu$), we are in the high-energy regime in which we can limit ourselves to the leading WKB terms but need to add higher orders in the EFT.

The two regimes are separated by a linear zone where $q_0 \approx \sqrt{\varpi\mu}$ in which one can use the LO EFT and LO WKB (see Fig. 2).

The WKB expansion. As we have seen above, in general, we can only find an approximate solution to the EOM using the WKB expansion. Here, we provide more detail. For ease of exposition, we consider a spherically symmetric potential $V = V(r)$ and focus on the low-energy regime, where we can confine ourselves to the LO in the EFT. Taking the time-translation and SO(3) rotational symmetries into account, the field takes the form

$$\pi(t,\mathbf{r}) = e^{iq_0t}\pi(r)Y_{\ell m}(\vartheta,\phi). \tag{3.19}$$

The EOM (3.12) now separates and, for the radial component in terms of the dimensionless variable u, we find

$$\delta^2\frac{\mathrm{d}}{\mathrm{d}u}\left(u^2\left(1-\bar{V}\right)^{3/2}\frac{\mathrm{d}}{\mathrm{d}u}\pi(u)\right) - \delta^2\ell(\ell+1)(1-\bar{V})^{3/2}\pi(u)$$
$$+ 3u^2\sqrt{1-\bar{V}}\pi(u) = 0. \tag{3.20}$$

As we have observed above, the leading derivative is controlled by the parameter δ. In the limit $\delta \to 0$, we can look

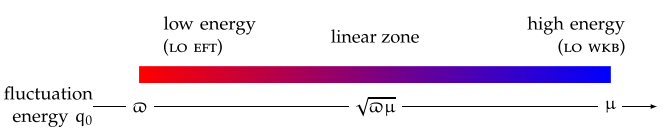

FIG. 2. Scale separation for the fluctuation energy q_0. The dynamics is controlled by the low-energy scale ϖ, fixed by the confining potential and the chemical potential μ, acting as a high-energy scale. In the intermediate region ($q_0 = O(\sqrt{\varpi\mu})$), the physics is well described by the physical optics approximation for the LO EFT and the measured dispersion is linear. At lower energies $\varpi \ll q_0 \ll \sqrt{\varpi\mu}$, higher orders in the WKB expansion are needed. At higher energies $\sqrt{\varpi\mu} \ll q_0 \ll \mu$, higher orders in the EFT are needed.

for a WKB perturbative solution of the form

$$\pi(u) = \exp\left[\frac{i}{\delta}\mathcal{S}_0(u) + \mathcal{S}_1(u) + i\delta\mathcal{S}_2(u) + \delta^2\mathcal{S}_3(u) + \cdots\right]. \tag{3.21}$$

The factors of $\sqrt{-1}$ in the ansatz are chosen such that all the equations are real. It is then apparent that the even functions $\mathcal{S}_{2n}$ control the frequency of the oscillations, while the odd ones $\mathcal{S}_{2n+1}$ control the amplitude of the fluctuations $\pi(u)$.

Expanding order by order, we find a hierarchy of algebraic equations for the derivatives $\mathcal{S}_p'(u)$. The leading (eikonal) equation is

$$[1 - \bar{V}(u)]\mathcal{S}_0'(u)^2 = 3 \tag{3.22}$$

and can be solved immediately for $\mathcal{S}_0'$. The next orders are

$$[3u\bar{V}' - 4(1-\bar{V})]\mathcal{S}_0' - 2u(1-\bar{V})(\mathcal{S}_0'' + 2\mathcal{S}_0'\mathcal{S}_1') = 0, \tag{3.23}$$

$$[3u\bar{V}' - 4(1-\bar{V})]\mathcal{S}_1' - 2u(1-\bar{V})[\mathcal{S}_1'' - 2\mathcal{S}_0'\mathcal{S}_2' + (\mathcal{S}_1')^2] + \frac{2}{u}\ell(\ell+1)(1-\bar{V}) = 0. \tag{3.24}$$

The angular momentum ℓ only appears in the equation for $\mathcal{S}_2$, and the general equation for $\mathcal{S}_{p+1}'$, $p = 2, 3, \ldots$, is

$$[3u\bar{V}' - 4(1-\bar{V})]\mathcal{S}_p' - 2u(1-\bar{V}) \times \left(\mathcal{S}_p'' + \sum_{k=0}^{\lfloor p/2\rfloor} I_{p,k}\mathcal{S}_k'\mathcal{S}_{p+1-k}' - \sin\left(\tfrac{3p\pi}{2}\right)(\mathcal{S}_{(p+1)/2}')^2\right) = 0, \tag{3.25}$$

where $I_{p,k} = 1 - (-1)^k + (-1)^p + (-1)^{k+p}$.

The WKB expansion can also be understood as a gradient expansion for the potential. From the structure of the equations, we see that $\mathcal{S}_0$ and $\mathcal{S}_1$ only depend on V, while $\mathcal{S}_2$ and $\mathcal{S}_3$ depend on the first and second derivatives V' and V'', and so on.

The same analysis can be performed for the WKB expansion of the NLO EOM, which must be used in the high-energy regime. Thanks to scale separation, the result is that at each order in the WKB hierarchy, there will be a new term, suppressed by $\eta = q_0/\mu$. We discuss this in detail in Sec. IV B.

IV. FLUCTUATION SPECTRUM

Our aim in this section is to compute the spectrum of the system in the presence of a trapping potential $V(r)$, i.e., the allowed values of the energy of small fluctuations. The natural expectation is for the spectrum to be discrete because of the confining nature of the potential. Here, we make this mathematically precise, showing that the operator that controls the fluctuation is self-adjoint both at LO and NLO in the EFT expansion. The EOM then turns into a Sturm–Liouville problem (LO) and its fourth-order generalization (NLO).

A. The linear regime

Let us start by considering the regime

$$\varpi \ll q_0 \ll \mu, \tag{4.1}$$

which—for reasons that will become clear in the following—we call the linear regime. In this approximation, both control parameters $\delta = \varpi/q_0$ and $\eta = q_0/\mu$ are very small, and the dynamics is well described by the LO EFT, whose EOM we can solve in the physical optics approximation (defined below).

Structure of the problem. The LO linearized EOM reads

$$3(\mu - V)^{1/2}\ddot{\pi} - \boldsymbol{\nabla}[2(\mu - V)^{3/2}\boldsymbol{\nabla}\pi] = 0. \tag{4.2}$$

The general structure becomes clearer if we rewrite the equation for a more general action $\mathcal{L}$, which is only a function of the Galilean-invariant object

$$X = \partial_t\theta - \tfrac{1}{2}(\nabla\theta)^2 - V. \tag{4.3}$$

In our case, $\mathcal{L}[X] = c_0X^{5/2}$. So we consider the equation

$$\mathcal{L}''[\mu - V]\ddot{\pi} - \boldsymbol{\nabla}(\mathcal{L}'[\mu - V]\boldsymbol{\nabla}\pi) = 0. \tag{4.4}$$

If the potential has spherical symmetry $V = V(r)$, the system is invariant under time translation and SO(3) rotations, so we can write $\pi(t, \mathbf{r})$ as (3.19). The EOM separates and reduces to an ordinary differential equation (ODE) for the radial mode $\pi(r)$:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\mathcal{L}'[\mu - V]\frac{\mathrm{d}}{\mathrm{d}r}\pi\right) - \mathcal{L}'[\mu - V]\ell(\ell+1)\pi = -r^2\mathcal{L}''[\mu - V]q_0^2\pi. \tag{4.5}$$

This is a standard Sturm–Liouville problem on the interval $0 < r < R_{cl}$ with eigenvalue q_0^2 and weight function

$$w(r) = r^2\mathcal{L}''[\mu - V(r)]. \tag{4.6}$$

It follows that the spectrum is discrete. For each allowed value of q_0^2 and fixed ℓ, there is a single eigenfunction $\pi_{n,\ell}$, and these eigenfunctions form an orthonormal basis for the Hilbert space L^2 with inner product

$$\langle n|m\rangle = \int_0^{R_{cl}} w(r)\mathrm{d}r\pi_n(r)\pi_m(r) = \delta_{nm}. \tag{4.7}$$

The weight function $w(r)$ has two parts. The r^2 factor is purely geometrical and corresponds to a three-dimensional system. The nontrivial part results from the fact that we are considering fluctuations in a nonhomogeneous medium.

A physical interpretation of the weight can be obtained as follows: For $V = 0$, the compressibility sum rule reads

$$\frac{\mathrm{d}^2\mathcal{L}}{\mathrm{d}\mu^2} = \frac{\rho}{c_s^2}, \tag{4.8}$$

where ρ is the density, c_s is the speed of sound, and we have identified the action at the saddle with the pressure as a function of the chemical potential. For the case of a confining potential, we can identify the weight function with the ratio of density and the square of the speed of sound, which are now both position dependent:

$$w(r) = r^2\frac{\rho(r)}{c_s^2(r)}. \tag{4.9}$$

This is precisely the same relation that one would have obtained in the LDA by trading the chemical potential μ in the $V = 0$ expressions for the combination $\mu - V(r)$. In our case,

it is, however, the result of a calculation and not of having imposed an approximation.

The same analysis can be performed for a cylinder geometry, where the system is confined in $x^2+y^2<R_{cl}^2$ and $0<z<L$, and the potential depends only on $r=(x^2+y^2)^{1/2}$. In this case, the fluctuations take the form

$$\pi(t,\mathbf{r})=e^{iq_0t}\sin\left(2\pi\frac{n_z z}{L}\right)e^{in_\vartheta\vartheta}\pi(r), \tag{4.10}$$

where n_z and n_ϑ are integers. The EOM is again a Sturm–Liouville problem for the radial field:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\mathcal{L}'[\mu-V]\frac{\mathrm{d}}{\mathrm{d}r}\pi\right)-\mathcal{L}'[\mu-V]\left(\frac{n_\vartheta^2}{r}+\frac{r}{L^2}(2\pi n_z)^2\right)\pi = -r\mathcal{L}''[\mu-V]q_0^2\pi. \tag{4.11}$$

The corresponding weight function only differs in the geometric factor since now the problem is effectively two-dimensional:

$$w(r)=r\mathcal{L}''[\mu-V]=r\frac{\rho(r)}{c^2(r)}. \tag{4.12}$$

The geometry introduces yet another new scale L, which generically changes the EFT description.

Physical optics approximation for the LO EFT. Having understood the general structure of the problem, we can try to find an approximate solution. In the linear regime, it is appropriate to use the physical optics approximation (the two leading WKB equations). The equations to solve are

$$[1-\bar{V}(u)]\mathcal{S}_0'(u)^2=3, \tag{4.13}$$

$$[3u\bar{V}'-4(1-\bar{V})]\mathcal{S}_0'-2u(1-\bar{V})(\mathcal{S}_0''+2\mathcal{S}_0'\mathcal{S}_1')=0, \tag{4.14}$$

which are, respectively, the eikonal and transport equations.

One finds immediately

$$\mathcal{S}_0(u)=\pm\sqrt{3}\int_{u_0}^{u}\frac{\mathrm{d}w}{\sqrt{1-\bar{V}(w)}}, \tag{4.15}$$

$$\mathcal{S}_1(u)=-\log\left(u\sqrt{1-\bar{V}(u)}\right)+k_1, \tag{4.16}$$

where u_0 and k_1 are constants. The solution to the EOM at this order is then

$$\pi(u)=\frac{D_+\exp\left[i\sqrt{3}\frac{q_0}{\varpi}\int_{u_0}^{u}\frac{\mathrm{d}w}{\sqrt{1-\bar{V}(w)}}\right]+D_-\exp\left[-i\sqrt{3}\frac{q_0}{\varpi}\int_{u_0}^{u}\frac{\mathrm{d}w}{\sqrt{1-\bar{V}(w)}}\right]}{u\sqrt{1-\bar{V}(u)}}, \tag{4.17}$$

where $D_\pm$ are constants.

For general values of the parameters, there is a singularity in $u=0$ (the center of the drop) and at $u=1$, where $\bar{V}(u)=1$ (the edge of the drop). These singularities are not physical since we want to describe small fluctuations. Eliminating them will fix the form of the spectrum, which we expect to be discrete due to the presence of a confining potential.

Around the origin $u=0$, $\mathcal{S}_0$ is given by

$$\mathcal{S}_0(u)\underset{u\to0}{\sim}\pm\sqrt{3}\left(\int_{u_0}^{0}\frac{\mathrm{d}w}{\sqrt{1-\bar{V}(w)}}+\frac{u}{\sqrt{1-\bar{V}(0)}}+\cdots\right), \tag{4.18}$$

so that the field π is

$$\pi(u)\underset{u\to0}{\sim}\frac{D_+\exp\left[i\sqrt{3}\frac{q_0}{\varpi}\int_{u_0}^{0}\frac{\mathrm{d}w}{\sqrt{1-\bar{V}(w)}}\right]+D_-\exp\left[-i\sqrt{3}\frac{q_0}{\varpi}\int_{u_0}^{0}\frac{\mathrm{d}w}{\sqrt{1-\bar{V}(w)}}\right]}{u\sqrt{1-\bar{V}(0)}}+O\left(u^0\right), \tag{4.19}$$

which is singular unless the quantity in the bracket vanishes. Setting, without loss of generality, $u_0=0$ and $D_+=-D_-=D/(2i)$ we obtain a solution that is regular in $u=0$:

$$\pi(u)=D\frac{\sin\left(\sqrt{3}\frac{q_0}{\varpi}\int_0^u\frac{\mathrm{d}w}{\sqrt{1-\bar{V}(w)}}\right)}{u\sqrt{1-\bar{V}(u)}}. \tag{4.20}$$

We still need to deal with the singularity in $u=1$. Asking for the sin to vanish, we obtain a quantization condition for q_0:

$$\frac{1}{\delta}\mathcal{S}_0(1)=\frac{q_0}{\varpi}\sqrt{3}\int_0^1\frac{\mathrm{d}w}{\sqrt{1-\bar{V}(w)}}=n\pi,\quad n\in\mathbb{Z}. \tag{4.21}$$

At leading order, the states in the spectrum are equidistant. This was to be expected, because at leading order in the WKB expansion the system behaves qualitatively as if the potential were constant [see Eq. (4.56)].

Our approximation is consistent for $q_0\gg\varpi$, which is the same as $n\gg1$. From the general analysis in the previous section, we know that these solutions form an orthogonal basis, which can be made orthonormal with an appropriate choice of the constant D:

$$\int\mathrm{d}u u^2\sqrt{1-\bar{V}(u)}\pi_r^n(u)\pi_r^m(u)=\delta_{mn} \tag{4.22}$$

if

$$\frac{1}{D^2}=\frac{1}{2}\int_0^1\frac{\mathrm{d}w}{\sqrt{1-\bar{V}(w)}}. \tag{4.23}$$

For a power-law potential of the form

$$V(r)=\mu\left(\frac{r}{R_{cl}}\right)^{2k},\quad R_{cl}=\frac{\sqrt{2\mu}}{\varpi}, \tag{4.24}$$

the reduced potential is just $\bar{V}=u^{2k}$. In this case, we can solve the integral in Eq. (4.20) analytically and obtain the spectrum

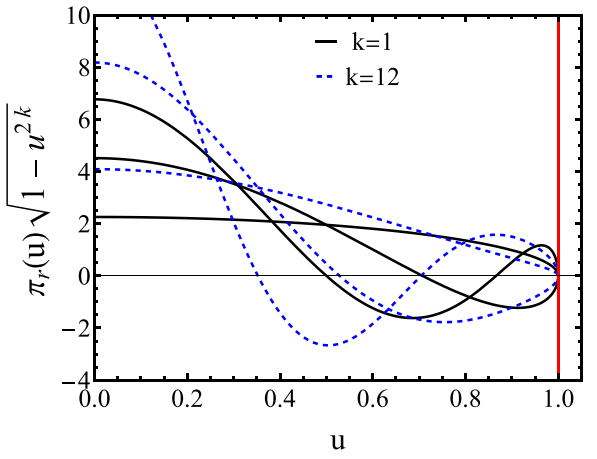

FIG. 3. Fluctuations of the charge density, taken from Eq. (4.26), with $k = 1$ (solid lines) and $k = 12$ (dashed lines) for the first three levels. The level counts the number of nodes.

equation

$$q_0(n) = \frac{2}{\sqrt{3}\binom{1/(2k)}{1/2}}\varpi n. \tag{4.25}$$

The slope of q_0 becomes steeper for flatter potentials (larger values of k) and varies between $q_0(n) = 2n\omega/\sqrt{3}$ for the harmonic potential ($k = 1$), and $q_0(n) = \pi n\varpi/\sqrt{3}$ in the $k \to \infty$ (spherical box) limit.

The fluctuation can be written explicitly in terms of hypergeometric functions:

$$\pi(u) = \sqrt{2k\binom{1/2k-1/2}{1/2}}\frac{\sin\left[\frac{2n}{\binom{1/(2k)}{1/2}}u\ {}_2F_1\left(\frac{1}{2},\frac{1}{2k};1+\frac{1}{2k};u^{2k}\right)\right]}{u\sqrt{1-u^{2k}}}, \tag{4.26}$$

which for the harmonic oscillator is simply[6]

$$\pi(u) = \frac{2}{\sqrt{\pi}}\frac{\sin[2n\arcsin(u)]}{u\sqrt{1-u^2}}. \tag{4.27}$$

The fluctuations of the charge density are illustrated in Fig. 3. Note that the result does not depend on the angular momentum ℓ of the fluctuations. To see the effect of the angular momentum, we need to add the next order in the WKB expansion, see Sec. IV D.

B. High-energy regime

Let us now consider the high-energy region

$$\frac{\varpi}{q_0} \ll \frac{q_0}{\mu} \ll 1, \tag{4.28}$$

in which we have to take into account the NLO corrections to the EFT—which are controlled by q_0/μ—but we can limit ourselves to the physical optics terms in the WKB expansion.

[6]The EOM for the fluctuations in the LO EFT with a harmonic potential can be solved exactly, and our LO result is consistent with the appropriate limit of the exact solution (see Sec. IV E).

The EOM for the fluctuations described by the action in Eq. (3.8) are of fourth order. Assuming a radially symmetric potential, the equations separate, and the EOM for the radial function $\pi(u)$ takes the form

$$\delta^4\frac{\mathrm{d}^2}{\mathrm{d}u^2}\left(p(u)\frac{\mathrm{d}^2\pi(u)}{\mathrm{d}u^2}\right) - \delta^2\frac{\mathrm{d}}{\mathrm{d}u}\left(s(u)\frac{\mathrm{d}\pi(u)}{\mathrm{d}u}\right) = w(u)\pi(u), \tag{4.29}$$

where

$$p(u) = -4u^2\sqrt{1-\bar{V}}q_0^2c_2, \tag{4.30}$$

$$\begin{aligned} s(u) = {} & -5u^2\mu^2(1-\bar{V})^{3/2}c_0 \\ & + u^2\left(\frac{\delta^2q_0^2(\bar{V}')^2}{2(1-\bar{V})^{3/2}} + \frac{4q_0^2}{\sqrt{1-\bar{V}}}\right)c_1 \\ & - 4q_0^2\delta^2c_2\left(4\sqrt{1-\bar{V}} - \frac{u\bar{V}'}{\sqrt{1-\bar{V}}}\right), \end{aligned} \tag{4.31}$$

$$w(u) = -15u^2\mu^2(1-\bar{V})^{1/2}c_0 - 8q_0^2\frac{\frac{\mathrm{d}}{\mathrm{d}u}\left[u^2\frac{\mathrm{d}}{\mathrm{d}u}(1-\bar{V})^{1/4}\right]}{(1-\bar{V})^{3/4}}\delta^2c_1. \tag{4.32}$$

We have omitted the ℓ dependence because it contributes only at higher order in the WKB approximation. Imposing appropriate boundary conditions, the fourth-order differential operator is self-adjoint, so we have a generalization of the Sturm–Liouville problem encountered at LO in Sec. IV A. This means that the spectrum of $q_0(n)^2$ remains discrete and there is a basis of eigenvectors $\pi_n(r)$ for the Hilbert space of fluctuations with weight $w(r)$.

The highest derivative term is again controlled by the small parameter $\delta = \varpi/q_0$, and we can look for a perturbative WKB solution. As pointed out above, in this regime, it is sufficient to correct the eikonal and transport equations in Eqs. (4.13) and (4.14), which become

$$\begin{aligned} & 5\mu^2(1-\bar{V})c_0\left(3-(1-\bar{V})\mathcal{S}_0'^2\right) \\ & \quad + 4c_1q_0^2\mathcal{S}_0'^2 - 4c_2q_0^2(1-\bar{V})\mathcal{S}_0'^4 = 0, \end{aligned} \tag{4.33}$$

$$\begin{aligned} & 5c_0\mu^2(1-\bar{V})[3u\bar{V}'\mathcal{S}_0' - (1-\bar{V})(4\mathcal{S}_0'(1+u\mathcal{S}_1') + 2u\mathcal{S}_0'')] \\ & + 4c_1q_0^2\left[\mathcal{S}_0'\left(4+4u\mathcal{S}_1' + \frac{u\bar{V}}{1-\bar{V}}\right) + 2u\mathcal{S}_0''\right] \\ & + 8c_2q_0^2\mathcal{S}_0'^2\{u\bar{V}'\mathcal{S}_0' - (1-\bar{V})[4\mathcal{S}_0'(1+u\mathcal{S}_1') + 6u\mathcal{S}_0'']\} = 0. \end{aligned} \tag{4.34}$$

Expanding at the appropriate order in q_0/μ, we find the solution

$$\begin{aligned} \mathcal{S}_0 = {} & \pm\sqrt{3}\int_{u_0}^{u}\mathrm{d}w\frac{1}{\sqrt{1-\bar{V}(w)}} \\ & \times\left[1 + \frac{2}{5c_0}\frac{c_1-3c_2}{[1-\bar{V}(w)]^2}\frac{q_0^2}{\mu^2} + O\left(\frac{q_0^4}{\mu^4}\right)\right], \end{aligned} \tag{4.35}$$

$$\mathcal{S}_1 = k_1 - \log(u\sqrt{1-\bar{V}}) + \frac{c_1-9c_2}{5c_0(1-\bar{V})^2}\frac{q_0^2}{\mu^2} + O\left(\frac{q_0^4}{\mu^4}\right). \tag{4.36}$$

At leading order, we have seen that the solution is in general singular at the edge of the droplet, $u = 1$, and picking only

regular solutions leads to a discrete spectrum. Adding the next term in the WKB expansion can shift the singularity inwards by an amount proportional to $O(q_0^2/\mu^2)$. This is reminiscent of the fact that the size of the droplet decreases, compared with the LO one, when including NLO corrections to the EFT. This effect becomes manifest if we write

$$\mathcal{S}_1(u) = k_1 - \log u - \frac{1}{4}\log\left[(1-\bar{V}(u))^2 - \frac{4}{5c_0}(c_1-9c_2)\frac{q_0^2}{\mu^2}\right] + O\left(\frac{q_0^3}{\mu^3}\right). \tag{4.37}$$

There are however two possible cases, depending on the sign of the combination $c_1 - 9c_2$.

Positive sign. If the sign is positive, which is what is found in mean-field theory (large N [7,15,31]), the $\bar{V}=1$ singularity is shifted to $u=u_1$, where u_1 is the solution to

$$[1-\bar{V}(u_1)]^2 - \frac{4}{5c_0}(c_1-9c_2)\frac{q_0^2}{\mu^2} = 0. \tag{4.38}$$

Once more, we are looking for regular solutions for our small oscillations. Imposing regularity at the origin, we find

$$\pi(u) = D\frac{\sin\left[\sqrt{3}\frac{q_0}{\varpi}\int_0^u \mathrm{d}w \frac{1}{\sqrt{1-\bar{V}(w)}}\left(1+\frac{2}{5c_0}\frac{c_1-3c_2}{[1-\bar{V}(w)]^2}\frac{q_0^2}{\mu^2}\right)\right]}{u\left[[1-\bar{V}(u)]^2 - \frac{4}{5c_0}(c_1-9c_2)\frac{q_0^2}{\mu^2}\right]^{1/4}}, \tag{4.39}$$

and regularity at $u=u_1$ leads to the quantization condition:

$$\sqrt{3}\frac{q_0}{\varpi}\int_0^{u_1}\mathrm{d}w\frac{1}{\sqrt{1-\bar{V}(w)}}\left(1+\frac{2}{5c_0}\frac{c_1-3c_2}{[1-\bar{V}(w)]^2}\frac{q_0^2}{\mu^2}\right) = n\pi. \tag{4.40}$$

In the case of a power-law potential $\bar{V}(u)=u^{2k}$, we can solve the integral explicitly in terms of gamma functions. First, we need to solve for the location of the singularity:

$$u_1 = 1 - \frac{1}{2k}\sqrt{\frac{4}{5c_0}(c_1-9c_2)}\frac{q_0}{\mu}. \tag{4.41}$$

The integral in the spectrum equation can be solved explicitly to find the first correction to LO in Eq. (4.25):

$$\frac{q_0}{\varpi}\left[\sqrt{3}\frac{\pi}{2}\binom{1/(2k)}{1/2} + \frac{51c_2-5c_1}{\sqrt{6}(5c_0)^{1/4}k(c_1-9c_2)^{3/4}}\sqrt{\frac{q_0}{\mu}}\right] = n\pi, \tag{4.42}$$

which can be inverted to give

$$\frac{q_0(n)}{\varpi} = \frac{2}{\sqrt{3}\binom{1/(2k)}{1/2}}n + \frac{4(5c_1-51c_2)}{3^{7/4}\pi k(5c_0)^{1/4}(c_1-9c_2)^{3/4}\binom{1/(2k)}{1/2}^{5/2}} \times\sqrt{\frac{\varpi}{\mu}}n^{3/2} + O\left(\frac{\varpi}{\mu}n^2\right). \tag{4.43}$$

Negative sign. If the sign of c_1-9c_2 is negative, as is found in the ϵ expansion [32], the singularity remains in $u=1$. However, the EFT expansion breaks down close to $u=1$ because the NLO term becomes larger than the LO one. We can only trust our approximation up to distances of the order of $u \lessapprox 1 - O(q_0/\mu)$, which is consistent with the value of u_1 we found in the other case in Eq. (4.41). The final result is that the spectrum takes again the form

$$\frac{q_0(n)}{\varpi} = \frac{2}{\sqrt{3}\binom{1/(2k)}{1/2}}n + \lambda\sqrt{\frac{\varpi}{\mu}}n^{3/2} + O\left(\frac{\varpi}{\mu}n^2\right), \tag{4.44}$$

where now λ is some new parameter depending on the edge physics, which is not accessible within our approximation.

The NLO in the EFT is sensitive to the effects from the cloud edge. This is expected, since the control parameter of the EFT is ultimately the charge density, which vanishes at the edge. Already in the study of the ground-state energy, this leads to the appearance of new fractional powers in the charge [12,26]. Here, we see that the leading order of the spectrum equation receives a correction that scales like $n^{3/2}$, with a coefficient that is computable in terms of the low-energy constants in the case of $c_1-9c_2>0$ or requires the inclusion of edge terms for $c_1-9c_2<0$.

C. No potential

As a limit case of the high-energy regime, we can consider the case of zero potential $V=0$, which amounts to $\varpi=0$. Here, the only remaining expansion parameter is $\eta = q_0/\mu$. We can also think of this as the limit $k\to\infty$ for the confining potential u^{2k}. To retain a confining effect, we need to impose Dirichlet boundary conditions at $r=R_{cl}$, which results in a confining bucket.

The quadratic action for the Goldstone $\pi(t,\mathbf{r})$ in Eq. (3.8) reduces to

$$\mathcal{L}^{(2)}[\pi] = -\frac{5}{8}c_0\sqrt{\mu}[2\mu(\nabla\pi)^2 - 3\dot{\pi}^2] + \frac{c_1}{\sqrt{\mu}}(\nabla\dot{\pi})^2 + c_2\sqrt{\mu}[(\Delta\pi)^2 - 3(\nabla\otimes\nabla\pi)^2], \tag{4.45}$$

and the corresponding EOM is

$$5c_0\mu(3\ddot{\pi} - 2\mu\Delta\pi) - 8c_1\Delta\ddot{\pi} + 16c_2\mu\Delta^2\pi = 0, \tag{4.46}$$

where Δ^2 is the biharmonic operator

$$\Delta^2\pi = \sum_{i,j=1}^{3}\partial_i\partial_i\partial_j\partial_j\pi. \tag{4.47}$$

In this configuration the symmetries of the problem change and we have the full Poincaré group ISO(1, 3); moreover the spectrum is continuous, so the natural Ansatz is a simple plane wave,

$$\pi(t,\mathbf{r}) = \exp[iq_0 t + i\mathbf{q}\cdot\mathbf{r}]. \tag{4.48}$$

Substituting this ansatz into Eq. (4.46), we find the dispersion relation for the plane wave:

$$5c_0\mu(-3q_0^2 + 2q^2\mu) - 8c_1q_0^2q^2 + 16\mu c_2q^4 = 0. \tag{4.49}$$

Equivalently, solving for q_0, we find

$$q_0(q) = \sqrt{\frac{2\mu}{3}}q - \frac{4}{15c_0}\sqrt{\frac{2}{3\mu}}(c_1-3c_2)q^3 + O(\mu^{-3/2}), \tag{4.50}$$

which can be parametrized as

$$q_0(q) = c_s q\left(1 + \frac{\gamma}{8c_s^2}q^2\right) \tag{4.51}$$

with

$$c_s = \sqrt{\frac{2\mu}{3}}, \quad \gamma = \frac{64}{45}\frac{3c_2 - c_1}{c_0}, \tag{4.52}$$

as already found above. In this simple case, the Goldstone nature of the fluctuation becomes completely manifest. Since the low-energy scale is zero, we can see here that q_0 is zero when $q = 0$ (which is not the case when $q = 0$ is not accessible with the EFT due to the presence of a potential). The value of c_s is completely fixed by scale invariance. For a flat potential at criticality, the low-energy spectrum must be linear with the slope fixed by the value of μ. We find this slope again when computing the dynamical structure factor in Sec. V.

To make contact with the WKB solution, we could have also looked for a spherically symmetric solution of the form

$$\pi_{q_0,q}(t, r) = e^{iq_0 t}\pi(r) \tag{4.53}$$

for which we would have found the EOM

$$c_0\left[-\frac{5}{2}\mu^{3/2}\pi'' - 5\sqrt{\mu}\left(\frac{\mu}{r}\pi' + \frac{3}{4}q_0^2\pi\right)\right] + 2\frac{c_1 q_0}{\sqrt{\mu}}\left(\pi'' + \frac{2}{r}\pi'\right) + 4c_2\sqrt{\mu}\left(\pi'''' + \frac{4}{r}\pi'''\right) = 0. \tag{4.54}$$

This admits a regular spherical wave solution

$$\pi_{q_0,q}(t, r) = e^{iq_0 t}\frac{\sin(qr)}{r\sqrt{\mu}}, \tag{4.55}$$

where q_0 and q are related by the same dispersion relation as for the plane wave.

In the case of a spherical box in which we impose Dirichlet boundary conditions $\pi(R_{cl}) = 0$ at $R_{cl} = \sqrt{2\mu}/\varpi$, the spectrum is discrete and reads

$$\frac{q_0}{\varpi} = \frac{\pi}{\sqrt{3}}n, \tag{4.56}$$

which is the large-k limit of the relation that we have found in Eq. (4.25) (see also Fig. 4). This is consistent with our results from the WKB expansion: the solution given in Eq. (4.20) reduces to this form for $\bar{V} = 0$. In this case, the eikonal approximation is exact, as the higher-order terms in the WKB expansion take into account the effects of a potential.

D. Low-energy regime

To go beyond the linear approximation in the low-energy regime, we need to add higher-order corrections to the WKB expansion. We study the low-energy region, where

$$\frac{q_0}{\mu} \ll \frac{\varpi}{q_0} \ll 1, \tag{4.57}$$

where we can neglect the NLO terms in the EFT.

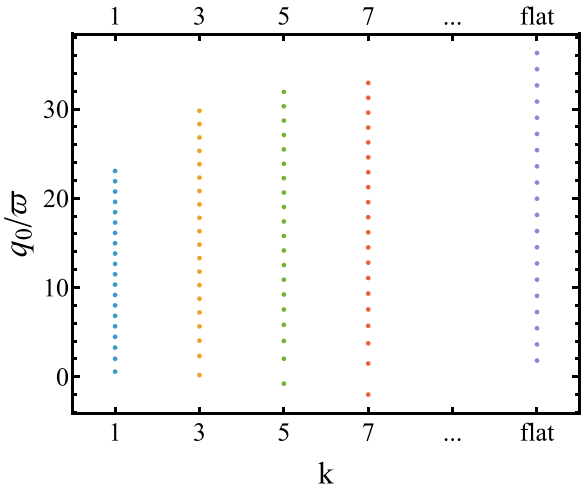

FIG. 4. First twenty allowed energy eigenvalues in the spectrum at low energy for different choices of the potential $\bar{V} = u^{2k}$. The eigenvalues are less dense for flatter potentials, and for $k \to \infty$ asymptote to the spectrum of the spherical box $q_0/\varpi = \pi n/\sqrt{3}$.

The new equations to solve are

$$[3u\bar{V}' - 4(1-\bar{V})]\mathcal{S}_1' - 2u(1-\bar{V})[\mathcal{S}_1'' - 2\mathcal{S}_0'\mathcal{S}_2' + (\mathcal{S}_1')^2] + \frac{2}{u}\ell(\ell+1)(1-\bar{V}) = 0, \tag{4.58}$$

$$[3u\bar{V}' - 4(1-\bar{V})]\mathcal{S}_2' - 2u(1-\bar{V})(\mathcal{S}_2'' + 2\mathcal{S}_0'\mathcal{S}_3' + 2\mathcal{S}_1'\mathcal{S}_2') = 0. \tag{4.59}$$

The solution can be written in terms of a function of $\bar{V}$:

$$\mathcal{S}_2(u) = \frac{1}{4\sqrt{3}}\int_{u_0}^{u}\frac{F_\ell[\bar{V}(w)]}{\sqrt{1-\bar{V}(w)}}\mathrm{d}w, \tag{4.60}$$

$$\mathcal{S}_3(u) = -\frac{1}{24}F_\ell[\bar{V}(u)], \tag{4.61}$$

where

$$F_\ell[\bar{V}(u)] = \bar{V}''(u) + \frac{3}{u}\bar{V}'(u) - \frac{2\ell(\ell+1)}{u^2}[1-\bar{V}(u)]. \tag{4.62}$$

At NLO we see the contribution of the angular momentum ℓ that in the physical optics approximation did not change the energy and only accounted for a large degeneracy. While here the degeneracy is lifted, the $1/u^2$ behavior of the new term proportional to $\ell(\ell+1)$ is problematic. For $u \to 0$ it becomes so large that it breaks the WKB approximation and leads to an apparent singularity. At this order, we can estimate the region of validity of the expansion by requiring the NLO term to be smaller than the leading term and find that for $\ell > 0$ our approximation is valid for distances larger than order δ from the origin. We expect the divergences to disappear when all orders of the expansion are resummed. In the case of the harmonic potential, treated in Sec. IV E, we have an exact solution which does not display any singularities. Here we concentrate on the s-wave approximation; in other words we only consider the lower rotational-invariant modes ($\ell = 0$).

Now we can repeat the same computation that we had performed in the previous section, fixing the integration constants and the allowed values for q_0 to avoid singular solutions. At

this order we find

$$\mathcal{S}_1(u)+\frac{\varpi^2}{q_0^2}\mathcal{S}_3(u)=-\log(u)-\frac{1}{2}\log[1-\bar{V}(u)] - \frac{\varpi^2}{24q_0^2}F_0[\bar{V}(u)]+O\left(\frac{\varpi^4}{q_0^4}\right), \quad (4.63)$$

that we can rewrite as

$$\mathcal{S}_1(u)+\frac{\varpi^2}{q_0^2}\mathcal{S}_3(u)=-\frac{1}{2}\log\left(u^2+\frac{u^2\varpi^2}{12q_0^2}F_0[\bar{V}(u)]\right) - \frac{1}{2}\log[1-\bar{V}(u)]+O\left(\frac{\varpi^4}{q_0^4}\right). \quad (4.64)$$

(1) Imposing regularity at $u=0$, we find that the solution must take the form

$$\pi(u)=D\frac{\sin\left(\frac{q_0}{\varpi}\sqrt{3}\int_0^u\frac{\mathrm{d}w}{\sqrt{1-\bar{V}(w)}}+\frac{\varpi}{q_0}\frac{1}{4\sqrt{3}}\int_0^u\frac{F_\ell[\bar{V}(w)]}{\sqrt{1-\bar{V}(w)}}\mathrm{d}w\right)}{u\sqrt{\left(1+\frac{\varpi^2}{12q_0^2}F_0[\bar{V}(u)]\right)[1-\bar{V}(u)]}}. \quad (4.65)$$

(2) Now we impose regularity in $u=1$ and find the quantization condition

$$\frac{q_0}{\varpi}\sqrt{3}\int_0^1\frac{\mathrm{d}w}{\sqrt{1-\bar{V}(w)}}+\frac{\varpi}{q_0}\frac{1}{4\sqrt{3}}\int_0^1\frac{F_0[\bar{V}(w)]}{\sqrt{1-\bar{V}(w)}} \times \mathrm{d}w=n\pi, \quad n=1,2,3,\ldots. \quad (4.66)$$

For a power-law potential $\bar{V}=u^{2k}$ it is possible to solve the integrals analytically to find

$$\frac{q_0(n)}{\varpi}=\frac{2}{\sqrt{3}\binom{1/(2k)}{1/2}}n-\frac{2(k+1)}{2\sqrt{3}\pi\binom{(k-1)/2k}{1/2}}\frac{1}{n}+\cdots. \quad (4.67)$$

In the case of the harmonic potential $k=1$, this reduces to

$$\frac{q_0(n)}{\omega}=\frac{2}{\sqrt{3}}n-\frac{4}{4\sqrt{3}}\frac{1}{n}+\cdots. \quad (4.68)$$

The result for some values of k is sketched in Fig. 4. For larger values of k (flatter potentials) the spectrum becomes less and less dense, and for $k\to\infty$ it asymptotes the spectrum of the spherical box in Eq. (4.56): $q_0/\varpi=\pi n/\sqrt{3}$.

In the next section we study in detail the case of the harmonic trap, that can be solved exactly in the LO EFT, and recover this result in the appropriate limit.

E. Harmonic potential

In the low-energy regime, where the controlling parameter η is very small, and it is sufficient to use the LO EFT, there is a special choice for the potential that allows us to solve the EOM exactly: the harmonic potential

$$V=\frac{\omega^2}{2}r^2. \quad (4.69)$$

In this case, the low-energy parameter is $\varpi=\omega$. For this section, we keep the ratio $\delta=\varpi/q_0$ finite.

The EOM for the radial mode is

$$\delta^2\frac{\mathrm{d}}{\mathrm{d}u}\left(u^2(1-u^2)^{3/2}\frac{\mathrm{d}}{\mathrm{d}u}\pi(u)\right)-\delta^2\ell(\ell+1)(1-u^2)^{3/2}\pi(u) + 3u^2\left(1-u^2\right)^{1/2}\pi(u)=0. \quad (4.70)$$

As already observed in Ref. [12], after the rescaling $\pi(u)=u^\ell w(u)$ and the change of variable $u^2=z$, the EOM takes the form of a hypergeometric equation:

$$z(1-z)\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+\left(\frac{3}{2}+\ell-(3+\ell)z\right)\frac{\mathrm{d}w}{\mathrm{d}z} - \frac{3}{4}\left(\ell-\frac{1}{\delta^2}\right)w=0. \quad (4.71)$$

Its solution is

$${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|z\right)$$

with parameters

$$a=\frac{\ell+2-\sqrt{\ell^2+\ell+4+3/\delta^2}}{2}, \quad b=\frac{\ell+2+\sqrt{\ell^2+\ell+4+3/\delta^2}}{2}, \quad c=\frac{3}{2}+\ell. \quad (4.72)$$

We want a regular solution in $z=1$. The general theory of Gaussian hypergeometric functions requires that, if $a+b>c$, a must be a nonpositive integer (i.e.,$a=-n$, where $n=0,1,\ldots$). This condition is precisely what causes the hypergeometric function to degenerate into a polynomial, and it directly leads to the expected quantization condition for δ:

$$\delta^2=\frac{3}{4n^2+4n\ell+8n+3\ell}, \quad n,\ell=0,1,\ldots. \quad (4.73)$$

The final result is that the fluctuations are written as

$$\pi(t,\mathbf{u})=e^{iq_0t}\,{}_2F_1\left(\begin{matrix}-n,n+\ell+2\\3/2+\ell\end{matrix}\middle|u^2\right)Y_{\ell m}(\vartheta,\phi) \quad (4.74)$$

with the spectrum

$$q_0(n,l)=\omega\sqrt{\frac{4n}{3}(n+\ell+2)+\ell}, \quad n,\ell=0,1,\ldots. \quad (4.75)$$

In the special case of $\ell=0$, the s-wave approximation on which we concentrate also in the following, the radial part is a Chebyshev polynomial of the second kind:

$$\pi_r^{n,0}(u)={}_2F_1\left(\begin{matrix}-n,n+2\\3/2\end{matrix}\middle|u^2\right)=U_n(1-2u^2), \quad (4.76)$$

which can be written in terms of trigonometric functions as

$$\pi_r^{n,0}(u)=U_n(1-2u^2)=\frac{2\sin[2(n+1)\arcsin(u)]}{u\sqrt{1-u^2}}. \quad (4.77)$$

This expression is precisely the same as the one found in Eq. (4.27), up to the shift $n\to n+1$. In the special case of the harmonic oscillator, the functional form obtained using the physical optics WKB approximation turns out to be exact. The quantization condition shows that δ becomes small when n is large, allowing us to approximate n as $n+1$ in this limit.

V. THE DYNAMIC STRUCTURE FACTOR

The phonon field and its fluctuations are not observable quantities, and therefore it is imperative to identify an observable that encodes the acoustic excitation spectrum of the superfluid. In the language of field theory, observables are matrix elements of local operators. Here, however, we are in the nonrelativistic regime, and therefore the probability to excite the superfluid many-body system away from the ground state is, via Fermi's golden rule, proportional to the so-called dynamic structure factor [36]

$$S(q_0, \mathbf{q}) = \sum_{\mathbf{n}} |\langle 0|\delta\rho(\mathbf{q})|\mathbf{n}\rangle|^2 \delta(q_0 - q_0(\mathbf{n})), \tag{5.1}$$

where $\delta\rho(\mathbf{q})$ is the Fourier transform of the small fluctuations induced in the charge density about the ground state, $\mathbf{n}$ is the set of quantum numbers that identify the corresponding quantum states and $q_0(\mathbf{n})$ is the spectrum equation.[7]

A. LO EFT

For ease of exposition we start from the LO EFT. It is convenient to rescale the field $\pi(t, \mathbf{u})$ so that its kinetic term is canonically normalized, which in this case means that the coefficient of the time derivative part of the Lagrangian is $1/2$. The rescaled field $\tilde{\pi}(t, \mathbf{u})$ is then

$$\tilde{\pi}(t, \mathbf{u}) = \mu^{1/4} \frac{\sqrt{15c_0}}{2} [1 - \bar{V}(r)]^{1/4} \pi(t, \mathbf{u}), \tag{5.2}$$

where $\pi(t, \mathbf{u})$ is the solution that we have found in various approximations in the previous section. The charge density is the variation of the action with respect to $\dot{\pi}$, so in terms of the fluctuations $\tilde{\pi}(t, \mathbf{u})$, the leading-order fluctuations of the charge are written as

$$\delta\rho = \frac{\sqrt{15c_0}}{2} \mu^{1/4} (1 - \bar{V})^{1/4} \frac{\partial}{\partial t} \tilde{\pi}(t, \mathbf{u}). \tag{5.3}$$

Now we need to canonically quantize starting with the field $\tilde{\pi}(t, \mathbf{r})$. The first step consists in identifying a basis of solutions for the EOM. We have performed a position-dependent rescaling from $\pi(t, \mathbf{q})$: this changes the form of the Sturm–Liouville problem (Sec. IV A) and the weight function in the associated scalar product. The (nongeometric part of the) weight is just the coefficient of the second time derivative in the EOM, which we have now rescaled to one. It follows that the $\tilde{\pi}(\mathbf{u})$ are orthogonal with respect to the usual flat-space product:

$$\begin{aligned} \langle \tilde{\pi}^{nlm}, \tilde{\pi}^{n'l'm'} \rangle &= \int_0^1 u^2 \mathrm{d}u \int_{S^2} \mathrm{d}\Omega \tilde{\pi}^{n\ell m}(u, \theta, \phi) \tilde{\pi}^{n'\ell'm'}(u, \theta, \phi) \\ &= \delta_{nn'} \delta_{\ell\ell'} \delta_{mm'}. \end{aligned} \tag{5.4}$$

To add the time component, we observe that the system is invariant under time translation. We can thus use the standard equal-time product associated with the EOM:

$$(\phi_1(t, \mathbf{u}), \phi_2(t, \mathbf{u})) = i \int_{u \leqslant 1} \mathrm{d}^3\mathbf{u} \phi_1(t, \mathbf{u})^* \overleftrightarrow{\partial}_t \phi_2(t, \mathbf{u}), \tag{5.5}$$

where $f \overleftrightarrow{\partial}_t g \equiv f(\partial_t g) - (\partial_t f) g$. With this we can identify a basis of orthonormal solutions. In the physical optics approximation they are

$$\phi_{n\ell m}(t, \mathbf{u}) = \frac{D}{2\sqrt{q_0}} e^{-iq_0 t} \frac{\sin[\mathcal{S}_0(u)/\delta]}{u(1 - \bar{V})^{1/4}} Y_{\ell m}(\theta, \phi), \tag{5.6}$$

where the q_0 are quantized as in Eq. (4.21):

$$\frac{q_0}{\varpi} \mathcal{S}_0(1) = n\pi, \quad n \in \mathbb{Z}, \tag{5.7}$$

and D is the normalization constant in Eq. (4.23). From here on, the procedure is standard. We introduce standard creator and annihilator operators and a Fock space starting from the vacuum $|0\rangle$ corresponding to the ground state

$$[a_{n\ell m}, a^\dagger_{n'\ell'm'}] = \delta_{nn'} \delta_{\ell\ell'} \delta_{mm'}, \quad |n\ell m\rangle = a^\dagger_{n\ell m} |0\rangle. \tag{5.8}$$

The quantized field reads

$$\hat{\pi}^{n\ell m}(t, \mathbf{u}) = \sum_{n\ell m} \phi_{n\ell m}(t, \mathbf{u}) a_{n\ell m} + \phi^*_{n\ell m}(t, \mathbf{u}) a^\dagger_{n\ell m}. \tag{5.9}$$

With this we can read off the small fluctuations:

$$\delta\hat{\rho}(t, \mathbf{u}) = \sum_{n\ell m} e^{-iq_0 t} F_{n\ell m}(\mathbf{u}) a_{n\ell m} + e^{iq_0 t} F^*_{n\ell m}(t, \mathbf{u}) a^\dagger_{n\ell m}, \tag{5.10}$$

where

$$F_{n\ell m}(\mathbf{u}) = \frac{i}{2} \mu^{1/4} \sqrt{\frac{15c_0 q_0}{Y}} \frac{\sin[\mathcal{S}_0(u)/\delta]}{u} Y_{\ell m}(\theta, \phi), \tag{5.11}$$

with the normalizations chosen so that

$$\begin{aligned} &\langle 0|\delta\rho(\mathbf{q})|n\ell m\rangle \\ &= i\mu^{1/4} D \sqrt{15c_0 q_0} \int \frac{\mathrm{d}^3\mathbf{u}}{(2\pi)^3} e^{iR_{cl}\mathbf{q}\cdot\mathbf{u}} \frac{\sin[\mathcal{S}_0(u)/\delta]}{u} Y_{\ell m}(\theta, \phi). \end{aligned} \tag{5.12}$$

The problem has radial symmetry and the angular part of the integral boils down to the decomposition of a plane wave in terms of spherical harmonics

$$\int \mathrm{d}\Omega Y_{\ell m} e^{i\mathbf{q}\cdot\mathbf{r}} = 4\pi \delta_{\ell\ell'} \delta_{mm'} i^{\ell'} j_{\ell'}(qr) Y_{l'm'}, \tag{5.13}$$

where the j_ℓ are Bessel functions. In the special case of $\ell = 0$, this is simply

$$\int \mathrm{d}\Omega Y_{00} e^{i\mathbf{q}\cdot\mathbf{r}} = 4\pi j_0(qr) Y_{00} = 2\sqrt{\pi} \frac{\sin(qr)}{qr}, \tag{5.14}$$

so, in the s-wave approximation, the matrix element of the charge fluctuation operator is the Fourier sine transform of the phase of the fluctuations:

$$\langle 0|\delta\rho(\mathbf{q})|n\rangle \propto \frac{\mu^{1/4}\sqrt{c_0 q_0}}{q} \int_0^1 \mathrm{d}u \sin(qR_{cl}u) \sin\left(\frac{\mathcal{S}_0(u)}{\delta}\right). \tag{5.15}$$

This expression relates the study of the fluctuations of the previous section directly to an experimentally accessible

[7] For a review, see Ref. [36]. In the appendix we provide a field-theoretic definition of the dynamic structure factor and response function.

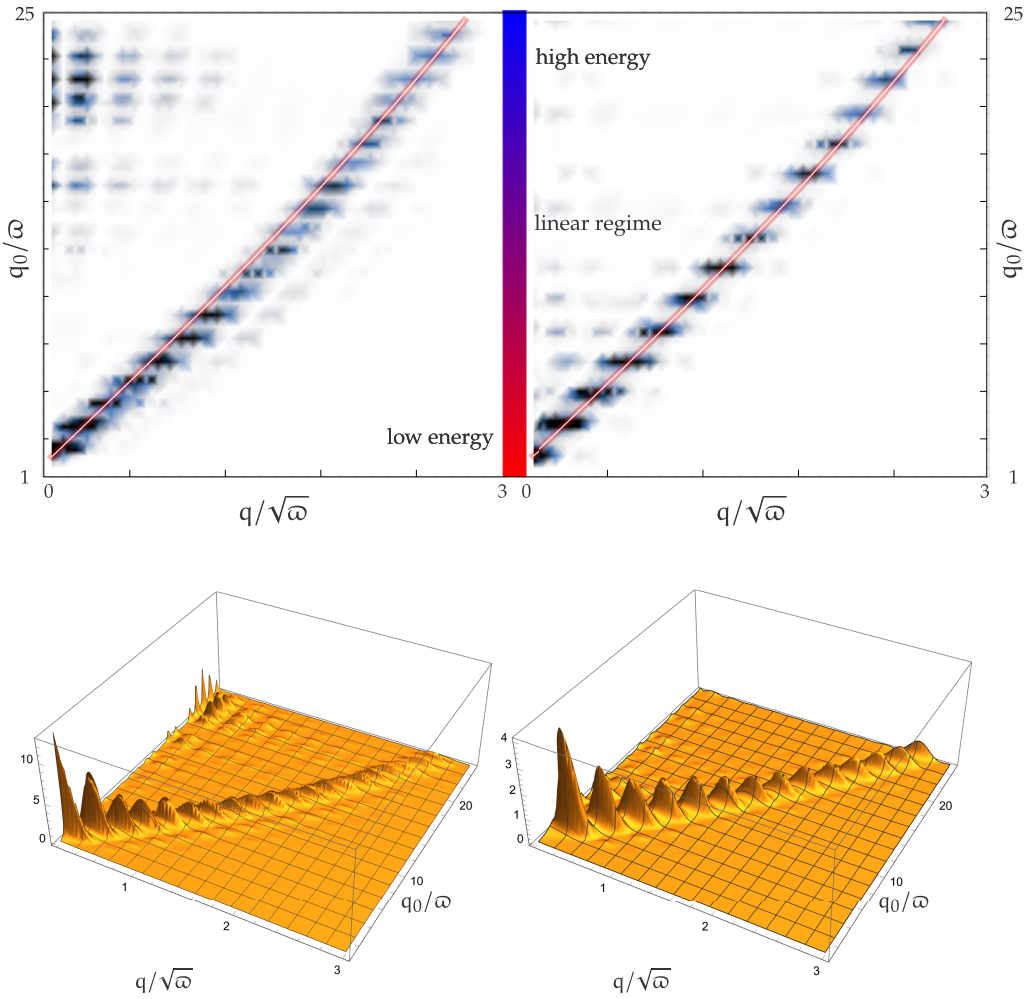

FIG. 5. Dynamic structure factor for a harmonic trap (left) and a superharmonic confining potential $\bar{V} = u^{16}$ (right) for $\mu/\varpi = 100$. The continuous lines show the curves along which $S(q_0, q)$ is peaked (dispersion relations). The discrete structure reflects the finite size of the system. The width of the peaks in the q direction results from the breaking of translational invariance. Note that the peaks are more sharply defined in the superharmonic case. A Lorentzian smoothing with parameter $\Delta q_0 = 0.5\varpi$ has been applied in the q_0 direction since the system does not break time-translation invariance.

quantity. Unfortunately, the integral cannot be solved analytically in general. One possible way out, then, is to perform a numerical analysis (see Fig. 5). Alternatively, as we do in the following section, one can look for approximate solutions that, in different energy regimes are controlled by the small parameter η or δ. These approximate solutions allow us to define a notion of dispersion relation even in the absence of translation invariance and to discuss the effects of both a nontrivial potential and of higher-order terms in the EFT.

B. High-energy regime

If $qR_{cl} \gg 1$ with $qR_{cl} = O(1/\delta)$, the main contribution to the integral comes from the saddle. Since $q > 0$ and $u > 0$, for $\mathcal{S}_0' > 0$, this is the value $u = \bar{u}$ such that

$$qR_{cl} = \frac{1}{\delta}\mathcal{S}_0'(\bar{u}) \tag{5.16}$$

and the integral is well approximated by

$$\langle 0|\delta\rho(\mathbf{q})|n\ell m\rangle \approx \mu^{1/4}\sqrt{\frac{2\pi c_0}{\varpi}}\frac{q_0}{q}\frac{1}{\sqrt{|\mathcal{S}''(\bar{u})|}}\sin\left(qR_{cl}\bar{u}\right) \times \sin\left(\frac{\mathcal{S}_0(u)}{\delta}\right). \tag{5.17}$$

At this order, the dynamic structure factor becomes

$$S(q_0, \mathbf{q}) \approx \mu^{1/2}\frac{c_0}{\varpi q^2}\sum_n \frac{q_0^2}{|\mathcal{S}_0''(\bar{u})|}\sin^2(qR_{cl}\bar{u}) \times \delta\left(q_0 - \frac{n\pi\varpi}{\mathcal{S}_0(1)}\right)\Big|_{q=\frac{q_0}{\sqrt{2\mu}}\mathcal{S}_0'(\bar{u})}. \tag{5.18}$$

As expected, we find a discrete sum of oscillating functions. We are interested in the position of the peaks of $S(q_0, \mathbf{q})$. Up to subleading correction in $1/\delta$, these functions in the sum are peaked for the values of (q_0, q) such that $\mathcal{S}_0''(\bar{u}(q_0, q)) = 0$.

Linear regime. In the linear regime, where it is sufficient to study the physical optics approximation of the LO EFT, we have found that $\mathcal{S}_0$ is given by Eq. (4.20):

$$\mathcal{S}_0(u) = \sqrt{3}\int_0^u \frac{\mathrm{d}w}{\sqrt{1-\bar{V}(w)}}, \tag{5.19}$$

so the saddle-point equation is

$$q = \frac{q_0}{\sqrt{2\mu}}\frac{\sqrt{3}}{\sqrt{1-\bar{V}(u)}} = \sqrt{\frac{3}{2[\mu - V(r)]}}q_0. \tag{5.20}$$

This equation is reminiscent of a local density approximation for the dispersion relation in a slowly varying potential (remember that in vacuum $q_0 = \sqrt{2\mu/3}q$). However, one should not read it as a position-dependent speed of sound because it is simply the curve $u = u(q_0, \mathbf{q})$ on which the Fourier integral in Eq. (5.15) localizes.

In a system with translation invariance, the Fourier integral would yield delta functions and the dynamic structure factor would be completely localized along the curve of the dispersion relation. The potential breaks translational invariance, so S is a more complicated function. It is natural to define an approximate dispersion relation as the curve in the (q_0, q) plane along which the dynamic structure factor is peaked. To see where this happens, we need to find the zeros of $\mathcal{S}_0''(\bar{u})$:

$$\mathcal{S}_0''(\bar{u}) = -\frac{\sqrt{3}\bar{V}'(\bar{u})}{2\big[1-\bar{V}(\bar{u})\big]^{3/2}} = 0. \tag{5.21}$$

So we have peaks corresponding to the stationary points of $\bar{V}(u)$. For a power-law potential, this means $\bar{u} = 0$. The result is that, at this order, the dynamic structure factor is peaked for

$$q = \sqrt{\frac{3}{2[\mu - V(r)]}}q_0\bigg|_{r=0} = \sqrt{\frac{3}{2\mu}}q_0, \tag{5.22}$$

which is independent of the potential (that vanishes at the origin). As expected, the leading behavior of the spectrum is linear, since the fluctuations around the ground state are described by a type-I Goldstone boson [6,11].

The width of the peak Δq will depend on the potential. To estimate it, start from the approximate expression for the matrix element of the density fluctuations in Eq. (5.17). We can just approximate the width of the peak in the dynamic structure factor with the position of the first zero in the sine function,

$$\Delta q \Delta u R_{cl} = \pi. \tag{5.23}$$

For a power-law potential $V = u^{2k}$, the variations Δq and Δu are related by

$$(\Delta u)^{2k} \approx \Delta q R_{cl}\delta, \tag{5.24}$$

and we find that

$$\Delta q = O\left[\frac{1}{R_{cl}}\left(\frac{q_0 R_{cl}}{\sqrt{\mu}}\right)^{1/(1+2k)}\right]. \tag{5.25}$$

The larger k, the flatter the potential and the narrower the peak.

NLO in the EFT. The construction generalizes to higher orders in the EFT. If we go to NLO in the EFT and remain at LO in WKB, consistently with the limit

$$\frac{\varpi}{q_0} < \frac{q_0}{\mu} \ll 1, \tag{5.26}$$

there are two main effects to take into account:

(1) The normalization of the fluctuations now becomes q dependent, reflecting the fact that the EOM is of higher order in derivatives.

(2) The function $\mathcal{S}_0(u)$ now contains a higher-order term in the q_0/μ expansion [see Eq. (4.35)].

Here we limit ourselves to studying how the NLO corrections in the EFT affect the dispersion relation. A detailed analysis of the behavior of the dynamic structure factor at NLO requires a different approach based on the path-integral formulation of this problem as outlined in Appendix.

The saddle-point equation now becomes

$$\begin{aligned} q &= \frac{q_0}{\sqrt{2\mu}}\mathcal{S}_0'(0) \\ &= \sqrt{\frac{3}{2\mu}}q_0\left[1 + \frac{2}{5c_0}(c_1 - 3c_2)\frac{q_0^2}{\mu^2} + O\left(\frac{q_0}{\mu}\right)^4\right], \end{aligned} \tag{5.27}$$

and the dynamic structure factor is peaked again for $\bar{u} = 0$, i.e., along the curve

$$q_0(q) = \sqrt{\frac{2\mu}{3}}q\left(1 - \frac{4}{15}\frac{c_1 - 3c_2}{c_0}\frac{q^2}{\mu}\right), \tag{5.28}$$

consistent with Eq. (4.50). In the high-energy regime that we are discussing here, the control parameter is $\eta = q_0/\mu$, which depends only on the energy of the fluctuations and on the chemical potential. It is not surprising, then, that the leading correction to the linear behavior of the dispersion relation in this region is independent of the details of the trapping potential and is in this sense universal. With the mean-field value of the low-energy constant given in Sec. II, the curvature is always convex. Other aspects of the dynamic structure factor (for example the width of the peak) do depend on V, but this goes beyond the scope of this work. The situation is quite different in the low-energy regime, as we see in the following section.

C. Low-energy regime

There are two sources of corrections to the position of the peaks of the dynamical structure factor in the low-energy regime ($\delta \ll 1$). The NLO term $\mathcal{S}_2$ in the WKB expansion of the wave function that we have discussed in Sec. IV D and the expansion of the integral in Eq. (5.17) to NLO beyond the saddle-point approximation. The quantity to compute is

$$I = \int_0^1 \mathrm{d}u \exp\left[i\left(qR_{cl}u - \frac{\mathcal{S}_0(u)}{\delta} - \mathcal{S}_2(u)\delta\right)\right]. \tag{5.29}$$

We know that the most important contribution comes from the saddle. For ease of exposition, we introduce

$$\xi = qR_{cl}\delta, \quad \phi_\xi(u) = \xi u - \mathcal{S}_0(u) - \mathcal{S}_2(u)\delta^2, \tag{5.30}$$

so that the integral takes the form

$$I = \int_0^1 \mathrm{d}u e^{i/\delta\phi_\xi(u)}, \tag{5.31}$$

which we expand at NLO around the saddle $u = u_\xi$ fixed by

$$\phi'_\xi(u) = \xi - \mathcal{S}'_0(u_\xi) - \mathcal{S}'_2(u_\xi)\delta^2 = 0. \tag{5.32}$$

The integral expanded at quartic order in $\hat{u} = u - u_\xi$ is given by

$$I \approx \int_{-\infty}^{\infty} \mathrm{d}\hat{u} \exp\left[\frac{i}{\delta}\left(\phi_\xi(u_\xi) + \frac{\phi''_\xi(u_\xi)}{2}\hat{u}^2 + \frac{\phi^{(4)}_\xi(u_\xi)}{4!}\hat{u}^4\right)\right]$$
$$\approx \sqrt{\frac{2\pi\delta}{|\phi''_\xi|}} e^{\frac{i}{\delta}\phi_\xi(u_\xi)}\left(1 - \frac{i\delta}{8}\frac{\phi^{(4)}_\xi(u_\xi)}{\phi''_\xi(u_\xi)^2}\right). \tag{5.33}$$

We exponentiate the parentheses to find

$$I \approx \sqrt{\frac{2\pi\delta}{|\phi''_\xi|}} \exp\left[\frac{i}{\delta}\left(\phi_\xi(u_\xi) - \frac{\delta^2}{8}\frac{\phi^{(4)}_\xi(u_\xi)}{\phi''_\xi(u_\xi)^2}\right)\right]. \tag{5.34}$$

Once more we are interested in the position of the peaks, which arise when the denominator vanishes. The condition in Eq. (5.21) becomes

$$\phi''_\xi = -\mathcal{S}''_0(u) - \mathcal{S}''_2(u)\delta^2 = 0. \tag{5.35}$$

Going back to the initial variables, the integral giving the expectation value of the density fluctuations localizes around the curve

$$q = \sqrt{\frac{3}{2(\mu - V)}} q_0 \left[1 + \frac{1}{6q_0^2}\left(V''(r) + 3\frac{V'(r)}{r}\right)\right]. \tag{5.36}$$

The peaks of the dynamic structure factor appear at the roots of the equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\sqrt{\frac{3}{2(\mu - V)}} q_0 \left[1 + \frac{1}{6q_0^2}\left(V''(r) + 3\frac{V'(r)}{r}\right)\right]\right] = 0. \tag{5.37}$$

For a power-law potential r^{2k} the roots are still found at $r = 0$. Plugging this back into the saddle-point equation, we see that for $k > 1$, at NLO in the WKB expansion, the position of the peaks remains unchanged with respect to the physical optics approximation. The only exception at this order is the harmonic potential $k = 1$, where we find the dispersion relation

$$q_0 = \sqrt{\frac{2\mu}{3}} q \left(1 - \frac{\omega^2}{q^2\mu} + \cdots\right) \text{ (harmonic potential)}, \tag{5.38}$$

which is concave.

To determine the parametric dependence of the first nonvanishing correction for a flatter potential, we estimate how many derivatives of $V(r)$ are required to obtain a nonzero result at $r = 0$. For $V \propto r^{2k}$, this occurs at the $2k$th derivative. Since R_{cl} sets the characteristic length scale, we expect a term of order $O(R_{\mathrm{cl}}^{2k})$. Substituting $R_{\mathrm{cl}} = \sqrt{2\mu}/\varpi$ and using the only remaining scale q to construct a dimensionless correction, we conclude that for a superharmonic potential the leading low-energy correction to the dispersion relation must take the form

$$q_0 = \sqrt{\frac{2\mu}{3}} q \left[1 + O\left(\frac{\omega^{2k}}{q^{2k}\mu^k}\right)\right] \text{ (superharmonic potential)}. \tag{5.39}$$

In the low-energy regime, the control parameter is $\delta = \varpi/q_0$, which depends explicitly on the scale fixed by the trapping potential. In contrast with the result of the previous section, the leading correction to the linear behavior of the dispersion relation depends explicitly on the details of $V(r)$. The flatter the potential (larger values of k), the more the deviation from the linear behavior is suppressed.

D. Harmonic potential

In Sec. IV E we have seen that the EOM for the harmonic trap can be solved exactly in the low-energy regime, and that in the s-wave limit, we can write the radial part of the wave function as

$$\pi_r^{n,0}(u) = U_n(1 - 2u^2) = \frac{2\sin[2(n+1)\arcsin(u)]}{u\sqrt{1-u^2}},$$
$$q_0 = \omega\sqrt{\frac{4n}{3}(n+2)}. \tag{5.40}$$

The low-energy regime $\omega/q_0 \ll 1$, coincides with the large-n limit, and we can follow the steps in Sec. V B to estimate the value the integral in the density fluctuations with a saddle-point approximation.

The integral in the expectation value for the density fluctuations localizes around the saddle at

$$q = \frac{2(n+1)}{R_{cl}\sqrt{1-u^2}}, \tag{5.41}$$

or, in terms of q_0:

$$q_0 = \sqrt{\frac{2\mu(1-u^2)}{3}\left(q^2 - \frac{2\omega^2}{\mu(1-u^2)}\right)}. \tag{5.42}$$

The peak of the dynamic structure factor is at $u = 0$, leading to the dispersion relation

$$q_0 = \sqrt{\frac{2\mu}{3}\left(q^2 - \frac{2\omega^2}{\mu}\right)} = \sqrt{\frac{2\mu}{3}} q\left(1 - \frac{\omega^2}{q^2\mu} + \cdots\right), \tag{5.43}$$

which is consistent with what we had found above at NLO in the WKB expansion.

VI. CONCLUSIONS

Experiments with ultracold atomic gases invariably impose a trapping mechanism in order to confine the atoms to a controlled, finite region of space. As a consequence, measurements of atomic observables carry an imprint of the trapping potential, denoted by V, and conclusions regarding the homogeneous system—i.e., the cold atomic gas in the absence of the trap—require a detailed quantitative understanding of the trapping effects. A common and very-successful method for managing these effects is the local density approximation, which effectively replaces everywhere the chemical potential with a position-dependent quantity that incorporates the effect of the trapping potential. In this paper, we have performed a careful study of the trap dependence of the superfluid EFT which describes fermions at unitarity. The sole degree of freedom in this EFT is the phonon field, whose fluctuations are strongly modified by V. This particular system is of

tremendous topical interest experimentally and is consequently also of great interest to theorists, who are able to exploit the large degree of symmetry of the system.

At unitarity, the Schrödinger-invariant superfluid system has a single scale, the chemical potential μ, generated by spontaneous breaking of the particle number, and, consequently, the nonrelativistic conformal symmetry. The trapping potential introduces a new scale, which can be taken either as the size of the superfluid droplet, R_{cl}, or the energy scale ϖ. These scales suggest a convenient hierarchy where the phonon field fluctuations can be treated in three distinct regimes characterized by the energy q_0 where $\varpi \ll q_0 \ll \mu$.

The main conclusions of this paper are the following:

(1) The low-energy regime corresponds to $\varpi \ll q_0 \ll \sqrt{\varpi\mu}$ and the phonon dynamics is described by the LO EFT and higher-order terms in the WKB expansion.

(2) The high-energy regime corresponds to $\sqrt{\varpi\mu} \ll q_0 \ll \mu$ and the dynamics is described by LO WKB and higher-order terms in the EFT.

(3) The low- and high-energy regimes are separated by an intermediate linear zone where $q_0 \approx \sqrt{\varpi\mu}$; this is described by LO EFT and LO WKB (see Fig. 2).

(4) Through a study of the phonon field fluctuations and dynamic structure factor in the various regimes, we find that the phonon dispersion relation receives significant trap-induced modifications only in the low-energy regime; the strongest effect appears for a harmonic potential. The corrections are small but calculable in the other regimes.

In this paper, we have treated in detail the special case of power-law confining potentials. However, the fundamentals of the analysis persist for general potentials that may be implemented experimentally. In addition, the results obtained here, while perhaps currently most relevant to the study of the BCS-BEC crossover regime in gases of fermionic atoms, have universal applicability. For instance, it may be the case that the surfaces, or skins, of neutron-rich nuclei (including neutron stars) experience squeezing of the three-dimensional superfluid neutron matter into a quasi-two-dimensional region, as well as other confining scenarios, which can be conveniently modeled by using anisotropic trapping potentials [37] and analyzed via the methods introduced here.

It is a straightforward matter to include Schrödinger-symmetry breaking effects due to a finite scattering length in the superfluid EFT. This enables a study of the fluctuations across the BCS-BEC crossover. The introduction of yet another scale generalizes the simple EFT hierarchy treated here in a straightforward manner. As the low-energy constants that appear in the superfluid EFT are somewhat uncertain, we postpone the discussion of symmetry breaking to a separate paper, which will address this issue, as well as finite-temperature corrections, in the context of the large-N expansion.

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DATA AVAILABILITY

There are currently no publicly available research data or software supporting this work. Requests for further information or data should be sent to the authors.

APPENDIX: RESPONSE FUNCTION IN THE PATH INTEGRAL FORMALISM

We can compute the response function and the dynamic form factor by using a different formalism, based on the path integral quantization. By definition, the response function is the two-point function for the charge density,

$$\chi(\mathbf{x},\mathbf{x}';t-t') = \langle \rho(\mathbf{x},t)\rho(\mathbf{x}',t')\rangle_T, \tag{A1}$$

where $\rho(\mathbf{x},t)$ is the charge density. Note that we are assuming time-translation invariance but no space-translation invariance, so to accommodate a potential. In the path integral formalism, the two-point function is computed by coupling the system to an external electric potential A_0, which is by construction the current dual to the charge density:

$$\chi(\mathbf{x},\mathbf{x}';t-t') = \frac{\delta^2 W[A_0]}{\delta A_0(\mathbf{x},t), A_0(\mathbf{x}',t')}, \tag{A2}$$

where $W[A_0]$ is the generating functional

$$W[A_0] = -\log\left(\int \mathcal{D}\theta e^{-S[\theta,A_0]}\right), \tag{A3}$$

where we assume Euclidean signature.

We are interested in the study of small fluctuations, so we expand $S[\theta, A_0]$ at quadratic order both in the Goldstone field $\pi = \theta - \mu t$ and in A_0, to write the action in the form

$$\begin{aligned} S[\pi, A_0] = \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}\mathbf{x}[&\pi(\mathbf{x},t)\mathcal{O}_{\pi\pi}\pi(\mathbf{x},t) \\ &+ 2\pi(\mathbf{x},t)\mathcal{O}_{\pi A}A(\mathbf{x},t) + A_0(\mathbf{x},t)\mathcal{O}_{AA}A_0(\mathbf{x},t)], \end{aligned} \tag{A4}$$

where $\mathcal{O}_{ij}$ are local differential operators. In the NLO EFT, the operator $\mathcal{O}_{\pi\pi}$ depends on the three low-energy constants c_0, c_1, and c_2, while $\mathcal{O}_{\pi A}$ and $\mathcal{O}_{AA}$ only depend on c_0 and c_1, since the term proportional to c_2 in the action vanishes in the ground state.

The fluctuations are quadratic, and we can perform the path integral to find

$$\begin{aligned} W[A_0] = &\frac{1}{2}\log[\det(\mathcal{O}_{\pi\pi})] \\ &- \frac{1}{2}\int \mathrm{d}x\mathrm{d}x' A_0(\mathbf{x})\mathcal{O}^{\dagger}_{\pi A}(x)\mathcal{O}^{\dagger}_{\pi A}(x')D(x;x')A_0(x') \\ &+ \frac{1}{2}\int \mathrm{d}x A_0(x)\mathcal{O}_{AA}(x)A_0(x), \end{aligned} \tag{A5}$$

where $D(x;x') = \mathcal{O}^{-1}_{\pi\pi}$ is the propagator of the small fluctuations. The response function is then[8]

$$\chi(\mathbf{x},\mathbf{x}';t-t') = -\mathcal{O}^{\dagger}_{\pi A}(x)\mathcal{O}^{\dagger}_{\pi A}(x')D(x;x') + \mathcal{O}^{\dagger}_{AA}(x)\delta(x-x'). \tag{A7}$$

To relate it to an experimentally measurable quantity, it is convenient to introduce relative and center-of-mass coordinates,

$$\mathbf{r} = \mathbf{x} - \mathbf{x}', \quad \mathbf{R} = \frac{\mathbf{x}+\mathbf{x}'}{2}, \tag{A8}$$

and Fourier transform with respect to $\mathbf{r}$ and $\Delta t = t - t'$:

$$\chi(\mathbf{q},\mathbf{R};q_0) = \int \mathrm{d}\Delta t\,\mathrm{d}\mathbf{r} e^{i\mathbf{q}\cdot\mathbf{r} - iq_0\Delta t}\chi(\mathbf{r},\mathbf{R};\Delta t). \tag{A9}$$

Due to the fluctuation-dissipation theorem, the imaginary part of the average over the droplet is the dynamic structure factor

$$S(q_0,\mathbf{q}) = -\frac{1}{\pi}\int \mathrm{d}\mathbf{R}\,\mathrm{Im}[\chi(\mathbf{q},\mathbf{R};q_0)]. \tag{A10}$$

This function will be peaked along a curve in the (q_0, q) plane. The curve $q_0 = q_0(q)$ is the dispersion relation.

[8]The second term is the static response function

$$\chi(\mathbf{x},\mathbf{x}') = \int \mathrm{d}t\,\chi(\mathbf{x},\mathbf{x}';t) = \delta(\mathbf{x}-\mathbf{x}')\mathcal{O}_{AA}(\mathbf{x}) \tag{A6}$$

and, by what we said above, only depends on c_0 and c_1.

In the LO EFT, the operators are

$$\mathcal{O}_{\pi\pi} = -\partial_0^2 + L, \quad \mathcal{O}_{\pi A} = -\partial_0, \quad \mathcal{O}_{AA} = 1, \tag{A11}$$

where L is the Sturm–Liouville operator,

$$L\pi = \frac{\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\mathcal{L}'[\mu-V]\frac{\mathrm{d}}{\mathrm{d}r}\pi\right) - \mathcal{L}'[\mu-V]\ell(\ell+1)\pi}{r^2\mathcal{L}''[\mu-V]}. \tag{A12}$$

Using the general theory outlined in Sec. IV, we can decompose the propagator on a basis of eigenfunctions of L:

$$L\pi_n = E_n^2\pi_n, \tag{A13}$$

$$D(\mathbf{x},\mathbf{x}';q_0) = \sum_n \frac{\pi_n(\mathbf{x})^*\pi_n(\mathbf{x})}{q_0^2 - E_n^2}, \tag{A14}$$

and recover the same expression for $S(\mathbf{q}, q_0)$ as in Eq. (5.1). To go beyond LO, it becomes cumbersome to decompose the propagator on the basis of eigenfunctions, but it is convenient to use an alternative approach, as done in Ref. [15]. We leave this problem for future investigations.

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