Quantum speed limit in open quantum systems: A stochastic approach

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Quantum speed limit characterizes the fastest rate at which a quantum system can evolve in the quantum-state space. It plays a significant role in the search for more efficient quantum control technologies. The previous work demonstrated that, within a stochastic quantum dynamical framework for open quantum systems, the quantum speed limit bound can be geometrically established via the evolution of the joint system-environment state. However, its explicit expression and tightness are generally unknown. In this paper, going beyond the usual Born-Markovian approximation, we provide a unified scheme to derive the exact expression of the quantum speed limit for a general discrete-variable open quantum system. More importantly, we demonstrate that the quantum speed limit based on the joint system-environment state can be tighter than that obtained from the traditional convolutionless master equation approach. Enriching the characterization schemes of the quantum speed limit, our result may have potential applications for some quantum manipulation tasks in practical noisy situations.

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I. INTRODUCTION

Quantum speed limit (QSL) imposes a lower bound on the evolution time between two distinguishable quantum states [1–7]. This concept was originally proposed to interpret the celebrated Heisenberg's energy-time uncertainty relation. In recent years, the QSL has become of significance, because the QSL has been employed to estimate the computation limits in quantum computing [8], the ultimate precision in quantum metrology [9–12], as well as the maximal power or the minimal energy cost in quantum engines and other quantum devices [13–20].

For closed quantum systems described by a time-independent Hamiltonian $\hat{H}_s$, Mandelstam and Tamm first reported a QSL bound for the shortest evolution time from $|\psi_s(0)\rangle$ to $|\psi_s(\tau)\rangle$ [1]. This QSL bound is called the Mandelstam-Tamm bound, which states $\tau \geqslant \tau_{\rm MT} = \arccos[\langle\psi(0)|\psi(\tau)\rangle]/\Delta E$ with $\Delta E = [\langle\psi_s(0)|\hat{H}_s^2|\psi_s(0)\rangle - \langle\psi_s(0)|\hat{H}_s|\psi_s(0)\rangle^2]^{1/2}$ being the energy variance. After the work of Mandelstam and Tamm, other different QSL bounds, such as the Margolus-Levitin bound [3] and the Sun-Zheng bound [4,21], were established sequentially for the unitary dynamics. By combining these independent bounds, a tight QSL for closed quantum systems can be well quantified.

The generalization of the QSL to open quantum systems has been discussed in Refs. [22–30]. However, almost all the existing studies rely on the tool of the quantum master equation, which describes the equation of motion for the reduced density matrix of the system itself by tracing out the degrees of freedom of the environment. In sharp contrast to this traditional master equation approach, in our preceding article, namely Ref. [31], by making use of the non-Markovian quantum state diffusion (QSD) equation method [32–34], we proposed an alternative scheme to characterize the QSL based on the evolution of the joint system-environment state. This scheme requires neither measuring the quantum jumps nor tracing over the degrees of freedom of the environment. Thus it can provide a global perspective for the QSL from the total Hamiltonian without *a priori* knowledge of the quantum master equation. A similar idea of characterizing the physical properties of an open system from the joint system-environment state has been widely employed in the studies of the geometric phase [35–37] and the non-Markovianity [38].

Unfortunately, some issues remain unresolved in our preceding article. For example, Ref. [31] only provides a coarse framework of deriving the QSL by using the stochastic wave function of the non-Markovian QSD equation method. How to obtain the stochastic wave function and derive the corresponding expression of the QSL for a general discrete-variable open quantum system is unknown. Moreover, it is important to determine whether the QSL derived from the joint system-environment state can provide a tighter bound compared to the traditional quantum master equation approach. By addressing these questions, the purpose of this paper is to extend and enrich our previous studies in Ref. [31], and to demonstrate our proposed scheme indeed provides an elegant way to characterize the QSL in open quantum systems.

This paper is organized as follows. In Sec. II, we briefly review the non-Markovian QSD equation method, which provides a powerful tool to handle the dissipative dynamics of an open quantum system from a stochastic dynamical viewpoint. In Sec. III, we show the details of characterizing the geometric QSL from the evolution of the joint system-environment state. The Feshbach projection-operator partitioning technique [39–42] is employed to derive the exact expression

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of the QSL beyond the Born-Markovian and the secular approximations, which are usually adopted in the traditional quantum master equation approach. A pure dephasing system and a dissipative $(N+1)$-level system are respectively used in Secs. IV and V as the illustrative examples to demonstrate the feasibility of our proposed scheme. In Sec. V, we show that our proposed QSL can be tighter than the previous results. The main conclusions of this paper are drawn in Sec. VI. In several Appendixes, we provide some additional materials about the main text. Throughout the paper, for the sake of simplicity, we set $\hbar = k_B = 1$.

II. NON-MARKOVIAN QSD EQUATION APPROACH

Let us consider a generic quantum system $\hat{H}_s$, which is embedded into a dissipative bosonic environment. The whole system-plus-environment Hamiltonian reads

$$\hat{H}_{\text{tot}} = \hat{H}_s + \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k + \sum_k (g_k \hat{L}^\dagger \hat{b}_k + \text{H.c.}), \tag{1}$$

where $\hat{L}$ is the coupling operator of the system to the environment, $\hat{b}_k$ is the annihilation operator of the kth environmental mode with the frequency ω_k, and g_k quantifies the system-environment coupling strengths. In the theory of open quantum systems, the frequency dependence of the interaction strengths is generally encoded into the so-called spectral density, which is defined as $J(\omega) \equiv \sum_k |g_k|^2 \delta(\omega - \omega_k)$. The explicit expressions of $\hat{H}_s$, $\hat{L}$, and $J(\omega)$ are not addressed here, because our employed formula is universal to their forms. Later, we will provide some examples of them to verify the feasibility of our proposed scheme.

Starting from a product initial state $|\Psi_{\text{tot}}(0)\rangle = |\psi_s(0)\rangle \otimes |\mathbf{0}\rangle$ with $|\mathbf{0}\rangle \equiv \bigotimes_k |0_k\rangle$ and $\hat{b}_k|0_k\rangle = 0$ being the Fock vacuum state of the bosonic environment, the evolution of the whole wave function is governed by the Schrödinger equation $|\dot{\Psi}_{\text{tot}}(t)\rangle = -i\hat{H}_{\text{tot}}|\Psi_{\text{tot}}(t)\rangle$. Practically, due to the large number of degrees of freedom in the environment, it is commonly impossible to directly solve this Schrödinger equation. However, by introducing the basis of the Bargmann coherent state $|z\rangle \equiv \bigotimes_k |z_k\rangle$ with $\hat{b}_k|z_k\rangle = z_k|z_k\rangle$, one can recast the Schrödinger equation into the non-Markovian QSD equation as (see Appendix A and Refs. [32–34] for details)

$$|\dot{\psi}_{\bar{z}}(t)\rangle = -i\hat{\mathcal{H}}_s|\psi_{\bar{z}}(t)\rangle, \tag{2}$$

where $|\psi_{\bar{z}}(t)\rangle \equiv \langle z|\Psi_{\text{tot}}(t)\rangle$ with $\bar{o} \equiv o^*$ is the total wave function in the Bargmann coherent state representation, and $\hat{\mathcal{H}}_s$ is an effective Hamiltonian defined as

$$\hat{\mathcal{H}}_s = \hat{H}_s + i\hat{L}\bar{z}_t - i\hat{L}^\dagger\hat{\mathcal{O}}(t, \bar{z}). \tag{3}$$

Here, the variable $z_t \equiv i\sum_k g_k z_k e^{-i\omega_k t}$ is a Gaussian color noise satisfying $\mathcal{M}\{z_t\} = \mathcal{M}\{\bar{z}_t\} = 0$ and $\mathcal{M}\{z_t\bar{z}_s\} = \alpha(t-s)$, where the notation $\mathcal{M}\{o(z_k)\} \equiv \prod_k \int \frac{d^2 z_k}{\pi} e^{-|z_k|^2} o(z_k)$ denotes the ensemble average over all the possible quantum trajectories of noises and $\alpha(t) \equiv \int_0^\infty d\omega J(\omega) e^{-i\omega t}$ is the environmental correlation function at zero temperature. The noise-dependent operator $\hat{\mathcal{O}}(t, \bar{z})$ is introduced as $\hat{\mathcal{O}}(t, \bar{z}) \equiv \int_0^t ds \alpha(t-s) \frac{\delta}{\delta \bar{z}_s}$, which is a functional derivative with respect to the stochastic wave function.

Commonly, there are three ways to obtain the expression of $\hat{\mathcal{O}}(t, z)$ [43–52]. (i) If $[\hat{L}, \ldots [\hat{L}, [\hat{L}, \hat{H}_s]] \ldots] = 0$, an exact expression for $\hat{\mathcal{O}}(t, z)$ can be analytically derived in principle [33,43–45]: (ii) If $[\hat{L}, \ldots [\hat{L}, [\hat{L}, \hat{H}_s]] \ldots] \neq 0$, such as the spin-boson model with the counterrotating-wave terms, one can expand the operator $\hat{\mathcal{O}}(t, z)$ in terms of the coupling strength, and then use the standard perturbation theory to compute the approximate expression [48]. (iii) For certain special cases, where the environmental correlation function $\alpha(t)$ can be (or at least approximately) written as a finite sum of exponentials [49,50,52], $\hat{\mathcal{O}}(t, z)$ can be numerically determined in a hierarchical equation form [51]. Thus the computational complexity of the non-Markovian QSD equation approach depends critically on the commutation structure between $\hat{L}$ and $\hat{H}_s$, which encodes the structure of the system-environment coupling. As long as $\hat{\mathcal{O}}(t, \bar{z})$ can be analytically or numerically determined, the time evolution of the open quantum system can be accordingly ascertained.

It should be pointed out that Eq. (3) from the non-Markovian QSD equation approach is totally different from the non-Hermitian Hamiltonian in Refs. [53–57]. In these references, the non-Hermitian Hamiltonian is commonly obtained from a Lindblad-type quantum master equation by ignoring the quantum-jump terms. In this sense, this non-Hermitian Hamiltonian only works in the Born-Markovian regime and may lead to the problem of non-normalization. In sharp contrast to this, the employed non-Markovian QSD equation method can provide a full description of the quantum dissipative dynamics with the quantum-jump terms self-consistently included. Taking the ensemble average over all the possible quantum trajectories, one can fully recover the exact non-Markovian quantum master equation [43–45].

III. QSL IN OPEN QUANTUM SYSTEMS

In the traditional scheme, the geometric QSL bound is built in the space of the reduced density matrix of the system $\tilde{\Theta}_s$. In our preceding article, we proposed that a geometric QSL bound for open quantum systems can be established via the joint system-environment state. As displayed in Fig. 1, in the space of the total system-plus-environment state $\Theta_s \bigotimes \Theta_b$, the shortest distance, or geodesic, connecting $|\Psi_{\text{tot}}(0)\rangle$ and $|\Psi_{\text{tot}}(\tau)\rangle$ can be characterized by the Bures angle [2,6]

$$\mathcal{L}_B \equiv \arccos|\langle\Psi_{\text{tot}}(0)|\Psi_{\text{tot}}(\tau)\rangle|. \tag{4}$$

However, the actual evolution path, which is quantified by the line integral $\ell \equiv \int_0^\tau dt \sqrt{g_{tt}}$ with

$$g_{tt}^{\text{FS}} \equiv \langle\dot{\Psi}_{\text{tot}}(t)|\dot{\Psi}_{\text{tot}}(t)\rangle - |\langle\dot{\Psi}_{\text{tot}}(t)|\Psi_{\text{tot}}(t)\rangle|^2 \tag{5}$$

being the famous Fubini-Study metric, dramatically deviates from the geodesic one. Thus, by introducing the average speed of the evolution as $\bar{v} = \ell/\tau$, a geometric QSL bound can be established as $\tau \geqslant \tau_{\text{QSL}}$ with [6,58]

$$\tau_{\text{QSL}} = \frac{\mathcal{L}_B}{\bar{v}} = \frac{\mathcal{L}_B \tau}{\int_0^\tau dt \sqrt{g_{tt}^{\text{FS}}}}. \tag{6}$$

Within the framework of the non-Markovian QSD equation approach, we found $\mathcal{L}_B$ and g_{tt}^{FS} can be evaluated via the stochastic wave function $|\psi_{\bar{z}}(t)\rangle$ as follows [31]:

$$\mathcal{L}_B = \arccos|\mathcal{M}\{\langle\psi_z(0)|\psi_{\bar{z}}(\tau)\rangle\}|, \tag{7}$$

$$g_{tt}^{\text{FS}} = \mathcal{M}\{\langle\dot{\psi}_z(t)|\dot{\psi}_{\bar{z}}(t)\rangle\} - |\mathcal{M}\{\langle\dot{\psi}_z(t)|\psi_{\bar{z}}(t)\rangle\}|^2. \tag{8}$$

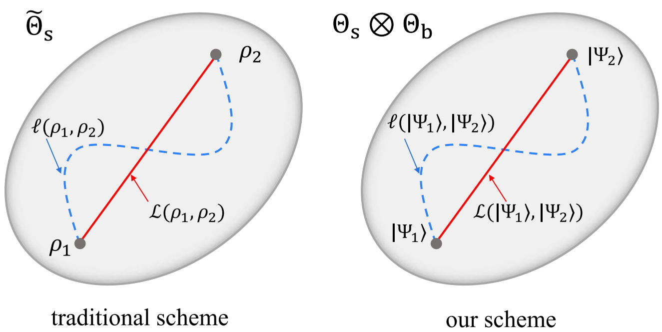

FIG. 1. Sketch of the geometric QSL. The red solid curves are geodesic distances, while the blue dashed lines are the actual evolution path. In the traditional quantum master equation approach, the geometric QSL bound is built in the state space of the reduced density matrix $\tilde{\Theta}_s$, i.e., $\rho_{1,2} \in \tilde{\Theta}_s$. Completely different from this approach, in our proposed scheme, geometric QSL is derived in the state space of the total system-plus-environment total wave function, namely, $|\Psi_{1,2}\rangle \in \Theta_s \otimes \Theta_b$ with Θ_s and Θ_b being the Hilbert spaces for the system and the environment, respectively.

Using Eq. (2), one finds the metric can be expressed in an alternative way as $g_{tt}^{\rm FS} = \mathcal{M}\{\langle\psi_z(t)|\hat{\mathcal{H}}_s^\dagger\hat{\mathcal{H}}_s|\psi_{\bar{z}}(t)\rangle\} - |\mathcal{M}\{\langle\dot{\psi}_z(t)|\psi_{\bar{z}}(t)\rangle\}|^2$. When the quantum system is completely decoupled from the surrounding environment, we have $|\psi_{\bar{z}}(t)\rangle = |\psi_s(t)\rangle$ and $\hat{\mathcal{H}}_s = \hat{H}_s$, which means $g_{tt}^{\rm FS}$ returns to the common energy variance and our proposed QSL bound naturally reduces to the Mandelstam-Tamm bound [1].

Physically, $\tau_{\rm QSL}$ quantifies the shortest evolution time from $|\Psi_{\rm tot}(0)\rangle$ to $|\Psi_{\rm tot}(\tau)\rangle$. One easily sees that $0 < \tau_{\rm QSL}/\tau = \mathcal{L}_B/\ell \leqslant 1$. As discussed in many previous studies [6,59–64], the ratio of $\tau_{\rm QSL}/\tau$ can be regarded as an indicator of the tightness, namely, how much a certain geometric QSL bound is saturated [6]. If the actual evolution path fully saturates the geodesic one, i.e., $\ell = \mathcal{L}$, we have $\tau_{\rm QSL}/\tau = 1$. This means the corresponding QSL bound is fully saturated and the quantum system has no more space for speedup. On the contrary, when $\tau_{\rm QSL}/\tau < 1$, there is still room for improvement in tightness. One thus concludes that the closer $\tau_{\rm QSL}/\tau$ is to 1, the better the tightness. Finding the tightest QSL bound plays an important role in estimating the fastest evolution speed.

Next, we display how to use the Feshbach PQ partitioning technique [39–42] to derive the expressions of $\mathcal{L}_B$ and $g_{tt}^{\rm FS}$. Assuming the system is a general $(N+1)$-level discrete-variable system, the stochastic wave function $|\psi_{\bar{z}}(t)\rangle$ and the effective Hamiltonian $\hat{\mathcal{H}}_s$ can be written in the following matrix form:

$$|\psi_{\bar{z}}(t)\rangle = \begin{bmatrix} P(t) \\ \hline Q(t)\end{bmatrix}, \quad \hat{\mathcal{H}}_s = \left[\begin{array}{c|c} h(t) & R(t) \\ \hline W(t) & D(t)\end{array}\right]. \tag{9}$$

Here, $P(t)$ is a scalar function in the P space spanned by an arbitrary chosen basis, $Q(t)$ is an N-dimensional vector in the residual Q space. Accordingly, $h(t)$ and $D(t)$ are self-Hamiltonians living in the P subspace and the Q subspace, respectively; $R(t)$ and $W(t)$ are their mutual correlation terms. Consequently, the non-Markovian QSD equation can be formally reexpressed into two coupled equations, i.e.,

$$i\dot{P}(t) = h(t)P(t) + R(t)Q(t), \tag{10}$$

$$i\dot{Q}(t) = W(t)P(t) + D(t)Q(t). \tag{11}$$

The formal solution of $Q(t)$ is

$$Q(t) = -i\int_0^t dsG(t,s)W(s)P(s) + G(t,0)Q(0), \tag{12}$$

where $\partial_t G(t,s) = -iD(t)G(t,s)$ with the initial condition $G(t,t) = 1$. Substituting Eq. (12) into Eq. (10), the equation of motion for $P(t)$ is given by

$$\begin{aligned} i\dot{P}(t) = h(t)P(t) - iR(t)\int_0^t dsG(t,s)W(s)P(s) \\ + R(t)G(t,0)Q(0). \end{aligned} \tag{13}$$

Once $P(t)$ and $Q(t)$ are solved, the expressions of $\mathcal{L}_B$ and $g_{tt}^{\rm FS}$ can be evaluated via

$$\mathcal{L}_B = \arccos|\mathcal{M}\{\bar{P}(0)P(\tau) + Q^\dagger(0)Q(\tau)\}|, \tag{14}$$

$$\begin{aligned} g_{tt}^{\rm FS} =& \mathcal{M}\{\dot{\bar{P}}(t)\dot{P}(t) + \dot{Q}^\dagger(t)\dot{Q}(t)\} \\ &- |\mathcal{M}\{\bar{P}(t)\dot{P}(t) + Q^\dagger(t)\dot{Q}(t)\}|^2. \end{aligned} \tag{15}$$

Next, we use two illustrative examples to verify the feasibility of the above scheme.

IV. PURE DEPHASING SYSTEM

In the first example, we consider an exactly solvable model: the pure dephasing two-level system with $\hat{H}_s = \frac{1}{2}\omega_0\hat{\sigma}_z$ and $\hat{L} = \lambda\hat{\sigma}_z$ with λ being a dimensionless real constant. This model is of interest, because its total wave function $|\Psi_{\rm tot}(t)\rangle$ can be exactly derived in an analytical way [65]. The Bures angle $\mathcal{L}_B \equiv \arccos|\langle\Psi_{\rm tot}(0)|\Psi_{\rm tot}(\tau)\rangle|$ and the metric $g_{tt}^{\rm FS} \equiv \langle\dot{\Psi}_{\rm tot}(t)|\dot{\Psi}_{\rm tot}(t)\rangle - |\langle\dot{\Psi}_{\rm tot}(t)|\Psi_{\rm tot}(t)\rangle|^2$ can be straightforwardly obtained from the expression of total wave function $|\Psi_{\rm tot}(t)\rangle$ (see Appendix B for more details). This particular characteristic can be employed as the benchmark to check the correctness of our proposed scheme based on the non-Markovian QSD equation method.

Within the framework of the non-Markovian QSD equation method, the effective Hamiltonian for the pure dephasing model is given by

$$\hat{\mathcal{H}}_s = \frac{1}{2}\omega_0\hat{\sigma}_z + i\lambda\bar{z}_t\hat{\sigma}_z - i\lambda^2\int_0^t ds\alpha(t-s)\mathbf{1}_2. \tag{16}$$

Choosing the initial state of the quantum system as $|\psi_s(0)\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$ with $|e\rangle$ ($|g\rangle$) being the excited (ground) state of $\hat{\sigma}_z$, one finds

$$P(t) = \frac{1}{\sqrt{2}}\exp\left[-\frac{i}{2}\omega_0 t + \lambda\int_0^t ds\bar{z}_s - \lambda^2\Lambda(t)\right], \tag{17}$$

$$Q(t) = \frac{1}{\sqrt{2}}\exp\left[\frac{i}{2}\omega_0 t - \lambda\int_0^t ds\bar{z}_s - \lambda^2\Lambda(t)\right], \tag{18}$$

where $\Lambda(t) = \int_0^t ds\int_0^s ds'\alpha(s-s')$ denotes the decay factor of the model.

If $J(\omega)$ is chosen as the Lorentzian spectral density, the environmental correlation function is of the Ornstein-Uhlenbeck type, i.e.,

$$\alpha(t) = \tfrac{1}{2}\Gamma\gamma e^{-\gamma|t| + i\Omega t}, \tag{19}$$

where Γ is the coupling strength and γ^{-1} is proportional to the scale of environmental memory time. Assuming $\Omega = 0$ for the

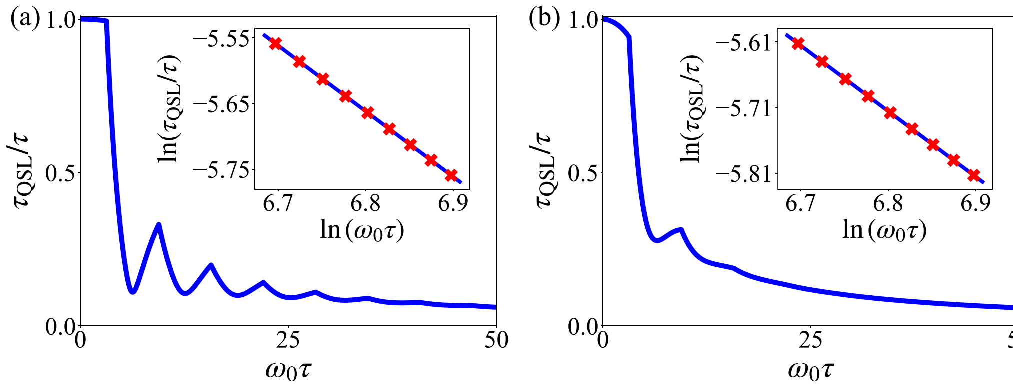

FIG. 2. $\tau_{\rm QSL}/\tau$ vs $\omega_0\tau$ for the pure dephasing model with different environmental memory times: (a) $\gamma = 0.01\omega_0$ and (b) $\gamma = 0.1\omega_0$. The blue solid lines are numerical simulations. The insets show $\tau_{\rm QSL}/\tau \propto 1/\tau$ in the large-τ region with the red crosses being analytical results from Eq. (26). Other parameters are $\omega_0 = 1\ {\rm cm}^{-1}$, $\lambda = 0.8$, and $\Gamma = \omega_0$.

sake of simplicity, $\alpha(t)$ then becomes a real function. In this case, one easily finds

$$\Lambda(t) = \frac{\Gamma}{2\gamma}(\gamma t - 1 + e^{-\gamma t}). \tag{20}$$

Substituting the above expressions into Eqs. (14) and (15) and after a lengthy calculation, we finally arrive at

$$\mathcal{L}_B = \arccos\left|\cos\left(\tfrac{1}{2}\omega_0\tau\right)e^{-\lambda^2\Lambda(\tau)}\right|, \tag{21}$$

$$g_{tt}^{\rm FS} = \tfrac{1}{4}\omega_0^2 + \lambda^2\alpha(0). \tag{22}$$

To derive the above expression, we have used the following relations:

$$\mathcal{M}\{e^{\lambda\int_0^t ds(z_s+\bar{z}_s)}\} = e^{2\lambda^2\Lambda(t)}, \tag{23}$$

$$\mathcal{M}\{z_t e^{\lambda\int_0^t ds(z_s+\bar{z}_s)}\} = \lambda\int_0^t ds\alpha(t-s)e^{2\lambda^2\Lambda(t)}, \tag{24}$$

$$\mathcal{M}\{z_t\bar{z}_t e^{\lambda\int_0^t ds(z_s+\bar{z}_s)}\} = \alpha(0)e^{2\lambda^2\Lambda(t)} + \left[\lambda\int_0^t ds\alpha(t-s)\right]^2 e^{2\lambda^2\Lambda(t)}. \tag{25}$$

These equations can be straightway derived by making use of the cumulant expansion technique combined with the statistical characteristics of the Gaussian noise (see Appendix C for derivations).

From Eq. (20), we have $e^{-\lambda^2\Lambda(t)} \simeq 0$ in the large-t regime. This long-time asymptotic behavior leads to

$$\lim_{\tau\to\infty}\frac{\tau_{\rm QSL}}{\tau} = \frac{\varkappa}{\tau}, \tag{26}$$

where $\varkappa = \pi/\sqrt{\omega_0^2 + 2\lambda^2\Gamma\gamma}$. Equation (26) is confirmed by the numerical simulations displayed in Fig. 2. This relation is exactly the same as the result from the direct computation via the evolution of the total wave function $|\Psi_{\rm tot}(t)\rangle$ (see Appendix B for the proof). A similar long-time asymptotic behavior is also reported in Ref. [66], which uses the exact convolutionless master equation approach to derive the QSL in the same model. These results convince us that the non-Markovian QSD equation method truly captures the QSL in open quantum systems. On the other hand, from Eq. (26), one sees the QSL tightness becomes very poor, namely, $\tau_{\rm QSL}/\tau \to 0$, in the large-$\tau$ regime. How to improve the tightness in this situation would be a very interesting potential research topic.

V. DISSIPATIVE $(N+1)$-level SYSTEM

Next, in the second example, we extend our analysis to the case of quantum dissipation in which the system can exchange both the information and the energy with its surrounding environment. We consider a general model of a dissipative $(N+1)$-level system. The Hamiltonian is $\hat{H}_s = \sum_{n=0}^N E_n|n\rangle\langle n|$ and the coupling operator is given by $\hat{L} = \sum_{n=1}^N |0\rangle\langle n|$, where we set $E_0 = 0$ and $E_{n\neq 0} = E$ for the sake of simplicity. This model is usually used to describe the relaxation dynamics of multiple qubit or qutrit systems in a noisy environment [41,44]. Within the non-Markovian QSD equation approach, the effective Hamiltonian is given by (see Appendix A)

$$\hat{\mathcal{H}}_s = \left[\begin{array}{c|cccc} 0 & i\bar{z}_t & i\bar{z}_t & \cdots & i\bar{z}_t \\ \hline 0 & E - iF(t) & -iF(t) & \cdots & -iF(t) \\ 0 & -iF(t) & E - iF(t) & \cdots & -iF(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -iF(t) & -iF(t) & \cdots & E - iF(t) \end{array}\right], \tag{27}$$

where $F(t) = \int_0^t ds\alpha(t-s)f(t,s)$ with $f(t,s)$ determined by (see Appendix A)

$$\dot{f}(t,s) = -iEf(t,s) + Nf(t,s)\int_0^t ds'\alpha(t-s')f(t,s') \tag{28}$$

and $f(t,t) = 1$. Taking the Ornstein-Uhlenbeck correlation function in Eq. (19) with $\Omega = E$, the equation of motion for $F(t)$ reduces to

$$\dot{F}(t) = \tfrac{1}{2}\Gamma\gamma - \gamma F(t) + NF^2(t) \tag{29}$$

with the initial condition $F(0) = 0$. This equation can be analytically solved and the solution is

$$F(t) = \frac{1}{2N}\left\{\gamma - d\tanh\left[\frac{1}{2}dt + {\rm arctanh}\left(\frac{\gamma}{d}\right)\right]\right\}, \tag{30}$$

where $d = \sqrt{\gamma(\gamma - 2N\Gamma)}$.

Assuming the initial state of the system is $|\psi_s(0)\rangle = \frac{1}{\sqrt{N+1}}\sum_{n=0}^{N}|n\rangle$, the expressions for $P(t)$ and $Q(t)$ are

$$P(t) = \frac{1}{\sqrt{N+1}}\left[1 + N\int_0^t ds\bar{z}_s\mathcal{F}^N(s)e^{-iEs}\right], \tag{31}$$

$$Q(t) = \frac{1}{\sqrt{N+1}}\mathcal{F}^N(t)e^{-iEt}\sum_{n=1}^{N}|n\rangle, \tag{32}$$

where $\mathcal{F}(t) = \exp[-\int_0^t dsF(s)]$. Substituting the above expressions into Eqs. (14) and (15), we finally obtain the exact expressions of $\mathcal{L}_B$ and $g_{tt}^{\rm FS}$ as

$$\mathcal{L}_B = \arccos\left[\frac{1}{N+1}\left|1 + Ne^{-iE\tau}\mathcal{F}^N(\tau)\right|\right], \tag{33}$$

and

$$g_{tt}^{\rm FS} = \mathcal{M}\{\dot{\bar{P}}(t)\dot{P}(t)\} + \mathcal{M}\{\dot{Q}^\dagger(t)\dot{Q}(t)\} - |\mathcal{M}\{\bar{P}(t)\dot{P}(t)\} + \mathcal{M}\{Q^\dagger(t)\dot{Q}(t)\}|^2, \tag{34}$$

where

$$\mathcal{M}\{\dot{\bar{P}}(t)\dot{P}(t)\} = \frac{N^2}{N+1}\alpha(0)|\mathcal{F}(t)|^{2N}, \tag{35}$$

$$\mathcal{M}\{\dot{Q}^\dagger(t)\dot{Q}(t)\} = \frac{N}{N+1}[E^2|\mathcal{F}(t)|^{2N} + N^2|\dot{\mathcal{F}}(t)|^2|\mathcal{F}(t)|^{2(N-1)} - iEN\bar{\mathcal{F}}^{N-1}\dot{\bar{\mathcal{F}}}(t)\mathcal{F}^N(t) + iNE\bar{\mathcal{F}}^N(t)\mathcal{F}^{N-1}\dot{\mathcal{F}}(t)], \tag{36}$$

$$\mathcal{M}\{\bar{P}(t)\dot{P}(t)\} = \frac{N^2}{N+1}\int_0^t ds\bar{\alpha}(t-s)e^{-iE(t-s)}\bar{\mathcal{F}}^N(s)\mathcal{F}^N(t), \tag{37}$$

$$\mathcal{M}\{Q^\dagger(t)\dot{Q}(t)\} = \frac{N}{N+1}[-iE|\mathcal{F}(t)|^{2N} + N\bar{\mathcal{F}}^N(t)\mathcal{F}^{N-1}(t)\dot{\mathcal{F}}(t)]. \tag{38}$$

For the special case of $N = 1$, Eq. (27) reduces to the dissipative two-level system considered in Refs. [31,59,63,67,68]. In this case, one easily finds

$$P(t) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\int_0^t ds\bar{z}_s u(s), \tag{39}$$

$$Q(t) = \frac{1}{\sqrt{2}}u(t), \tag{40}$$

where

$$u(t) = e^{-\frac{1}{2}\gamma t - i\omega_0 t}\left[\cosh\left(\frac{1}{2}\sqrt{\gamma^2 - 2\Gamma\gamma t}\right) + \frac{\gamma}{\sqrt{\gamma^2 - 2\Gamma\gamma}}\sinh\left(\frac{1}{2}\sqrt{\gamma^2 - 2\Gamma\gamma t}\right)\right], \tag{41}$$

with $E = \omega_0$ being the frequency of the two-level system. The corresponding expressions for geodesic length and metric are given by

$$\mathcal{L}_B = \arccos\left|\tfrac{1}{2} + \tfrac{1}{2}u(\tau)\right|, \tag{42}$$

$$g_{tt}^{\rm FS} = \tfrac{1}{2}[|\dot{u}(t)|^2 + \alpha(0)|u(t)|^2] - \tfrac{1}{4}|\bar{u}(t)\dot{u}(t) - \dot{\bar{u}}(t)u(t) + i\omega_0|u(t)|^2|^2. \tag{43}$$

The above expressions exactly reproduce the previous results reported in Ref. [31].

With these results of the dissipative two-level system at hand, one easily demonstrates that, in the large-t regime, $|u(t)|$ exhibits an exponential decay behavior, i.e., $|u(t)| \propto e^{-\kappa t}$ with κ being the decay rate. This long-time behavior leads to $\mathcal{L}_B \simeq \pi/3$ and $g_{tt}^{\rm FS} \simeq e^{-2\kappa t}$ in the long-time regime. These asymptotic results mean

$$\lim_{\tau\to\infty}\frac{\tau_{\rm QSL}}{\tau} \propto \frac{\pi\kappa}{3} = \text{const.} \tag{44}$$

Such a long-time relation is in qualitative agreement with the QSL from the traditional convolutionless quantum master equation approach (see the discussions in Appendix D as well as Refs. [59,63,67]), but in sharp contrast to that of the dephasing case. In Fig. 3, we compare the tightness of the QSL bound obtained by our non-Markovian QSD method and that of the traditional convolutionless master equation approach. It is revealed that our proposed QSL bound can be tighter, namely, the value of $\tau_{\rm QSL}/\tau$ is larger [4,6,69], with suitable parameters. This result highlights an advantage of our approach over previous studies.

VI. CONCLUSION

In summary, by making use of the non-Markovian QSD equation method along with the Feshbach projection-operator partitioning technique, we provide a unified framework to derive the explicit expression of the QSL based on the evolution of the joint system-environment wave function for a general discrete-variable open quantum system. Totally different from the traditional quantum master equation approach, our geometric QSL is established based on the geodesic and the metric of the total system-environment Hilbert space. Two typical examples, i.e., the pure dephasing system and the dissipative multilevel system, are displayed to demonstrate the feasibility of our proposed scheme. Moreover, it is demonstrated that our proposed QSL bound can be tighter than that of the previous convolutionless quantum master equation approach.

In our manuscript, we concentrate on the situation of few-level quantum systems. It still remains an open question whether our proposed scheme can be feasible to interacting many-body systems, such as the one-dimensional transverse field Ising Hamiltonian considered in Ref. [70]. This may serve as an interesting direction for future research. Enriching the search field of the QSL characterization, our result may have potential applications for some quantum-state control tasks in practical noisy situations.

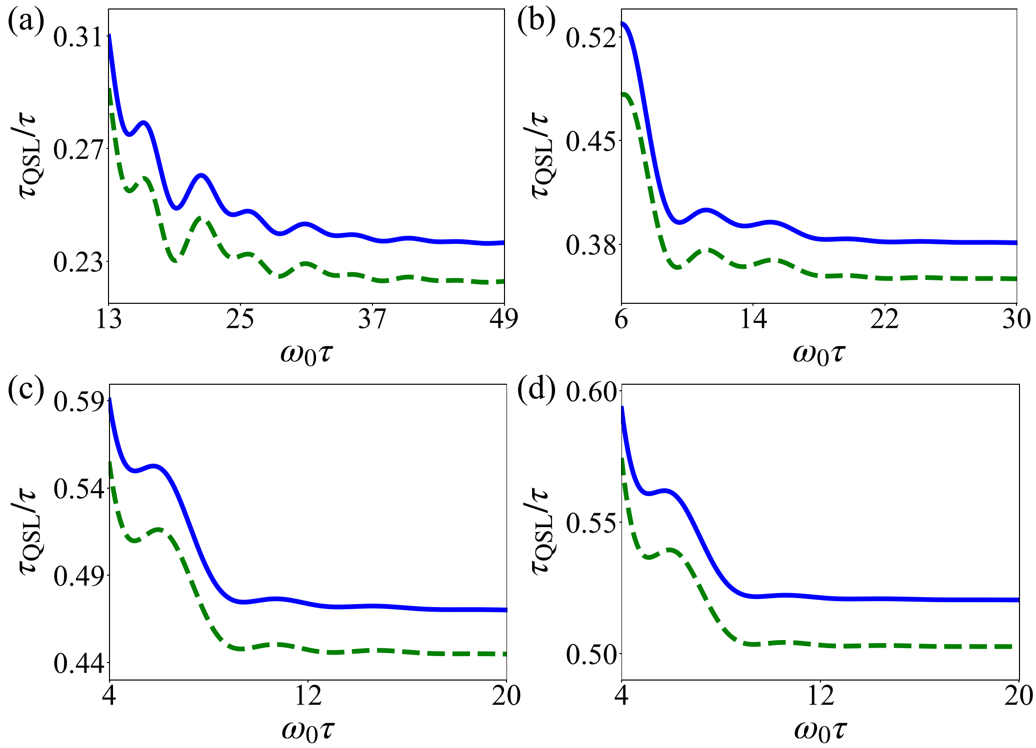

FIG. 3. The QSL $\tau_{\rm QSL}/\tau$ vs $\omega_0\tau$ in the dissipative two-level system with different environmental memory times: (a) $\gamma = 0.25\omega_0$, (b) $\gamma = 0.5\omega_0$, (c) $\gamma = 0.75\omega_0$, and (d) $\gamma = \omega_0$. The blue solid lines are from our proposed QSL, while the green dashed lines are the results predicted by the traditional convolutionless master equation approach displayed in Appendix D. Other parameters are $\omega_0 = 1$ cm^{-1} and $\Gamma = \omega_0$.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: DERIVATION OF NON-MARKOVIAN QSD EQUATION

In this Appendix, we shall display the details of deriving the non-Markovian QSD equation. In the interaction picture, the evolution of the total wave function $|\Psi_{\rm tot}(t)\rangle$ is governed by the Schrödinger equation as

$$|\dot{\Psi}_{\rm tot}(t)\rangle = -i\left[\hat{H}_s + \sum_k (\bar{g}_k\hat{L}\hat{b}_k^\dagger e^{i\omega_k t} + g_k\hat{L}^\dagger\hat{b}_k e^{-i\omega_k t})\right] \times |\Psi_{\rm tot}(t)\rangle. \tag{A1}$$

By left-multiplying the Bargmann coherent state $\langle z|$ on both sides of the above equation, one finds

$$|\dot{\psi}_{\bar{z}}(t)\rangle = -i\hat{H}_s|\psi_{\bar{z}}(t)\rangle - i\langle z|\left[\sum_k (\bar{g}_k\hat{L}\hat{b}_k^\dagger e^{i\omega_k t} + g_k\hat{L}^\dagger\hat{b}_k e^{-i\omega_k t})\right]|\Psi_{\rm tot}(t)\rangle. \tag{A2}$$

Next, using the following properties of the Bargmann coherent state [32],

$$\hat{b}_k|z_k\rangle = z_k|z_k\rangle,\ \hat{b}_k^\dagger|z_k\rangle = \frac{\partial}{\partial z_k}|z_k\rangle, \tag{A3}$$

Eq. (A2) can be reexpressed as

$$|\dot{\psi}_{\bar{z}}(t)\rangle = -i\left[\hat{H}_s + i\bar{z}_t\hat{L} + \hat{L}^\dagger\sum_k g_k e^{-i\omega_k t}\frac{\partial}{\partial\bar{z}_k}\right]|\psi_{\bar{z}}(t)\rangle.$$

Following Refs. [32–34], the term of $\partial_{\bar{z}_k}|\psi_{\bar{z}}(t)\rangle$ can be cast as a functional derivative with respect to $\bar{z}_t$ by making use of the functional chain rule as follows:

$$\frac{\partial}{\partial\bar{z}_k}|\psi_{\bar{z}}(t)\rangle = \int_0^t ds\frac{\partial\bar{z}_s}{\partial\bar{z}_k}\frac{\delta}{\delta\bar{z}_s}|\psi_{\bar{z}}(t)\rangle. \tag{A4}$$

Noticing $\partial\bar{z}_s/\partial\bar{z}_k = -i\bar{g}_k e^{i\omega_k s}$, we finally arrive at

$$|\dot{\psi}_{\bar{z}}(t)\rangle = -i\left[\hat{H}_s + i\bar{z}_t\hat{L} - i\hat{L}^\dagger\int_0^t ds\alpha(t-s)\frac{\delta}{\delta\bar{z}_s}\right]|\psi_{\bar{z}}(t)\rangle, \tag{A5}$$

which reproduces Eq. (2) in the main text. Due to the fact that the effective Hamiltonian $\hat{\mathcal{H}}_s$ is a functional of $\bar{z}_t$, the corresponding solution for the stochastic wave function $|\psi_{\bar{z}}(t)\rangle$ should also be a functional of the noise.

For the dissipative $(N+1)$-level system considered in the main text, one can try an ansatz of the form [33]

$$\frac{\delta}{\delta \bar{z}_s}|\psi_{\bar{z}}(t)\rangle = f(t,s)\hat{L}|\psi_{\bar{z}}(t)\rangle, \tag{A6}$$

with $f(t,s)$ a function, depending on both t and s, to be determined. Applying the following so-called consistency condition [32–34],

$$\frac{\delta}{\delta \bar{z}_s}\frac{\partial}{\partial t}|\psi_{\bar{z}}(t)\rangle = \frac{\partial}{\partial t}\frac{\delta}{\delta \bar{z}_s}|\psi_{\bar{z}}(t)\rangle, \tag{A7}$$

one finds

$$\partial_t f(t,s)\hat{L} = -i[\hat{H}_s, f(t,s)\hat{L}] + \bar{z}_t[\hat{L}, f(t,s)\hat{L}] - \left[\int_0^t ds'\alpha(t-s')f(t,s')\hat{L}^\dagger\hat{L}, f(t,s)\hat{L}\right]. \tag{A8}$$

Noticing $[\hat{H}_s,\hat{L}] = -E\hat{L}$ and $[\hat{L}^\dagger\hat{L},\hat{L}] = -N\hat{L}$, we finally obtain the equation of motion for $f(t,s)$ as

$$\dot{f}(t,s) = -iEf(t,s) + Nf(t,s)\int_0^t ds'\alpha(t-s')f(t,s'), \tag{A9}$$

which reproduces Eq. (28) in the main text. The corresponding effective Hamiltonian is then given by

$$\begin{aligned}\hat{\mathcal{H}}_s &= \hat{H}_s + i\hat{L}\bar{z}_t - i\int_0^t ds\alpha(t-s)f(t,s)\hat{L}^\dagger\hat{L}\\ &= \sum_{n=0}^N E_n|n\rangle\langle n| + i\bar{z}_t\sum_{n=1}^N|0\rangle\langle n| - iF(t)\sum_{n=1}^N|n\rangle\langle n|,\end{aligned} \tag{A10}$$

which recovers Eq. (27) in its matrix form. Similarly, one finds the expression of effective Hamiltonian for the pure dephasing model.

APPENDIX B: QSL FROM THE TOTAL WAVE FUNCTION

From the total wave function viewpoint, for the chosen initial state $|\Psi_{\rm tot}(0)\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)|\mathbf{0}\rangle$, in the interaction picture, the wave function at $t=\tau$ is given by $|\Psi_{\rm tot}(\tau)\rangle = \frac{1}{\sqrt{2}}(e^{-i\omega_0\tau/2}|e\rangle|\xi(\tau)\rangle + e^{i\omega_0\tau/2}|g\rangle|-\xi(\tau)\rangle)$ with $|\xi(\tau)\rangle \equiv \bigotimes_k \exp[\xi_k(\tau)\hat{b}_k^\dagger - \bar{\xi}_k(\tau)\hat{b}_k]|0_k\rangle$ being a bosonic coherent state and $\xi_k(\tau) \equiv \lambda g_k(1-e^{i\omega_k\tau})/\omega_k$ [65]. Using these results, one easily finds

$$\begin{aligned}&\langle\Psi_{\rm tot}(0)|\Psi_{\rm tot}(\tau)\rangle\\ &= \cos\left(\frac{1}{2}\omega_0\tau\right)\exp\left[-\lambda^2\int d\omega J(\omega)\frac{1-\cos(\omega\tau)}{\omega^2}\right],\end{aligned} \tag{B1}$$

which vanishes in the large-τ region, and

$$\langle\Psi_{\rm tot}(t)|\dot{\Psi}_{\rm tot}(t)\rangle = 4i\lambda^2\sum_k\frac{|g_k|^2}{\omega_k}\sin^2\left(\frac{1}{2}\omega_k t\right), \tag{B2}$$

$$\begin{aligned}&\langle\dot{\Psi}_{\rm tot}(t)|\dot{\Psi}_{\rm tot}(t)\rangle\\ &= \frac{1}{4}\omega_0^2 + \sum_k\left[\lambda^2|g_k|^2 + 16\lambda^4\frac{|g_k|^4}{\omega_k^2}\sin^4\left(\frac{1}{2}\omega_k t\right)\right].\end{aligned} \tag{B3}$$

Using the inequality of $\sin^2 x \leqslant |\sin x|$, we find

$$\begin{aligned}0 &\leqslant \sum_k\frac{|g_k|^2}{\omega_k}\sin^2\left(\frac{1}{2}\omega_k t\right) \leqslant \left|\sum_k\frac{|g_k|^2}{\omega_k}\sin\left(\frac{1}{2}\omega_k t\right)\right|\\ &= \left|\frac{t}{2}\sum_k|g_k|^2\mathrm{sinc}\left(\frac{1}{2}\omega_k t\right)\right|,\end{aligned} \tag{B4}$$

where $\mathrm{sinc}(x) \equiv \sin x/x$. In the large-$t$ region, using the following identity of the Dirac-δ function,

$$\lim_{t\to\infty}\frac{t}{\pi}\mathrm{sinc}(tx) = \delta(x), \tag{B5}$$

one finds

$$\frac{t}{2}\sum_k|g_k|^2\mathrm{sinc}\left(\frac{1}{2}\omega_k t\right) = \pi\sum_k|g_k|^2\delta(\omega_k) = 0, \tag{B6}$$

from which we prove that $\langle\Psi_{\rm tot}(t)|\dot{\Psi}_{\rm tot}(t)\rangle = 0$. Via replacing the discrete summation in Eq. (B2) by a continuous integral, i.e., $\sum_k \to \int d\omega$, one can prove the same conclusion by directly using the Riemann-Lebesgue lemma.

On the other hand, one easily sees that

$$\begin{aligned}\lambda^2\sum_k|g_k|^2 &\leqslant \sum_k\left\{\lambda^2|g_k|^2 + 16\lambda^4\frac{|g_k|^4}{\omega_k^2}\sin^4\left(\frac{1}{2}\omega_k t\right)\right\}\\ &\leqslant \lambda^2\sum_k|g_k|^2 + 16\lambda^4\left|\frac{\pi}{2}\sum_k\frac{|g_k|^4}{\omega_k}\frac{t}{\pi}\mathrm{sinc}\left(\frac{1}{2}\omega_k t\right)\right|,\end{aligned} \tag{B7}$$

where the inequality of $\sin^4 x \leqslant |\sin x|$ has been employed. In the large-t region, using Eq. (B5), the faster-oscillating terms on the right-hand side of the above inequality vanish. Thus we find $\langle\dot{\Psi}_{\rm tot}(t)|\dot{\Psi}_{\rm tot}(t)\rangle = \frac{1}{4}\omega_0^2 + \lambda^2\alpha(0)$. From the above expression, one immediately sees $\mathcal{L}_B = \arccos|\langle\Psi_{\rm tot}(0)|\Psi_{\rm tot}(\infty)\rangle| = \pi/2$ and $g_{tt}^{\rm FS} = \langle\dot{\Psi}_{\rm tot}(t)|\dot{\Psi}_{\rm tot}(t)\rangle - |\langle\dot{\Psi}_{\rm tot}(t)|\Psi_{\rm tot}(t)\rangle|^2 = \mathrm{const}$, which leads to the asymptotic behavior of $\tau_{\rm QSL}/\tau \propto 1/\tau$ in the long-time regime.

APPENDIX C: NOISE AVERAGES

In this Appendix, we show the details of deriving Eqs. (23)–(25). Thanks to the Gaussian nature of the noise z_t, $e^{\lambda \int_0^t ds(z_s+\bar{z}_s)}$ can be computed as [71,72]

$$\begin{aligned}\mathcal{M}\{e^{\lambda \int_0^t ds(z_s+\bar{z}_s)}\} &= \exp\left[\lambda^2 \mathcal{M}\left\{\int_0^t dt_1 \int_0^t dt_2 (z_{t_1}+\bar{z}_{t_1})(z_{t_2}+\bar{z}_{t_2})\right\}\right]\\ &= \exp\left[\lambda^2 \int_0^t dt_1 \int_0^t dt_2 (\mathcal{M}\{z_{t_1}\bar{z}_{t_2}\}+\mathcal{M}\{\bar{z}_{t_1}z_{t_2}\})\right]\\ &= \exp\left\{\lambda^2 \int_0^t dt_1 \int_0^{t_1} dt_2 [\alpha(t_1-t_2)+\bar{\alpha}(t_1-t_2)]\right\}\\ &= \exp\left\{2\lambda^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \mathrm{Re}[\alpha(t_1-t_2)]\right\}. \end{aligned} \tag{C1}$$

If $\alpha(t)$ is the Ornstein-Uhlenbeck correlation function with $\Omega = 0$, Eq. (23) in the main text is immediately recovered.

To prove Eq. (24), we introduce a functional,

$$\mathcal{J}(\zeta) \equiv \mathcal{M}\{e^{\lambda \int_0^t dsZ_s+\zeta z_t}\}, \tag{C2}$$

where $Z_t \equiv z_t + \bar{z}_t$ is also a Gaussian noise. By using the cumulant expansion, $\mathcal{J}(\zeta)$ can be expressed as

$$\mathcal{J}(\zeta) = \exp\left[\frac{1}{2}\zeta^2\langle z_t z_t\rangle_c + \frac{1}{2}\lambda\zeta \int_0^t ds(\langle z_t Z_s\rangle_c + \langle Z_s z_t\rangle_c) + \frac{1}{2}\lambda^2 \int_0^t ds_1 \int_0^t ds_2 \langle Z_{s_1} Z_{s_2}\rangle_c\right], \tag{C3}$$

where $\langle\bullet\rangle_c$ denotes the cumulant averages [71]. Here, we have used the following characteristics for Gaussian noises: $\langle z_t\rangle_c = \langle Z_t\rangle_c = 0$ and $\langle z_{t_1} z_{t_2} \dots z_{t_l}\rangle_c = \langle Z_{t_1} Z_{t_2} \dots Z_{t_l}\rangle_c = 0$ for $l \geqslant 3$. With the help of Eq. (C3), we finally arrive at

$$\begin{aligned}\mathcal{M}\{z_t e^{\lambda \int_0^t dsZ_s}\} &= \left.\frac{d\mathcal{J}(\zeta)}{d\zeta}\right|_{\zeta=0}\\ &= \frac{1}{2}\lambda \mathcal{J}(0) \int_0^t ds(\langle z_t Z_s\rangle_c + \langle Z_s z_t\rangle_c)\\ &= \lambda \int_0^t ds\alpha(t-s) \exp\left\{\lambda^2 \int_0^t ds_1 \int_0^t ds_2 \mathrm{Re}[\alpha(t_1-t_2)]\right\}, \end{aligned} \tag{C4}$$

which reproduces Eq. (24) in the main text with $\Omega = 0$.

In the same manner, by introducing a functional,

$$\begin{aligned}\mathcal{G}(\zeta_1,\zeta_2) &\equiv \mathcal{M}\{e^{\lambda \int_0^t dsZ_s+\zeta_1 z_t+\zeta_2\bar{z}_t}\}\\ &= \exp\left[\frac{1}{2}\zeta_1^2\langle z_t z_t\rangle_c + \frac{1}{2}\zeta_2^2\langle\bar{z}_t\bar{z}_t\rangle_c + \frac{1}{2}\zeta_1\zeta_2(\langle z_t\bar{z}_t\rangle_c + \langle\bar{z}_t z_t\rangle_c) + \frac{1}{2}\lambda\zeta_1 \int_0^t ds(\langle z_t Z_s\rangle_c + \langle Z_s z_t\rangle_c)\right.\\ &\quad \left. + \frac{1}{2}\lambda\zeta_2 \int_0^t ds(\langle\bar{z}_t Z_s\rangle_c + \langle Z_s\bar{z}_t\rangle_c) + \frac{1}{2}\lambda^2 \int_0^t ds_1 \int_0^t ds_2 \langle Z_{s_1} Z_{s_2}\rangle_c\right], \end{aligned} \tag{C5}$$

we have

$$\begin{aligned}\mathcal{M}\{z_t\bar{z}_t e^{\lambda \int_0^t dsZ_s}\} &= \left.\frac{d^2\mathcal{G}(\zeta_1,\zeta_2)}{d\zeta_1 d\zeta_2}\right|_{\zeta_1=\zeta_2=0}\\ &= \left[\alpha(0) + \left|\lambda \int_0^t ds\alpha(t-s)\right|^2\right] \exp\left\{\lambda^2 \int_0^t ds_1 \int_0^t ds_2 \mathrm{Re}[\alpha(t_1-t_2)]\right\}, \end{aligned} \tag{C6}$$

which recovers Eq. (25) in the main text if one takes $\Omega = 0$.

APPENDIX D: QUANTUM MASTER EQUATION APPROACH

In this Appendix, we provide the expressions of the QSL of the dissipative two-level system from the traditional convolutionless quantum master equation approach. The Bures angle (the geodesic) and the corresponding Fubini-Study metric for mixed states are given by [23,69,73]

$$\mathcal{L}_B^M = \arccos\,[\mathrm{Tr}\sqrt{\sqrt{\rho_s(0)}\rho_s(\tau)\sqrt{\rho_s(0)}}], \tag{D1}$$

$$g_{tt}^{\mathrm{FS},M} = \tfrac{1}{4}\mathcal{I}_t, \tag{D2}$$

where $\mathcal{I}_t \equiv \mathrm{Tr}[\hat{\varsigma}^2\rho_s(t)]$ is the quantum Fisher information servingas a quantifier of statistical difference between two different quantum states, and $\hat{\varsigma}$ is determined by $\partial_t\rho_s(t) = \frac{1}{2}[\hat{\varsigma}\rho_s(t) + \rho_s(t)\hat{\varsigma}]$ [11]. For the two-level system considered in this paper, the reduced density matrix can be conveniently described by the Bloch representation, namely, $\rho_s(t) = \frac{1}{2}(\mathbf{1}_2 + \boldsymbol{r}\cdot\hat{\boldsymbol{\sigma}})$ with $\boldsymbol{r}$ being the Bloch vector and $\hat{\boldsymbol{\sigma}} \equiv (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ being the vector of the Pauli matrices. In this case, the quantum Fisher information can be easily calculated via [74]

$$\mathcal{I}_t = |\partial_t\boldsymbol{r}|^2 + \frac{(\boldsymbol{r}\cdot\partial_t\boldsymbol{r})^2}{1-|\boldsymbol{r}|^2}. \tag{D3}$$

When $\rho_s(t)$ is a pure state, Eq. (D3) reduces to $\mathcal{I}_t = |\partial_t\boldsymbol{r}|^2$.

The exact convolutionless quantum master equation for the dissipative two-level system can be obtained by solving the Schrödinger equation in the one-excitation subspace [75] or employing the stochastic decoupling method [76]. Using the same notations used in the main text, it is given by [75–77]

$$\begin{aligned}\dot{\rho}_s(t) = -i\mathrm{Im}\left[\frac{\dot{u}(t)}{u(t)}\right][\rho_s(t), \hat{\sigma}_+\hat{\sigma}_-] + \mathrm{Re}\left[\frac{\dot{u}(t)}{u(t)}\right][\hat{\sigma}_+\hat{\sigma}_-\rho_s(t)\\ + \rho_s(t)\hat{\sigma}_+\hat{\sigma}_- - 2\hat{\sigma}_-\rho_s(t)\hat{\sigma}_+].\end{aligned} \tag{D4}$$

Solving the above master equation with the initial state $\rho_s(0) = \frac{1}{2}\mathbf{1}_2$, one easily finds

$$\rho_s(t) = \frac{1}{2}\begin{bmatrix}|u(t)|^2 & u(t)\\ \bar{u}(t) & 2-|u(t)|^2\end{bmatrix} \tag{D5}$$

in the basis of $\{|e\rangle, |g\rangle\}$. With the exact expression at hand, we find

$$\mathcal{L}_B^M = \arccos\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\mathrm{Re}[u(\tau)]}, \tag{D6}$$

$$\begin{aligned}g_{tt}^{\mathrm{FS},M} = \frac{1}{4}\Bigg\{|\dot{u}(t)|^2 + [\dot{\bar{u}}(t)u(t) + \bar{u}(t)\dot{u}(t)]^2\\ + \frac{[\dot{\bar{u}}(t)u(t) + \bar{u}(t)\dot{u}(t)]^2[|u(t)|^2 - \frac{1}{2}]^2}{|u(t)|^2 - |u(t)|^4}\Bigg\}.\end{aligned} \tag{D7}$$

For the Ornstein-Uhlenbeck correlation function considered in this paper, one sees that $|u(t)|$ exhibits an exponential-decay behavior in the large-t region. Using this asymptotic relation, we find

$$\lim_{\tau\to\infty}\frac{\tau_{\mathrm{QSL}}}{\tau} \propto \mathrm{const} > 0. \tag{D8}$$

This result is consistent with the discussions in the main text as well as many previous studies [59,63,67].

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