Since $V_d(r')=0$ for $r'>a$, the r'' integral in Eq. (21) need only be evaluated for $r'\leqq a$. In the r'' integral, which was evaluated exactly, the term which arises from the $e^{-(\gamma+\beta)r''}$ term of ψ_S is small[8] compared to that which arises from the $e^{-\gamma r''}$ term of ψ_S for $r'\leqq a$. This small contribution was neglected in the subsequent r' integration. For the r' integration, the integrand was expanded in a power series, whose values differed from those of the original expression by less than 1 percent for $r'\leqq a$. With the resulting $\phi_S^{(2)}$, Eq. (7) yielded[9]

$$\alpha_{SS}^{(2)}=\alpha_{SS}^{(1)}+\delta\alpha_{SS},\quad \delta\alpha_{SS}=0.020\times10^{-39}\ \text{cm}^3. \tag{22}$$

The author is indebted to Professor L. I. Schiff for suggesting this problem and for his guidance throughout its solution.

[8] The ratio of the term which arises from $e^{-(\gamma+\beta)r''}$ to that which arises from $e^{-\gamma r''}$ is -0.112 at $r'=0.10a$, -0.061 at $r'=0.50a$, and -0.032 at $r'=a$. The contribution of this term to $\delta\alpha_{SS}$ would be about -0.001×10^{-39} cm^3. With this estimate, Eq. (22) would read $\delta\alpha_{SS}=0.019\times10^{-39}$ cm^3.

[9] *Note added in proof.*—J. Sawicki, Acta phys. Polon. **13**, 225 (1954), has used a different variation technique to obtain $\alpha_{SS}\approx0.32\times10^{-39}$ cm^3. His method yields a lower limit for α_{SS}.

PHYSICAL REVIEW VOLUME 98, NUMBER 1 APRIL 1, 1955

Derivation of the Feynman-Dyson Rules from Time-Independent Theory

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(Received December 15, 1954)

The Feynman-Dyson rules for computing the scattering matrix are derived starting with the time-independent Schrödinger equation. Although no essentially new results are obtained, it is interesting to trace the connection between the time-dependent and time-independent methods starting from the time-independent end rather than, as is usually done, starting from the time-dependent interaction representation. In particular, a theorem due to R. J. Eden plays an important role in our discussion.

I. INTRODUCTION

THE Feynman-Dyson rules[1] for the calculation of scattering matrix elements differ from the rules of the older formulations of perturbation theory in two essential ways: (A) One can immediately write down the net contribution of all virtual processes which can be represented by the same "Feynman diagram" whereas in the older theory each virtual process was treated separately. (B) Not only is momentum conserved at each "vertex" (as in the older theory), but also energy is conserved at each "vertex."

These rules were originally derived by Dyson[1] from the time-dependent Schrödinger equation in the interaction representation, and by Feynman[1] from his Lagrangian formulation of quantum mechanics. In this note we will derive the Feynman-Dyson rules starting from the time-independent Schrödinger equation. We feel that this derivation may be of interest for two reasons: (i) It is hoped that the derivation of the Feynman-Dyson rules from the time-independent Schrödinger equation for problems in the continuum may suggest ways of further clarifying the connection between the "relativistic two-body equation"[2] and the time-independent Schrödinger equation for bound states. (ii) Our work may also be considered as a contribution to the discussion[3] of the connection between the Feynman-Dyson and the time-independent definitions of the scattering matrix. Our approach is different from that of most authors in that we start from the time-independent theory rather than from the time-dependent theory, and although no essentially new results are obtained in this way it does seem of interest to look at the connection from the "other end." In particular, we will find that a theorem due to Eden[4] will play an important role in our discussion.

In the next section we will carry through a derivation of the Feynman-Dyson rules and in the final section we will make a connection with the "adiabatic switch-on, switch-off method."[5]

II. DERIVATION OF THE FEYNMAN-DYSON RULES

The state vector for the time-independent scattering problem with interaction V satisfies the well-known equation:[6]

$$\psi=\phi-2\pi i\delta_+(E-H_0)V\psi, \tag{1}$$

where $H_0\phi=E\phi$, and where

$$\delta_+(x)=\tfrac{1}{2}\delta(x)-\text{p.v.}\frac{1}{2\pi ix}, \tag{2}$$

[1] R. P. Feynman, Phys. Rev. **76**, 749 and 769 (1949); F. J. Dyson, Phys. Rev. **75**, 486 and 1736 (1949).
[2] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951); J. Schwinger, Proc. Nat. Acad. Sci. U. S. **37**, 452 and 455 (1951).
[3] See F. J. Belinfante and C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **28**, No. 6 (1954) for a detailed discussion of the problem and references to the literature.
[4] R. J. Eden, Proc. Roy. Soc. (London) **A198**, 540 (1949).
[5] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
[6] P. A. M. Dirac, *Quantum Mechanics* (Clarendon Press, Oxford, 1947), third edition, pp. 197–199.

p.v. standing for principal value. Some useful representations of these singular functions are

$$\delta_+(x)=\lim_{\eta\to 0}\frac{1}{2\pi i}\frac{1}{x+i\eta},\quad \delta(x)=\lim_{\eta\to 0}\frac{\eta}{\pi(x^2+\eta^2)} \tag{3a}$$

and

$$\delta_+(x)=\frac{1}{2\pi}\int_0^\infty d\tau e^{i\tau x},\quad \delta(x)=\frac{1}{2\pi}\int_{-\infty}^\infty d\tau e^{i\tau x}. \tag{3b}$$

What we are interested in computing is the matrix element $(\phi',V\psi)$ on the energy shell,[5] i.e., with $H_0\phi'=E\phi'$. In the usual way, we write this matrix element as $(\phi',T\phi)$ whence from (1), using the representation (3a), we see that

$$\begin{aligned}T&=V[1-(E+i\eta-H_0)^{-1}V]^{-1}\\&=V[E+i\eta-H_0-V]^{-1}[E+i\eta-H_0],\end{aligned}$$

which on the energy shell becomes

$$T=i\eta V[E+i\eta-H_0-V]^{-1}, \tag{4a}$$

or

$$T=\frac{2V}{\delta(0)}\delta_+(E-H_0-V). \tag{4b}$$

We now use the representation (3b) to write (4b) as

$$T=\frac{1}{\pi\delta(0)}\int_0^\infty d\tau e^{iE\tau}Ve^{-i(H_0+V)\tau},$$

which, on the energy shell, becomes

$$\begin{aligned}T&=\frac{1}{\pi\delta(0)}\int_0^\infty d\tau e^{iH_0\tau}Ve^{-i(H_0+V)\tau}\\&\equiv\frac{1}{\pi\delta(0)}\int_0^\infty d\tau V(\tau)U(\tau,0),\end{aligned} \tag{5}$$

where $V(\tau)=e^{iH_0\tau}Ve^{-iH_0\tau}$ and $U(\tau,0)=e^{iH_0\tau}e^{-i(H_0+V)\tau}$.

From its definition we see that $U(\tau,0)$ satisfies the equations

$$dU(\tau,0)/d\tau=-iV(\tau)U(\tau,0),\quad U(0,0)=1, \tag{6}$$

whence we may also write

$$U(\tau,0)=P\exp\left(-i\int_0^\tau V(\tau')d\tau'\right), \tag{7}$$

where P is Dyson's chronological ordering operator. Further, by comparing (5) and (6), we see that

$$T=\frac{i}{\pi\delta(0)}\int_0^\infty dU(\tau,0)=\frac{i}{\pi\delta(0)}[U(\infty,0)-1]. \tag{8}$$

With this result, we have completed the derivation of part (A) of the Feynman-Dyson rules, since by using the form (7) for $U(\infty,0)$ we can analyze the perturbation expansion of $U(\infty,0)-1$ into "Feynman diagrams."

To establish part (B) of the Feynman-Dyson rules, we must now show that we can replace $U(\infty,0)-1$ by $(U(\infty,-\infty)-1)/2$ since $U(\infty,-\infty)$ is the Feynman-Dyson scattering matrix and the factor $\frac{1}{2}$ is needed in order to get the proper connection between T and the scattering matrix.[5] However, in agreement with other authors,[3] we are thus far able to derive this result only by the use of perturbation theory. If we use perturbation theory then, as we stated earlier, we can analyze $U(\infty,0)-1$ into "Feynman diagrams" and we can also analyze $U(\infty,-\infty)-1$ into the *same* diagrams. The difference between the contribution of a diagram in the two cases arises from the fact that if E_k is (minus) the sum of the energies at the kth vertex in a diagram, then in the case of $U(\infty,0)-1$ one will get a factor of

$$(2\pi)^{-1}\int_0^\infty d\tau_k e^{iE_k\tau_k}=\delta_+(E_k),$$

while in the case of $U(\infty,-\infty)-1$ one will get a factor of

$$(2\pi)^{-1}\int_{-\infty}^\infty d\tau_k e^{iE_k\tau_k}=\delta(E_k),$$

i.e., the energy conservation mentioned in the introduction. Hence if there are N vertices in a diagram we must compare

$$\prod_{k=1}^N\delta_+(E_k)\quad\text{with}\quad\prod_{k=1}^N\delta(E_k).$$

This we do by use of a theorem due to Eden[4] according to which

$$\prod_{k=1}^N\delta_+(E_k)=Z\delta_+\left(\sum_{k=1}^N E_k\right),\quad \prod_{k=1}^N\delta(E_k)=Z\delta\left(\sum_{k=1}^N E_k\right),$$

where Z is a function which need not concern us here. Now the fact that we are on the energy shell means that $\sum_{k=1}^N E_k=0$ and therefore

$$\delta_+\left(\sum_{k=1}^N E_k\right)=\tfrac{1}{2}\delta\left(\sum_{k=1}^N E_k\right)$$

so that from Eden's theorem we may conclude that on the energy shell we may replace

$$\prod_{k=1}^N\delta_+(E_k)\quad\text{by}\quad\tfrac{1}{2}\prod_{k=1}^N\delta(E_k)$$

and hence that we may replace $U(\infty,0)-1$ by $[U(\infty,-\infty)-1]/2$, which completes the derivation of the Feynman-Dyson rules.

III. DISCUSSION

Equation (8) in the preceding section is apparently derived without approximation. However in its derivation we used the representation (3b) which is perhaps not as satisfactory as using a limiting process as in (3a). If in fact we do need to use such a process, then in order to derive (8) it appears that we must assume the validity of the representation:

$$\delta_+(E-H_0-V)=\lim_{\eta\to 0}\frac{1}{2\pi}\int_0^\infty d\tau e^{-\eta\tau+iE\tau}$$
$$\times P\exp\left(-i\int_0^\tau (H_0+Ve^{-\eta\tau'})d\tau'\right). \quad (9)$$

To see this, we introduce (9) into (4a) and, defining

$$V'(\tau)=V(\tau)e^{-\eta\tau},$$
$$U'(\tau,0)=e^{iH_0\tau}P\exp\left(-i\int_0^\tau (H_0+Ve^{-\eta\tau'})d\tau'\right),$$

we find

$$T=\eta\int_0^\infty d\tau V'(\tau)U'(\tau,0).$$

But from its definition $U'(\tau,0)$ satisfies

$$dU'(\tau,0)/d\tau=-iV'(\tau)U'(\tau,0),\quad U'(0,0)=1,$$

whence we can write

$$U'(\tau,0)=P\exp\left(-i\int_0^\tau V'(\tau')d\tau'\right)$$

and[7]

$$T=i\eta(U'(\infty,0)-1), \quad (8')$$

which just corresponds to (8). We have as yet been unable to give an *a priori* proof of the general validity of (9).

It is interesting to complete the derivation of the Feynman-Dyson rules by using this representation. To do this we consider the perturbation expansion of $U'(\infty,0)-1$. Now the $\prod_{k=1}^N \delta_+(E_k)$ will appear as

$$\prod_{k=1}^N\left(-\frac{1}{2\pi i}\right)\frac{1}{E_k+i\eta}$$

which, from Eden's theorem, we may replace by

$$\frac{1}{2}\prod_{k=1}^N\frac{1}{\pi}\frac{\eta}{E_k^2+\eta^2}.$$

One readily sees that this replacement is equivalent to replacing

$$\prod_{k=1}^N\frac{1}{2\pi}\int_0^\infty d\tau_k e^{iE_k\tau_k-\eta\tau_k}\quad\text{by}\quad \frac{1}{2}\prod_{k=1}^N\frac{1}{2\pi}\int_{-\infty}^\infty d\tau_k e^{iE_k\tau_k-\eta|\tau_k|},$$

which in turn is equivalent to replacing $U'(\infty,0)-1$ by $[U''(\infty,-\infty)-1]/2$, where

$$U''(\infty,-\infty)=P\exp\left(-i\int_{-\infty}^\infty V''(\tau)d\tau\right),$$

and where

$$V''(\tau)=V(\tau)e^{-\eta|\tau|}.$$

Thus we have established a connection with the "adiabatic switch-on, switch-off method"[5] but, in agreement with other authors,[3] only by the use of a perturbation expansion.

I wish to thank Professor H. Jehle for a very interesting discussion.

[7] If we would start with the representation

$$\delta_+(x)=\lim_{\eta\to 0}\frac{1}{2\pi}\int_0^\infty d\tau e^{ix\tau-\eta\tau},$$

which is equivalent to (3b) and presumably not subject to any doubt, then one can derive from (4a), after an integration by parts, that

$$T=i\eta\left(\eta\int_0^\infty d\tau U(\tau,0)e^{-\eta\tau}-1\right).$$

From this one can "derive" an expression which is in a certain sense intermediate between (8) and (8′) by writing

$$T=i\eta\left(\int_0^\infty dZU\left(\frac{Z}{\eta},0\right)e^{-Z}-V\right)$$

and saying that in the limit as $\eta\to 0$ this becomes what we might write, in a sort of "mixed representation" of singular functions as

$$T=i\eta(U(\text{“}\infty,\text{”}0)-1).$$