Quantization of a Nonlinear Field Theory

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A quantization procedure, involving the expansion of the field in a space lattice of classical particles, is suggested for a field theory in which classical particles appear as small regions in which the field is intense. Approximate energy eigenvalues are calculated for certain values of the parameters involved.

INTRODUCTION

It is known that one can construct classical nonlinear field theories in which all quantities of physical interest are finite.[1] In such theories, the nonlinearity in the field equations is so chosen as to ensure the existence of solutions which have a finite energy-momentum, most of which is concentrated in a finite volume, and which can therefore be interpreted as (classical) particles. The motion of these "particles" is thus fully defined by the field equations, and no separate equations of motion need (in fact, can consistently) be postulated. A simple example of such a field is the one defined by the Lagrangian density:

$$\mathfrak{L}=-\frac{\partial\psi}{\partial x_\lambda}\frac{\partial\psi^*}{\partial x_\lambda}-\sigma^2\psi\psi^*+\tfrac{1}{2}g\psi^2\psi^{*2}, \qquad (1)$$

$$(x_4=ict),$$

which is discussed classically in I.[1] Quantization of any nonlinear theory meets with difficulties of both computation and interpretation[2] (e.g., it may be impossible to find a straightforward method for diagonalizing the Hamiltonian even approximately). In this paper we suggest a quantization procedure which circumvents some of the former difficulties for a theory in which classical particles exist. The procedure consists in expanding the field ψ in a cubic space lattice of lattice constant D of real, classical, static, spherically symmetric, particle-like solutions $\theta(r)$,

$$\psi=\sum_{ijk} q_{ijk}\theta[|\mathbf{r}-(\mathbf{i}+\mathbf{j}+\mathbf{k})D|], \qquad (2)$$

and interpreting the expansion coefficients q_{ijk} as operators subject to the usual commutation relations. ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are integral multiples of mutually perpendicular unit vectors.) Equation (2) will perhaps appear plausible if we compare it with the well-known "expansion"

$$f(x)=\int f(x')\delta(x-x')dx',$$

and observe that the singular Dirac $\delta(x)$ is the way in which particle-like properties often appear in linear theories. Equation (2) enables us to compute approximate energy eigenvalues for certain values of the parameters σ, g, D. However, we are still faced with the fact that the Hamiltonian does not commute with the quantity that should be interpreted as the momentum of the field. (It should perhaps be noted that similar difficulties arise when the interaction of linear fields is considered.) Furthermore, the original relativistic invariance of the expression (1) is lost in the expansion (2). Finally, the parameter D has been introduced in an *ad hoc* fashion, and a criterion for fixing its value would have to be adopted.

SINGLE CLASSICAL PARTICLE

For simplicity, let us first formally work with one single classical particle, rather than the lattice (2). For that purpose, we recall from I that the field equation obtained from (1) in the spherically symmetric, static case [$\psi=\theta(r)$, θ real] is

$$d^2\theta/dr^2+(2/r)d\theta/dr-\sigma^2\theta=-g\theta^3, \qquad (3)$$

and has a countably infinite number of everywhere analytic, quadratically integrable solutions. Henceforth the symbol θ will refer exclusively to the simplest such solution, which is nodeless, has one finite maximum, and falls off exponentially outside a finite volume. The theory could presumably be extended to include other classical solutions as well.

Assume, then, that we have

$$\psi=q\theta. \qquad (4)$$

The Lagrangian density (1) then becomes

$$\mathfrak{L}=c^{-2}\dot{q}^2\theta^2-q^2\nabla\theta\cdot\nabla\theta-\tfrac{1}{2}gq^4\theta^4. \qquad (5)$$

Let us integrate this over space to get the Lagrangian $L=\int\mathfrak{L}dV$, using the abbreviations, more general than

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[1] N. Rosen, Phys. Rev. **55**, 94 (1939); Finkelstein, LeLevier, and Ruderman, Phys. Rev. **83**, 326 (1951); N. Rosen and H. B. Rosenstock, Phys. Rev. **85**, 257 (1952), henceforth referred to as I. The classical theory discussed by L. I. Schiff [Phys. Rev. **84**, 1 (1951)] differs from I by the sign of the nonlinear term and does not lead to isolated particle-like solutions.

[2] Other approaches to quantization of nonlinear theories have been given by R. J. Finkelstein, Phys. Rev. **75**, 1079 (1949); D. R. Yennie, Phys. Rev. **88**, 527 (1952); and L. I. Schiff, Phys. Rev. **92**, 764 (1953). Schiff utilizes a lattice expansion, although from a viewpoint different from the one here adopted. I wish to thank Professor Schiff for a prepublication copy of his paper.

needed at this point:

$$J_a/\sigma g=\int \theta(r)\theta(|\mathbf{r}-\mathbf{a}D|)dV,$$

$$J_{\nabla a}g/\sigma=\int \nabla\theta(r)\cdot\nabla\theta(|\mathbf{r}-\mathbf{a}D|)dV, \tag{6}$$

$$J_{abc}\sigma/g^2=\int \theta(r)\theta(|\mathbf{r}-\mathbf{a}D|)\theta(|\mathbf{r}-\mathbf{b}D|)\theta(|\mathbf{r}-\mathbf{c}D|)dV.$$

Here a, b, c are numbers and the dimensionless J's are thus presumably known. Using the relation $J_{\nabla a}+J_a=J_{00a}$ which follows from the field equation (3), we get, after a convenient change of scale $(\sigma g)^{-\frac{1}{2}}q\rightarrow q$,

$$L=M_0\dot{q}^2-K_{000}q^2+\tfrac{1}{2}\sigma g K_{000}q^4, \tag{7}$$

where we have put $M_a=c^{-2}J_a$ and $K_{abc}=\sigma^2 J_{abc}$. The procedure now is to define the conjugate momentum

$$p=\partial L/\partial q \tag{8a}$$

and the Hamiltonian

$$H=p\dot{q}-L, \tag{8b}$$

and to calculate the eigenvalues of the latter with the assumed commutation relations

$$[p,q]=i\hbar. \tag{9}$$

We find

$$H=p^2/4M_0+K_{000}q^2-\tfrac{1}{2}\sigma g K_{000}q^4, \tag{10}$$

representing a harmonic oscillator except for the q^4 term. The energy levels are known if the latter (i.e., its coefficient g) is small enough to be treated as a perturbation.[3] In that case

$$H_N=(\hbar\sigma c/2)[(2N+1)(J_{000}/J_0)^{\frac{1}{2}} - (2N^2+2N+1)3\hbar gc/32J_0]. \tag{11}$$

The number $N=0, 1, 2, \cdots$ may in the usual fashion be interpreted as the number of particles in the field. The values of J_0 and J_{000} are, by a rough numerical integration, found to be about 6.5π and 36π, respectively.

In view of the minus sign in front of the g term in (10), the potential is not strictly binding however small g may be. The number of levels in the potential valley will depend on the value of g. The maximum number of levels with spacing given by the first term in (11) that can get in below the peaks is found to be

$$N\lesssim 10/\hbar gc; \tag{12}$$

the number of levels with small probability of escape by tunneling is, of course, smaller than that. In particular, we get no binding in the "strong-coupling" case of large g.

[3] See, e.g., L. Pauling and E. B. Wilson, *Introduction to Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1935), p. 160.

LATTICE OF CLASSICAL PARTICLES

Now, instead of (4), take

$$\psi=\textstyle\sum_i q_i\theta(r-iD). \tag{13}$$

The field is now expanded in a linear chain of classical particles, each q_i is an operator, and (8) and (9) are to be replaced by

$$p_i=\partial L/\partial q_i,\quad H=\textstyle\sum_i p_i\dot{q}_i-L,\quad [p_i,q_j]=i\hbar\delta_{ij}. \tag{14}$$

It should be remarked that although a superposition of solutions of the nonlinear field equation (3) is generally not a solution, (13) is nevertheless acceptable if D is so large that the interaction potential between two adjacent classical particles is proportional to $e^{-\sigma D}/\sigma D$, as will be assumed to be the case. The superposition can then be shown to differ from a solution by a term of the order $(e^{-\sigma D}/\sigma D)^2$.

Proceeding as in the preceding section, we obtain, in place of the one oscillator in (7),

$$L=\sum_{ij} M_{ij}\dot{q}_i\dot{q}_j-\sum_{ij} K_{00i-j}q_iq_j + \tfrac{1}{2}\sigma g\sum_{ijkl} K_{i-j,\,i-k,\,i-l}q_iq_jq_kq_l, \tag{15}$$

an aggregate of such oscillators (the $i=j=k=l$ terms) coupled to each other (by the $i\neq j$ terms). The magnitude of the last sum is determined by the value of g, that of successive terms in each sum by the value of $e^{-\sigma D}$ (because, as is shown in I, $1/\sigma$ is a measure of the size of a single particle and for reasonably large D, K_{mna} and M_a fall off as $e^{-a\sigma D}/a\sigma D$). For certain values of the dimensionless constants $\hbar gc$ and $e^{-\sigma D}$, certain terms in (15) may therefore be dropped.

Thus, if $e^{-\sigma D}$ is so small that all interaction terms are negligible, we are left with a set of oscillators like (11):

$$H_{\{N_i\}}=(\hbar\sigma c/2)\textstyle\sum_i[(2N_i+1)(J_{000}/J_0)^{\frac{1}{2}} - (2N_i^2+2N_i+1)3\hbar gc/32J_0]. \tag{16}$$

If, in addition, g is negligibly small, this reduces to a set of harmonic oscillators. Extension of (16) to the case where the field is represented by the three-dimensional lattice (2) rather than by (13) is straightforward.

If, on the other hand, $\hbar gc$ is so small that the third sum in (15) may be neglected entirely, but $e^{-\sigma D}$ is large enough to require retention of several terms in the first two sums, i.e.,

$$L=\textstyle\sum_{ij} M_{i-j}\dot{q}_i\dot{q}_j-\sum_{ij} K_{00i-j}q_iq_j, \tag{17}$$

then we must, prior to quantization, bring this into normal form

$$L=\textstyle\sum_r(\dot{Q}_r^2-\omega_r^2Q_r^2), \tag{18}$$

by introducing normal modes, which, for boundary conditions requiring that the 0th and Nth particle be held fast, turn out to be standing waves of the form $\sin(i\pi r/N)\,(r=1, 2, \cdots, N)$. We find

$$H_{\{N_r\}}=\textstyle\sum_r(2N_r+1)\hbar\omega_r/2, \tag{19}$$

where

$$\omega_r^2 = [\sum_l K_{00l}\cos(l\pi r/N)]/[\sum_l M_l(\cos l\pi r/N)]. \quad (20)$$

If $e^{-\sigma D}$ is so small that only nearest neighbors need to be retained, (20) may be written as

$$\omega_r \cong \omega_0\left[1+\tfrac{1}{2}\left(\frac{K_{001}}{K_{000}}-\frac{M_1}{M_0}\right)\cos(\pi r/N)\right], \quad (21)$$

$\omega_0=(K_{000}/M_0)^{\frac{1}{2}}$ being the harmonic oscillator frequency, and each level is seen to be split into N levels by the coupling.

On account of the interaction in all conceivable directions, extension to three dimensions of (19) and (20) is more complicated than previously, but presents no serious difficulties,[4] and there is no need for writing down the corresponding expressions here. In any case, for all σD, the coefficients K and M in each series of (20) or its equivalent fall off exponentially; each series converges for all r.

From the conservation laws,[5] the quantity

$$\mathcal{G}_k = (\partial\mathcal{L}/\partial\dot{\psi})(\partial\psi/\partial x_k)$$

should be interpreted as the k component of the momentum density of the field, and the momentum then becomes, with (1) and (13),

$$G=\sum_{ij} q_i q_j \int \theta(r-iD)\nabla\theta(r-jD)dV.$$

Since it is generally possible simultaneously to diagonalize two quadratic forms (such as the two appearing in the Hamiltonian) but not three, it follows that G will not commute with H.

I wish to thank Professor Nathan Rosen for patiently guiding the preparation of the thesis[4] from which this paper is taken, and for making countless suggestions. I am also grateful to Professor Wayne A. Bowers for discussions.

[4] H. B. Rosenstock, thesis, University of North Carolina, Chapel Hill, North Carolina, 1951 (unpublished).

[5] G. Wentzel, *Quantentheorie der Wellenfelder* (J. W. Edwards, Ann Arbor, 1946), Sec. 2.

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Nucleon Isobars in Intermediate Coupling

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The energies of excited states of nucleons are calculated for the symmetric pseudoscalar meson theory with a fixed, extended source. A version of the Tomonaga method, which is shown to be approximately correct in the limit of strong coupling, is used. The calculation is carried out for moderate coupling by using trial functions with 0, 1, 2, and 3 mesons. Numerical results are presented for a source of the Yukawa shape. For coupling stronger than a critical strength which varies with state and source size, isobars are found for angular momenta $\frac{1}{2}$, $\frac{3}{2}$ and for isotopic spins $\frac{1}{2}$, $\frac{3}{2}$. The $(\frac{3}{2},\frac{3}{2})$ state always lies lowest in energy. A very large source ($\sim\frac{1}{2}$ the meson Compton wavelength) yields a $(\frac{3}{2},\frac{3}{2})$ isobar at $\sim$350–400 Mev excitation, and a degenerate $(\frac{3}{2},\frac{1}{2})$ and $(\frac{1}{2},\frac{3}{2})$ pair at $\geqslant$500 Mev. Other isobars lie considerably higher. Smaller sources correspond to higher excitation energies in this range of coupling strength.

I. INTRODUCTION

THE existence of nucleon isobars was first predicted by strong coupling meson theory. Since that time, isobars have been used in phenomenological theories in attempts to explain meson-nucleon[1] and nucleon-nucleon[2,3] scattering, and photomeson production.[4] Until now, the isobars of symmetric pseudoscalar theory have been examined only in the strong coupling limit.[5] The only relevant low-lying isobar predicted in that limit is the one for which $I=\frac{3}{2}$, $J=\frac{3}{2}$. As a result, this excited state is the one which has been considered in scattering problems, although there was no guarantee that at some smaller coupling another state could not lie lower in energy. We have, therefore, calculated the positions of the low-lying isobars using the intermediate-coupling theory of Tomonaga.[6,7]

There is no attempt in this paper to calculate the widths of the isobaric levels. The results should, nonetheless, be useful in predicting the order and approximate spacing of the excited nucleon levels. The motivation is comparable to that of theories of nuclear structure which attempt to predict the same features of the levels of complex nuclei.

One of the results of these calculations is that nucleon isobars do exist for surprisingly low coupling. The

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