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Thermal Energy in Almost Normal Modes

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A mode of motion that consists of a standing wave is usually assigned the thermal energy of a Planck oscillator in thermal equilibrium at the temperature of the medium carrying the standing wave. For a real standing wave which will be damped in time if it is excited beyond the thermal level, the energy of a Planck oscillator E_{p0} is not necessarily the correct energy to assign this mode at thermal equilibrium.

For a one-dimensional standing wave where a characteristic impedance for the medium $Z_0=R_0+iX_0$ exists, the correct thermal energy (average stored thermal energy) to assign a standing wave E_{sw} is related to the energy of a Planck oscillator by

$$E_{p0}/E_{sw}=1+X_0^2/R_0^2.$$

THE set of standing waves that can be set up in a continuous medium or a bounded region of space is often taken to be the normal modes of motion of the system. This picture is a precise one only when the coupling between the standing waves is practically nonexistent. This condition holds for electromagnetic waves in an idealized cavity, for pressure waves in an idealized fluid bounded by idealized walls, and for waves on a lossless transmission line.

For pressure waves in a real fluid or solid, for shear waves in a solid, for electromagnetic waves in a real cavity filled with a real medium, and for electromagnetic waves on a real transmission line, standing waves are more or less tightly coupled to one another. If any one standing wave is excited beyond the thermal level, it is damped in time. The damping is accounted for macroscopically by nonzero resistances and conductances in the case of electromagnetic waves, and by their analogs, viscosity, and "internal friction" for mechanical waves in fluids and solids.

On a microscopic scale there is no resistance or friction, but merely coupling to many other (usually high-frequency) modes of the system. Any mode consisting of a standing wave that is excited beyond its thermal-equilibrium level loses, on the average, energy to other less excited modes until the excitation is dissipated as thermal motion. Resistance and friction are concepts to account macroscopically for this energy transfer.[1]

Plane standing waves are concepts used for normal modes in the development of the specific heat of solids and the character of radiation in a closed cavity. Kirchhoff's law for the thermal radiation from a surface of known absorbing properties is derived by considering the energy exchange between the surface and radiation in a closed cavity.

Completely analogous to the derivation of Kirchhoff's law is the derivation of thermal noise generated by a resistor. Exchange of energy between an impedance containing the resistance and an ideal transmission line in thermal equilibrium is considered. The impedance is assumed to generate noise sufficient to maintain the balance of power in the exchange.

The following will deal with standing waves on a transmission line, but the results are applicable to any plane standing waves where one can characterize the transmission of a wave by a characteristic impedance and a propagation constant.

Consider a transmission line of length l, with a characteristic impedance

$$Z_0=R_0+iX_0=\left(\frac{r+ix}{g+ib}\right)^{\frac{1}{2}}.$$

The series impedance per unit length of the line is $r+ix$, and the parallel admittance per unit length of the line is $g+ib$. All these quantities may be frequency dependent. The propagation constant for such a line will

[1] Informative comments on the coupling between members of a set of harmonic oscillators accounting for damping and on normal modes with damping have been made by R. T. Cox, Revs. Modern Phys. 24, 320 (1952).

be

$$\gamma = [(r+ix)(g+ib)]^{\frac{1}{2}}.$$

The real part of γ measures the fractional amplitude attenuation per unit length for a wave propagated along the line, and the imaginary part is 2π times the reciprocal of the wavelength of the wave. Analogs of r, g, b, and x for other types of waves are defined by these expressions for Z_0 and γ, where Z_0 and γ are the quantities appropriate to the waves.

In the following, standing waves will be considered, even where there is damping and where a wave established to amplitude far above the thermal level would quickly die down. At the thermal level, the same coupling that produces damping for a wave of high amplitude contributes energy to the wave as often as it removes it, and, on the average, the level of the wave is maintained.

Standing waves on the line correspond to the condition

$$nc/\nu = 2l,$$

where n is a positive integer, c the velocity of waves on the line of frequency ν. This condition neglects possible changes of phase upon reflection at the ends of the line, but this introduces negligible error for large values of n. For every positive integer there is a corresponding standing wave, and for large n there are, in the frequency range $d\nu$, $(2l/c)d\nu$ standing waves.

In this frequency range $d\nu$ let the thermal energy associated with each standing wave be E_{sw}. A standing wave can be considered a running wave that continually traverses the line and reflects from the ends. The average power in such a running wave passing a point in one direction is $E_{sw}c/2l$, and this is just the power from each standing wave impinging on the end of the line. Power impinging on the end of the line belonging to the frequency range $d\nu$ is then $E_{sw}d\nu$.

If one end of the line is terminated by an impedance

$$Z_t = R_t + iX_t,$$

then the fraction of the power impinging on that end absorbed in Z_t is[2]

$$4R_tR_0(1+X_0^2/R_0^2)/|Z_0+Z_t|^2.$$

Of the thermal energy in the line in the frequency range $d\nu$, the power absorbed in Z_t is

$$P(l-Z_t) = E_{sw}4R_tR_0(1+X_0^2/R_0^2)d\nu/|Z_0+Z_t|^2.$$

If Z_t is in thermal equilibrium with the line, then it must put into the line just this much power. Z_t may be considered to house a source of emf such that its mean square, $\langle V^2\rangle_{\text{Av}}$, will transmit this power to the line. Such a source will put into the line power

$$P(Z_t-l) = \langle V^2\rangle_{\text{Av}}R_0/|Z_t+Z_0|^2.$$

Since the frequency structure of such noise voltage is incoherent, the source of emf drives an impedance Z_t+Z_0 instead of Z_t plus the steady state impedance of the line which depends upon the line length and how it is terminated at the other end.[3]

The condition $P(l-Z_t)=P(Z_t-l)$ requires

$$\langle V^2\rangle_{\text{Av}} = E_{sw}4R_t(1+X_0^2/R_0^2)d\nu. \tag{1}$$

For an imagined lossless line the standing waves will be the true normal modes so that $E_{sw}=E_{p0}$, and then also $X_0=0$.[4] For this case one gets the usual equation for the noise generated in a resistor.

$$\langle V^2\rangle_{\text{Av}} = 4E_{p0}R_td\nu. \tag{2}$$

For low frequencies E_{p0} is kT, Boltzmann's constant times the absolute temperature. Since the required noise emf generated in Z_t is independent of X_t, it is concluded that only the resistance is a noise source, and a resistor at temperature T can be treated like a resistor at absolute zero in series with a source of noise emf given by (2) for each frequency range $d\nu$.[5]

When Z_t terminates a real line, (2) still gives the noise generated in Z_t, but (1) is the noise that must be generated to maintain power balance. When these two expressions for $\langle V^2\rangle_{\text{Av}}$ are equated, the relation between the thermal energy in a standing wave on a real line and the thermal energy of a Planck oscillator is obtained.

$$E_{sw} = \frac{E_{p0}}{1+X_0^2/R_0^2} = E_{p0}\cos^2\phi_0, \tag{3}$$

where ϕ_0 is the phase angle of Z_0, $\tan^{-1}(X_0/R_0)$.

If the characteristic impedance is real, the thermal energy of a standing wave is just the thermal energy of a true normal mode, even though the propagation constant γ may have a real part, indicating that the line is not a lossless one.

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[2] F. E. Terman, *Radio Engineers Handbook* (McGraw-Hill Book Company, Inc., New York, 1943), pp. 172–197. This is an outline of the relations associated with the behavior of electric lines. References given there lead to more detailed discussions in papers and text books. This expression for fractional power absorbed in a terminal impedance is derived from the voltage ratio given on p. 179.

[3] In a small frequency range a thermal noise voltage can be pictured as sinusoidal but randomly modulated in phase and amplitude in a manner that produces sizeable changes only when extended over many periods. The line here considered is many wavelengths long; therefore the voltage at the end of the line reflected from the other end is randomly related in phase and amplitude to the voltage produced by the terminal noise source. Over a length of time long enough to average the power associated with these voltages, the average voltage of the reflected wave will be zero and its phase will be random with respect to the voltage of the source and hence produce no average effect on the impedance "seen" by the source.

[4] A lossless line must have $r=g=0$ and hence $Z_0=(x/b)^{\frac{1}{2}}$. If x and b have the same sign, Z_0 is real and the above statement holds. If x and b have different signs, $Z_0=i(|x|/|b|)^{\frac{1}{2}}$ is pure imaginary. In this case, though, the propagation constant $\gamma=j(xb)^{\frac{1}{2}}=\pm(|xb|)^{\frac{1}{2}}$ is real and no true wave propagation exists.

[5] These results were derived by H. Nyquist, Phys. Rev. **32**, 110 (1928). His derivation, with minor variations, appears now in texts that discuss thermal noise, e.g., S. Goldman, *Frequency Analysis, Modulation, and Noise* (McGraw-Hill Book Company, Inc., New York, 1948), pp. 388–394.

The mechanism of damping of a standing wave on a real line is that in every unit length, currents flowing through a series resistance, r, and parallel conductance g transfer energy from the wave into thermal energy. The mechanism by which energy in a standing wave is maintained at the thermal level given by (3) is that g and r act as generators, as indicated by (2), and return to the line just the energy they extract. The following detailed development of this idea results in a second derivation of (3).

These relations follow from the definitions above:

$$Z_0=\left(\frac{r+ix}{g+ib}\right)^{\frac{1}{2}}=\left(\frac{r^2+x^2}{g^2+b^2}\right)^{\frac{1}{4}}\frac{e^{\frac{1}{2}i\phi_1}}{e^{\frac{1}{2}i\phi_2}}=\left(\frac{r^2+x^2}{g^2+b^2}\right)^{\frac{1}{4}}e^{\frac{1}{2}i(\phi_1-\phi_2)},$$

where

$$\phi_1=\tan^{-1}(x/r),$$

$$\phi_2=\tan^{-1}(b/g),$$

$$\phi_0=\tfrac{1}{2}(\phi_1-\phi_2),$$

$$R_0=\left(\frac{r^2+x^2}{g^2+b^2}\right)^{\frac{1}{4}}\cos[\tfrac{1}{2}(\phi_1-\phi_2)],$$

$$\gamma=(r+ix)^{\frac{1}{2}}(g+ib)^{\frac{1}{2}}=(r^2+x^2)^{\frac{1}{4}}(g^2+b^2)^{\frac{1}{4}}e^{\frac{1}{2}i(\phi_1+\phi_2)}.$$

The unit length for which r, x, g, and b apply can be chosen very small so that r or x times one unit of length is small compared to Z_0, and g or b times one unit of length is small compared to $1/Z_0$. The fractional loss of power that a wave experiences in traversing this unit length is just twice the real part of γ times one unit of length. At thermal equilibrium, power $E_{sw}d\nu$ in frequency range $d\nu$ is traversing each unit length from each direction. The power extracted per unit length is

$$4E_{sw}(r^2+x^2)^{\frac{1}{4}}(g^2+b^2)^{\frac{1}{4}}\cos[\tfrac{1}{2}(\phi_1+\phi_2)]d\nu.$$

Resistance r acts like a voltage source in series with the line generating mean square voltage per unit length $4E_{p0}rd\nu$. It delivers into the line (considering that $r\ll Z_0$) power per unit length

$$\frac{4E_{p0}r2R_0d\nu}{|2Z_0|^2}=2E_{p0}rR_0d\nu\left(\frac{g^2+b^2}{r^2+x^2}\right)^{\frac{1}{2}}.$$

Conductance g acts like a current source across the line generating mean square current per unit length $4E_{p0}gd\nu$. Such a source delivers into the line power per unit length

$$4E_{p0}gd\nu R_0/2=2E_{p0}gR_0d\nu.$$

At thermal equilibrium the power extracted per unit length from the line by g and r in frequency range $d\nu$ must equal that delivered to the line. To satisfy this requires

$$\frac{E_{sw}}{E_{p0}}=\frac{(1/2)(gR_0+rR_0/|Z_0|^2)}{(r^2+x^2)^{\frac{1}{4}}(g^2+b^2)^{\frac{1}{4}}\cos[\frac{1}{2}(\phi_1+\phi_2)]}$$

$$=\frac{1}{2}\left(\frac{g}{(g^2+b^2)^{\frac{1}{2}}}+\frac{r}{(r^2+x^2)^{\frac{1}{2}}}\right)\frac{\cos[\frac{1}{2}(\phi_1-\phi_2)]}{\cos[\frac{1}{2}(\phi_1+\phi_2)]}$$

$$=\tfrac{1}{2}(\cos\phi_2+\cos\phi_1)\frac{\cos[\frac{1}{2}(\phi_1-\phi_2)]}{\cos[\frac{1}{2}(\phi_1+\phi_2)]}.$$

When ϕ_1 is substituted for $(A+B)$, and ϕ_2 is substituted for $(A-B)$ in the trigonometric identity:

$$\cos(A+B)+\cos(A-B)=2\cos A\cos B,$$

then

$$\tfrac{1}{2}(\cos\phi_2+\cos\phi_1)=\cos[\tfrac{1}{2}(\phi_1+\phi_2)]\cos[\tfrac{1}{2}(\phi_1-\phi_2)].$$

Thus,

$$E_{sw}/E_{p0}=\cos^2[\tfrac{1}{2}(\phi_1-\phi_2)]=\cos^2\phi_0=1/1+X_0^2/R_0^2, \quad (4)$$

which is the same as (3).

If it proves possible to construct the necessary characteristic impedance functions for mechanical waves in matter, Eqs. (3) and (4) will provide a new means for cutting off contributions to specific heat at high frequencies and at intermediate frequencies where cut-off bands occur. In a frequency region where relaxation phenomena occur, the characteristic impedance may be expected to be complex, and these expressions offer a means for ascertaining the contribution to specific heat of motion in such a range of frequency.

If suitable characteristic impedances cannot be constructed or if they are too complex, these concepts may have little application.