increased the rate 20 percent. Hence the amount subtracted is four times 4.6 percent of 20 percent, which is 3.7 percent. When this is added to the above 2.2 percent, the total correction is about 6 percent.

RESULTS

The ratio of electron-electron to positron-electron scattering is shown in Figs. 1 and 3 after all corrections have been applied. The theoretical ratio at 33.5° is 1.83 (with exchange), and 1.36 (no exchange). The experimental ratio is 1.82 ± 0.11. Most of this error is due to uncertainties in the corrections. The Bhabha theory with exchange is thus very definitely favored.

The coincidence rates quoted above are only 0.35 times the rates expected from absolute cross section calculations. When the theoretical rate is reduced by the previously mentioned factors of 16 percent and 20 percent successively, a counting rate for electrons of 0.22 coincidence/sec is obtained. The absolute experimental coincidence rate lies somewhere between 0.27 count/sec, which is the rate just above the noise level near zero bias, and 0.13 count/sec, which is the rate at the knee of the curve. Hence no conclusions can be drawn regarding absolute cross sections.

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The Radiation of a High Energy Electron in a Constant Magnetic Field

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The quantum-mechanical radiation formula for an electron in a constant magnetic field—rigorous within the approximation of the first power of the fine structure constant—is derived. With it, the quantum effects for energies $E_{\mathrm{Bev}}<4(10^4/H_{\mathrm{gauss}})$ are analyzed and their importance assessed. The electron and the photon severally are subject to fluctuations which produce effects that are quite significant relative to the difference between the velocity of light and that of the electron, the decisive quantity at high energies. The net result however is that they are inconsequential in the domain investigated and that in its radiative aspects the system behaves—for all practical purposes—classically.

I. INTRODUCTION

A CALCULATION is here presented to show that there are no quantum corrections (in the practical sense) to the radiation of an electron in a constant magnetic field up to energies $E<4(10^4/H)$ Bev. The magnetic field H in the formula (as elsewhere in this article) is to be expressed in gauss.

A detailed inquiry into a domain, where an elementary application of the correspondence principle renders it plausible that the classical results are valid, seems to require an explanation. It was not undertaken with the thought that this principle could somehow be circumvented. Nor was it deemed necessary or feasible to verify its operation in detail for systems of considerable complexity. In view, however, of persistent reports[1] that significant quantum corrections have been deduced from the theory, a possibility seemingly not inconsistent with the principle, as it has hitherto been applied, suggested itself.

As the electron energy assumes very high values, the difference between its velocity and the velocity of light becomes increasingly important in the description of the frequency spectrum and the angular distribution of the radiation. It is therefore perhaps not entirely fair to gauge the importance of quantum fluctuations by considering their effect on the group velocity of the particle. A relevant quantity in this very high energy domain might also be the difference between the two velocities.

It turned out in the course of the work that elements of competition are present between the quantum uncertainty in the velocity and the difference mentioned, and the result is quite sensitive to the relative magnitude of these two quantities. In the domain investigated, however, the latter quantity is overwhelmingly larger. The possibility that the situation may change at much higher energies, but still well within the range of correspondence of the two theories, cannot be ruled out completely on the basis of the present investigation, but it seems rather unlikely that such effects play any important role in this particular context.

On the practical side, the somewhat remote possibility of such effects had to be weighted against the important consequences their presence would have on future accelerator design. It was concluded that a detailed search for them was not an altogether unreasonable precaution.

In order to exhibit the quantum aspect of the problem

[1] See for example, L. Schiff, Am. J. Phys. **20**, 474 (1952), reference 9.

with some clarity relative to the classical theory, the calculation was based on the Lorentz force law rather than the more usual Hamiltonian scheme. The wave functions of the electron were constructed in a representation to which the operational methods recently emphasized by Feynman[2] may readily be applied. Thus the bulk of analytic operations—which are apparently difficult to handle—has been dispensed with and this computational aspect of the problem reduced to a few elementary manipulations. Inessential complications have been avoided by completely neglecting spin effects —which do not matter here—and by carrying out the calculation within the framework of the Schrödinger-Gordon theory. Moreover, an average over the polarization of the emitted photons is carried out at an early stage since information of this type is not of particular interest.

The investigation is limited to the energy range $0.1\times(10^4/H)$ Bev$<E<4\times(10^4/H)$ Bev. It is unncesessary to consider lower energies, since there the validity of the classical theory has already been established by Judd, Lepore, Ruderman, and Wolff,[3] and by Olsen and Wergeland.[4] Higher ones are avoided in order to remain at a safe distance from the analytic morass of an asymptotic transition region.[5] It was felt in this connection that a more definite statement limited to a narrower domain may be more useful than a vague one covering a broader range. Unlike the effects in the lower energy range considered in reference 3 and 4, a more subtle effect involving light quanta has to be taken into account here in order to establish the substantial identity of the two theories.

It is believed that the following may fairly be said. There are no corrections, in the practical sense, for $E<4\times(10^4/H)$ Bv. From this one may reasonably infer that no significant correction will come into play at energies that are of interest in connection with present or immediate future accelerator design. Because of the mathematical fact that an asymptotic transition region sets in above these energies, one may perhaps suspect that some corrections are called for well above the energies of present experimental interest but below the limit sometimes assigned to the validity of the classical radiation formula.[1] These suspicions should be considerably allayed by the fairly convincing wave packet arguments offered by Schiff[1] and mildly sharpened by the possibility mentioned earlier.

II. QUANTUM-MECHANICAL RADIATION FORMULA

This formula will now be derived within the approximation of the first power of the fine structure constant. The conservation law for the stress tensor $\theta_{\mu\nu}$ of the dynamical part of the electromagnetic field,

$$\partial_\mu \theta_{\mu\nu} + (1/c)F_{\nu\lambda} j_\lambda = 0, \tag{1}$$

implies that the field gains momentum

$$P_\nu(\sigma) = (1/c)\int_\sigma d\sigma_\mu \theta_{\mu\nu} \tag{2}$$

in a time interval T contained between two space like surfaces σ_1 and σ_2 at an average rate of

$$\Delta P_\nu/\Delta T = -(1/Tc^2)\int_{\sigma_1}^{\sigma_2} d^4x F_{\nu\lambda} j_\lambda. \tag{3}$$

The limits of integration in (3) may be extended from $+\infty$ to $-\infty$ with the understanding that an infinite time factor in the form of a redundant delta-function will subsequently be canceled against T.

The operators in (3) are specified in the Heisenberg representation. The current $j_\lambda(x)$ is therefore in part maintained by an external agency (the constant electromagnetic field) and in part induced by the vacuum fluctuations of the dynamic field. The two may effectively be separated by transforming to a bound state representation.[6] In it, the expectation value of (3) in a state with one electron and no photons appears as

$$(\Delta P_\nu/\Delta T)_{1,0} = -(i/\hbar c^2 T)\int_{-\infty}^{+\infty} dx \int_{-\infty}^{\sigma} dx' \times\langle 1,0|[j_\alpha(x)A_\alpha(x), F_{\nu\lambda}(x')j_\lambda(x')]|1,0\rangle, \tag{4}$$

where only the first nonvanishing term of an expansion has been retained. The momentum transfer in (4) due to radiation is separated from self-energy effects by employing an idea of Dirac,[7] which implies that the former quantity is given by the right side of (4) if the finite limit of integration is extended to infinity and a factor (1/2) appended.

The photon operators in (4) obey the free Maxwell equations and their expectation value in the vacuum state is readily expressed in the form[8]

$$(\Delta P_\nu/\Delta T)_{1,0}{}^R = (1/c^3 T)\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx' \times\langle 1|j_\lambda(x)j_\lambda(x')|1\rangle \partial_\nu D^+(x-x'). \tag{5}$$

Because of the time average incorporated in (3) transient effects were removed in going from (4) to (5) by the simple device of partial integration. It will be noticed that (5) is formally almost identical with the corresponding classical expression. For the case of commuting current operators only the odd part of $D^+(x)$, namely $\frac{1}{2}D(x)$, is effective, thus reducing (5) to its classical form.

[2] R. Feynman, Phys. Rev. **84**, 108 (1951).
[3] Judd, Lepore, Ruderman, and Wolff, Phys. Rev. **86**, 123 (1952).
[4] H. Olsen and H. Wergeland, Phys. Rev. **86**, 123 (1952).
[5] F. Tricomi, Ann. Mat. **28**, 263 (1949).
[6] W. Furry, Phys. Rev. **81**, 115 (1951).
[7] P. Dirac, Proc. Roy. Soc. (London) **167**, 148 (1938).
[8] J. Schwinger, Phys. Rev. **75**, 651 (1949).

The average power radiated, $\mathcal{P}$ is given by the time-like component of (5) multiplied by c:

$$\mathcal{P}=-(1/c^2T)\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dx' \times\langle 1|j_\lambda(x)j_\lambda(x')|1\rangle\partial_0 D^+(x-x'). \quad (6)$$

Even though more elegant representations of the one-particle expectation value in (6) are available, we shall express it as an infinite sum over final states. These are denoted by $n-\lambda$, l'. The initial state is assigned hte letters n, $-n$. (This—still somewhat vacuous—labelling scheme is introduced at the present point to save rewriting formulas.) Representing $D^+(x)$ by means of a Fourier integral and eliminating the time-like component of the current with the aid of the conservation law, we obtain from (6):

$$\mathcal{P}=(1/c^2T)(1/(2\pi)^3)\int d^3k\langle 1|\mathbf{n}\times\mathbf{j}(-\mathbf{k})\cdot\mathbf{n}\times\mathbf{j}(\mathbf{k})|1\rangle, \quad (7)$$

where

$$\mathbf{j}(\mathbf{k})=\int_{-\infty}^{+\infty}d^4x\exp(-ikx)\mathbf{j}(x);\quad k=[\mathbf{k},i|\mathbf{k}|];\quad \mathbf{n}=\mathbf{k}/|\mathbf{k}|. \quad (8)$$

For a Schrödinger-Gordon current $\mathbf{j}(\mathbf{k})$ is given by

$$\mathbf{j}(\mathbf{k})=(e/2m)\int_{-\infty}^{+\infty}d^4x\bar{\phi}(x)\{\boldsymbol{\pi}(x),\exp(-ikx)\}\phi(x)\}. \quad (9)$$

The curly bracket in (9) denotes an anticommutator and $\Pi=p-(e/c)A$. Equation (7) assumes the form

$$\mathcal{P}=(1/T)(e/mc)^2(2\pi)^{-3}\sum_{l,\lambda}\int d^3k \times\left|\int d^4x\bar{\phi}_{n,-n}(x)\mathbf{n}\times\boldsymbol{\pi}(x)\exp(-ikx)\phi_{n-\lambda,l}(x)\right|^2. \quad (10)$$

The wave function $\phi(x)$ obeys the equation:

$$(\Pi^2+m^2c^2)\phi=0. \quad (11)$$

A transformation of the wave function,

$$\phi(x)\rightarrow\exp\left[(ie/\hbar c)\int_0^1 d\lambda A_\mu(\lambda x)x_\mu\right]\phi(x), \quad (12)$$

changes

$$\Pi_\mu\rightarrow p_\mu+(e/c)\int_0^1 d\lambda\cdot\lambda F_{\mu\nu}(\lambda x)x_\nu, \quad (13)$$

leaving (10) and (11) otherwise unaltered. In this new gauge, in the presence of a constant electromagnetic field,

$$\Pi=p+(e/2c)Fx. \quad (14)$$

[The F and A in Eqs. (12)–(14) are parameters referring to the external field and are not to be confused with the photon operators used previously.] If in addition we specialize to a frame in which the electric field vanishes and the magnetic field has the z direction and is sensed upward, (11) may be written

$$[(p_1^2+p_2^2)+m^2c^2\kappa_L^2(x_1^2+x_2^2) -2mc\kappa_L L+p_3^2-p_0^2+m^2c^2]\phi=0, \quad (15)$$

where

$$L=x_1p_2-x_2p_1 \quad (16)$$

is a constant of motion for the system. The quantity κ_L is a relativistic invariant which in this frame reduces to

$$\kappa_L=eH/2mc^2, \quad (17)$$

(within a multiplicative constant) the Larmor frequency. Choosing $\phi(x)$ to be an eigenfunction of p_3 and p_0, we may write

$$\phi(x)=(\kappa_C/K_0)^{\frac{1}{2}}(1/Z)^{\frac{1}{2}}\exp[i(K_3x_3-K_0x_0)]\psi(x_1,x_2). \quad (18)$$

Here $\kappa_C=mc/\hbar$, Z is the length of the magnetic field in the Z direction and the factor $(\kappa_C/K_0)^{\frac{1}{2}}$ has been chosen to normalize the charge to unity.

Equation (18) is now substituted in (10), and the integrations over x_3 and x_0 are carried out. This gives rise to some redundant delta-functions which are canceled against Z and T. The components of the vector $\mathbf{n}$ are then expressed in spherical coordinates, with the Z direction chosen as a polar axis. A short calculation shows that (10) may be put in the following form:

$$\mathcal{P}=2c(e/mc)^2(1/4\pi)\sum_{\lambda,l}\int d^3k\delta(K_0^n-|\mathbf{k}|-K_0^{n-\lambda}) \times R_n(\lambda,l). \quad (19)$$

The function $R_n(\lambda,l)$ may be expressed in terms of two quantities $A_\pm^n(\lambda,l)$ which are defined as

$$A_\pm^n(\lambda,l)=\int dx_1dx_2\bar{\psi}_{n-\lambda,l}(x_1x_2) \times\exp[-i(k_1x_1+k_2x_2)]\Pi_\pm\psi_{n,-n}(x_1x_2), \quad (20)$$

where

$$\Pi_\pm=(1/2)(\Pi_1\pm i\Pi_2), \quad (21)$$

the explicit expression being

$$R=(1+\cos^2\theta)[\bar{A}_+A_++\bar{A}_-A_-] -\sin^2\theta[\exp(2i\phi)\bar{A}_+A_-+\exp(-2i\phi)\bar{A}_-A_+]. \quad (22)$$

We neglected to write out corresponding indices on both sides of (22). The assumption was made that $K_3=0$ in the initial state.

Going back to (15) and (16) it is seen that a change of variables

$$x_i=(2^{\frac{1}{2}}\kappa)^{-1}(a_i+a_i^*);\quad p_i=-i\hbar\kappa 2^{-\frac{1}{2}}(a_i-a_i^*);\quad [a_i,a_j^*]=\delta_{ij};\quad i=1,2, \quad (23)$$

followed by a transformation

$$a_1=2^{-\frac{1}{2}}(a_-+a_+);\quad a_2=+i2^{-\frac{1}{2}}(a_--a_+);\quad [a_+,a_+^*]=[a_-,a_-^*]=1, \tag{24}$$

where

$$\kappa=(\kappa_C\kappa_L)^{\frac{1}{2}}, \tag{25}$$

leaves them in the form

$$[4\kappa^2a_+^*a_++K_3^2-K_0^2+\kappa_C^2]\psi=0, \tag{26}$$

$$L=\hbar(a_-^*a_--a_+^*a_+). \tag{27}$$

The eigenfunction may then be written as

$$\psi_{n,\,l}=[n!(n+l)!]^{-\frac{1}{2}}a_+^{*n}a_-^{*n+l}\phi_0, \tag{28}$$

where ϕ_0 is the vacuum state. The degeneracy has been classified according to the eigenvalues of L. Thus

$$L\psi_{n,\,l}=\hbar l\psi_{n,\,l};\quad -n\leqslant l<\infty, \tag{29}$$

where it is understood that only integers enter the inequality in (29). By (26), the physical parameters are constrained through

$$4\kappa^2(n+\tfrac{1}{2})+K_3^2-K_0^2+\kappa_C^2=0. \tag{30}$$

It is not too difficult to see that $l=-n$ corresponds to a classical orbit centered about the origin of coordinates; a state with $-n<l<0$ to a family of orbits with the origin as an interior point; for one with $l>0$ the origin is exterior to the orbits.[9] The significance of the labeling scheme introduced earlier is now clear.

Substituting (23), (24), and (28) into (20) and using the relations $\Pi_+=-i\hbar\kappa a_+$, $\Pi_-=i\hbar\kappa a_+^*$, the quantities in question become

$$A_+^n(\lambda,l)=-i\hbar\kappa S_{n-\lambda+l}\times\{\psi_{n-\lambda},\exp[-i\alpha(a_+^*\exp(i\phi)+a_+\exp(-i\phi))]a_+\psi_n\}, \tag{31a}$$

$$A_-^n(\lambda,l)=i\hbar\kappa S_{n-\lambda+l}\{\psi_{n-\lambda},\exp[-i\alpha(a_+^*\exp(i\phi)+a_+\exp(-i\phi))]a_+^*\psi_n\}, \tag{31b}$$

where

$$S_{n-\lambda+l}=\{\psi_{n-\lambda+l},\exp[-i\alpha(a_-^*\exp(-i\phi)+a_-\exp(i\phi))]\phi_0\}, \tag{32}$$

and

$$\alpha=\tfrac{1}{2}(k/\kappa)\sin\theta. \tag{33}$$

In going from (21) to (31) the product space of a_+ and a_- has been factored. The a_- operators have been absorbed in the $S_{n-\lambda+l}$, the a_+, into the factor by which they are multiplied. An essential simplification is now achieved by observing that the representation of the a_+ and a_- operators is only determined to within a phase factor. The transformation,

$$a_\pm\rightarrow a_\pm\exp(\pm i\phi), \tag{34}$$

without altering the algebraic relations among them, will free the exponentials in (31) of its dependence on the angle and, in addition, induce a transformation on the products:

$$\bar{A}_\pm A_\pm\rightarrow\bar{A}_\pm A_\pm, \tag{35a}$$

$$\bar{A}_\pm A_\mp\rightarrow\exp(\mp 2i\phi)\bar{A}_\pm A_\mp. \tag{35b}$$

These are precisely the phase factors needed to free R [Eq. (22)] of its dependence on ϕ. Thus ϕ may effectively be put equal to zero.

The radial integration in (20) is now carried out. The delta-function in the integrand together with (30) implies that

$$k(\lambda)=k^m[1-(1-2k^C(\lambda)/k^m)^{\frac{1}{2}}]. \tag{36}$$

The symbols appearing on the right side of (36) denote:

$$k^m=K_0/\sin^2\theta, \tag{37}$$

$$k^C(\lambda)=2(\kappa/K_0)\kappa\lambda. \tag{38}$$

The quantity $k^C(\lambda)$ is harmonic to the classical rotation frequency of the electron and it may be inferred from its dependence on λ that it corresponds to the classical radiation frequency. If we let

$$(2k^C(\lambda)/k^m)^{\frac{1}{2}}=\text{arc}\,\sin\delta(\lambda), \tag{39}$$

then (36) may be written as

$$k(\lambda)=k^C(\lambda)\sec^2(\delta(\lambda)/2). \tag{40}$$

A short calculation shows that single term of the sum over λ in Eq. (19) is given by

$$\frac{d\mathcal{P}(\lambda)}{d\Omega}=ce^2\frac{4\kappa^2[n(n+1)]^{\frac{1}{2}}}{K_0^2}\cdot\frac{k^2(\lambda)}{2\pi}\cdot\frac{1-k(\lambda)/K_0}{1-\sin^2\theta k(\lambda)/K_0}H(n,\lambda), \tag{41}$$

where the angular integration has been represented in differential form. The explicit appearance of the n's is accounted for by the use of the well-known relations, $a^*\psi_n=(n+1)^{\frac{1}{2}}\psi_{n+1}$ and $a\psi_n=n^{\frac{1}{2}}\psi_{n-1}$, to reduce (31) to a standard form, which will soon be indicated. The fractional expression preceding the still undefined quantity $H(n,\lambda)$ arises from the evaluation of the derivative of the argument of the delta-function, *qua* function of k, at the relevant zero of the function. Finally $H(n,\lambda)$ may, after some rearrangements, be put in the form

$$\begin{aligned}H(n,\lambda)=P_1\{&\cos^2\theta\mathcal{L}_{n,\lambda}^{(+)2}(\alpha^2)+\mathcal{L}_{n,\lambda}^{(-)2}(\alpha^2)\\&+\tfrac{1}{2}[\{n/(n+1)\}^{\frac{1}{2}}-\{(n+1)/n\}^{\frac{1}{2}}]\\&\times\mathcal{L}_{n,\lambda}^{(+)}(\alpha^2)\mathcal{L}_{n,\lambda}^{(-)}(\alpha^2)+\tfrac{1}{4}[\{n/(n+1)\}^{\frac{1}{2}}\\&+\{(n+1)/n\}^{\frac{1}{2}}-2](1+\cos^2\theta)\\&\times[\mathcal{L}_{n,\lambda}^{(+)}(\alpha^2)+\mathcal{L}_{n,\lambda}^{(-)}(\alpha^2)]\},\end{aligned} \tag{42}$$

where $\mathcal{L}_{n,\lambda}^{(\pm)}$ are contractions for

$$\mathcal{L}_{n,\lambda}^{(\pm)}=\tfrac{1}{2}[\mathcal{L}_{n-\lambda}^{\lambda-1}\pm\mathcal{L}_{n-\lambda}^{\lambda+1}]. \tag{43}$$

[9] For a discussion of other representations of the eigenfunctions see M. Johnson and B. Lippman, Phys. Rev. **77**, 702 (1950).

The quantities $\mathfrak{L}$ and P_1 denote:[10]

$$\mathfrak{L}_{n-\lambda}{}^{\lambda}(\alpha^2)=i^{\lambda}\{\psi_{n-\lambda},\ \exp[-i\alpha(a+a^*)]\psi_n\}, \tag{44}$$

$$P_1=\sum_l |\{\psi_{n-\lambda+l},\ \exp[-i\alpha(a+a^*)]\phi_0\}|^2, \tag{45}$$

the symbols having been chosen appropriate to familiar objects in still unrecognized form. Thus, $\mathfrak{L}_{n-\lambda}{}^{\lambda}$ will turn out to be the well-known normalized Laguerre function (after some reduction). The distinction between a_+ and a_- is no longer necessary and has therefore been abandoned in the notation.

The only remaining task in this section is the evaluation of (44) and (45). These are of the form to which the Feynman methods[2] apply and are readily handled with the aid of two equalities:

$$\exp(\mu_1 a+\mu_2 a^*)\equiv\exp(\mu_1\mu_2/2)\exp(\mu_2 a^*)\exp(\mu_1 a),$$
$$\exp(\mu_1 a)\exp(\mu_2 a^*)\equiv\exp(\mu_1\mu_2)\exp(\mu_2 a^*)\exp(\mu_1 a).$$

The manipulations are trivial and only the result will be stated. The right side of (44) is what the symbol on the left alleges it to be.[11] In evaluating (45) it is found that

$$|\{\psi_{n-\lambda+l},\ \exp[-i\alpha(a+a^*)]\phi_0\}| =\exp(-\alpha^2/2)[(n-\lambda+l)!]^{-\frac{1}{2}}\alpha^{n-\lambda+l}.$$

Squaring and summing with respect to l, we obtain[12]

$$P_1\equiv 1. \tag{46}$$

With due regard to (46), Eqs. (41) and (42) represent the rigorous quantum-mechanical radiation formula (within the approximation of the first power of the fine structure constant). In the range of energies that will be considered here, it is permissible to neglect terms whose coefficients are $O(n^{-1})$ and $O(n^{-2})$ relative to those of order $O(1)$. Equation (42) therefore simplifies to

$$H(n,\lambda)=\cos^2\theta\,\mathfrak{L}_{n,\lambda}{}^{(+)2}(\alpha^2)+\mathfrak{L}_{n,\lambda}{}^{(-)2}(\alpha^2). \tag{47}$$

III. THE CORRESPONDENCE DOMAIN

Before comparing the radiation formulas[13] (41) and (42) with the one deduced from classical theory,[14] we should like to delineate, with some precision, the domain in which the comparison is to be made. A general orientation in the orders of magnitudes involved is helpful:

$$\kappa_L=2,\ 93(H/10^4)\ \text{cm}^{-1},$$
$$\kappa=2,\ 76\times10^5(H/10^4)\ \text{cm}^{-1},$$
$$\kappa_C/\kappa_L=0.88\times10^{10}(10^4/H).$$

Thus κ_L is for most fields of interest of order unity and κ—the geometric mean of the reciprocal Compton and Larmor lengths—is of the order of the reciprocal wavelength of visible light. From these relations one readily sees that

$$n=\tfrac{1}{4}\beta^2(E/mc^2)^2(\kappa_C/\kappa_L)=2.2\beta^2(E/mc^2)^2\times(10^4/H)\times10^9. \tag{48}$$

We shall now attempt to define a domain of correspondence of classical and quantum theory in this particular problem. This is not to be construed to imply substantial identity. Quantum effects here may be of some importance, but they are only corrections to an essentially classical theory. The frequency of the emitted radiation may be taken as a fairly sensitive indicator. Classically the emitted frequency is harmonic to the rotational frequency

$$\omega=2c(\kappa_C/K_0)\kappa_L \tag{49}$$

of the electron, thus being proportional to $k^C(\lambda)$ in (40). More quantitatively, expression (36) should be adequately represented by the first nonvanishing term obtained after expanding the square root. Numerically $(k/k_m)=0.500$ corresponds to $(k^C/k_m)=0.375$, hardly a close approximation. The ratio $(k/k_m=0.100$ calls for $(k^C/k_m)=0.095$, still a significant difference. The number $(k/k_m)=0.0100$ gives a more comfortable result $(k^C/k_m)=0.00995$. Thus a ratio of 1/137 is about the right order of magnitude. With this (not entirely unique) choice of an integer, the classical principal harmonic

$$k_p{}^C=3\kappa_L(K_0/\kappa_C)^2 \tag{50}$$

is situated comfortably within the range if

$$K_0/\kappa_C<\tfrac{1}{3}\kappa_B/\kappa_L, \tag{51}$$

where $\kappa_B=2\times10^8$ cm^{-1} has the magnitude of a reciprocal Bohr length. In terms of practical parameters this reads

$$E_{\text{Mev}}<1\times10^7(10^4/H). \tag{52}$$

Thus an argument based on the frequency of the emitted quanta rather than the electron energy seems to lead to a somewhat lower upper limit (by a factor of about 10^2) for the validity of the classical radiation formula.[1] That such frequency shifts $\delta(\lambda)$ must be taken into account in this very high energy domain will soon become apparent. The limit (52) should not be pushed much higher; otherwise radiation whose principal harmonic has a sizeable $\delta(\lambda)$ correction will be included.

The asymptotic behavior of Laguerre functions is too involved[5] in the entire range (52) to permit an easy comparison of the results of the two theories. A further

[10] The reader will overlook for a moment a notational inconsistency in (44).

[11] A normalized Laguerre function may be represented by the series $\mathfrak{L}_n{}^{\lambda}(x)=(n!/(n+\lambda)!)^{\frac{1}{2}}x^{\lambda/2}\exp(-x/2)\sum_s\binom{n+\lambda}{n-s}(1/s!)(-x)^s$.

[12] This has been shown, within a good approximation, in a less trivial analytical context in references 3 and 4. Equality (46) asserts that the specific effect then under discussion is not only immaterial to a particular energy range, but nonexistent. An effect P_2, unrelated to P_1, will be studied in the next section. It occurs within the framework of the correct radiation formula and turns out, on examination, to be insignificant, but is none the less extant.

[13] Some of its features have been recognized by G. Parzen, Phys. Rev. **84**, 235 (1951). References 3 and 4 should also be consulted.

[14] See, for example, J. Schwinger, Phys. Rev. **75**, 1912 (1949).

restriction,

$$\alpha^3/n^{\frac{1}{2}}<1, \qquad (53)$$

is therefore imposed for reasons of purely mathematical convenience. Within the limitation of (53) the substantial identity of the two theories will be established.

Before doing this, however, we shall indicate the physical implications of this inequality. It may be written with the aid of (33) as

$$\beta^3(\lambda/2)^3(1/n^2)\sin^3\theta<1, \qquad (54)$$

or, expressed differently,

$$k^C(\lambda)/K_0<(\beta/\sin\theta)(1/n^{1/3}). \qquad (55)$$

A glance at (48) makes clear the numerical significance of (55). The electron indeed loses only a very small fraction of its energy. The principal harmonic will be located in this range if

$$E/mc^2<[(4/27)\beta(\kappa_c/\kappa_L)^2\sin^{-3}\theta]^{1/5}, \qquad (56)$$

or, numerically,

$$E_{\mathrm{Mev}}<4\times10^3(10^4/H)^{2/5}\sin^{-3/5}\theta. \qquad (57)$$

These formulas have been written with a somewhat greater care than the circumstances call for in order to exhibit the fact that if the breakdown of the asymptotic representation should imply the appearance of significant quantum effect, then these would make themselves first felt in the neighborhood of the orbital plane.

Within this domain, a normalized Laguerre function may be approximated by[5]

$$\mathfrak{L}_n{}^\alpha(x)\cong J_\alpha[2(n_1x)^{\frac{1}{2}}];\quad n_1=n+(\alpha+1)/2. \qquad (58)$$

With the aid of the sum and difference rule for Bessel functions (47) can be represented as

$$H(n,\lambda)=\cot^2\theta[J_\lambda(\lambda\beta P_2\sin\theta)/\beta P_2]^2+[J_\lambda'(\lambda\beta P_2\sin\theta)]^2, \qquad (59)$$

where

$$P_2=(1-\lambda/2n)^{\frac{1}{2}}\sec^2(\delta(\lambda)/2). \qquad (60)$$

In writing down (59) the numbers 1 and 2 were neglected relative to n and $n-\lambda$. In terms of classical parameters $[R=(n+1)^{\frac{1}{2}}/\kappa$, ω defined in (49)] Eq. (41) may be written as

$$\begin{aligned}d\mathcal{P}(\lambda)/d\Omega=&(e^2/R)\omega\beta^3(\lambda^2/2\pi)\sec^4(\delta(\lambda)/2)\\&\times[1-\beta^2(\lambda/2n)\sec^2(\delta(\lambda)/2)]\\&\times[1-\beta^2(\lambda/2n)\sec^2(\delta(\lambda)/2)\\&\times\sin^2\theta]^{-1}H(n,\lambda).\end{aligned} \qquad (61)$$

Looking back at the numbers quoted earlier it is seen that (61) is adequately approximated by

$$\begin{aligned}dP(\lambda)/d\Omega=&(e^2/R)\omega\beta^3(\lambda^2/2\pi)\\&\times\{\cot^2\theta[J_\lambda(\lambda\beta P_2\sin\theta)/\beta]^2\\&+[J_\lambda'(\lambda\beta P_2\sin\theta)]^2\},\end{aligned} \qquad (62)$$

an expression identical with the classical radiation formula except for the presence of P_2. The importance of this quantity will now be assessed.

IV. THE QUANTUM EFFECT

The decisive variable in the behavior of Bessel functions in this asymptotic range is the difference between order and argument, or

$$1-\beta P_2\cong\tfrac{1}{2}(1-\beta^2)+(1-P_2)-\tfrac{1}{2}(1-P_2)(1-\beta^2)+\tfrac{1}{8}(1-\beta^2)^2, \qquad (63)$$

where we have confined ourselves to the orbital plane. As noted in the introduction, $1-P_2$ is being compared here with the difference between the velocity of the particle and that of light as expressed in $1-\beta^2\sim10^{-8}$. If indeed $1-P_2$ were significant relative to this small magnitude, the classical radiation formula would break down. According to (60), P_2 consists of a factor

$$(1-\lambda/2n)^{\frac{1}{2}}\cong1-\lambda/4n-\lambda^2/32n^2 \qquad (64)$$

arising from the lack of precision in the definition of the orbit, and of a factor

$$\csc^2(\delta(\lambda)/2)\cong1+\beta^2\lambda/4n+\beta^4\lambda^2/8n^2 \qquad (65)$$

expressing a very slight shift in the frequency of the photon relative to the classical frequency. Their product is

$$1-P_2\cong-\lambda^2/32n^2+(\lambda/4n)(1-\beta^2), \qquad (66)$$

and (63) becomes

$$1-\beta P_2\cong\tfrac{1}{2}(1-\beta^2)-\lambda^2/32n^2+(\lambda/4n)(1-\beta^2)+\tfrac{1}{8}(1-\beta^2)^2. \qquad (67)$$

(Some numbers to keep here in mind are $E\sim5\times10^3\,mc^2$, $1-\beta^2\sim4\times10^{-8}$, $\lambda/n\sim2\times10^{-6}$; the terms in (67) are roughly 10^{-8}, -10^{-13}, 10^{-14}, 10^{-16}.) The second term in (67) is about a hundred thousand times smaller than the first (classical) term and the third (the hopeful one) a million. A physically *un*interesting observation in this connection is that the sign of the leading quantum term is negative. Thus because of (insignificant) quantum effects the particle has an effective velocity (insignificantly) closer to that of light and radiates (insignificantly) more, contrary to some expectations.

It should be noted in this connection that had only the electron term (64) been taken into account, the quantum correction would have been $\lambda/4n\sim10^{-7}$; a quantity larger than the classical term $1-\beta^2\sim10^{-8}$. Unlike in the lower energy range, both of these effects must be considered, in order to establish the substantial identity of the two theories. This, admittedly crude, argument will now be put on a more quantitative basis.

It is convenient for this purpose to write (62) in the form

$$d\mathcal{P}(\lambda)/d\Omega=(e^2/R)\omega(\lambda^2/2\pi)\{[J_\lambda'(\lambda\beta_e)]^2+\tan^2\psi[J_\lambda(\lambda\beta_e)]^2\}, \qquad (68)$$

where

$$\beta_e=\beta P_2\cos\psi \qquad (69)$$

and

$$P_2=2[1+(1-\beta^2\cos^2\psi\lambda/n)^{\frac{1}{2}}]^{-1}(1-\lambda/2n)^{\frac{1}{2}}. \quad (70)$$

A change of variable $\psi=\pi/2-\theta$ has also been introduced. The interesting part of the spectrum,

$$\lambda\sim(1-\beta_e)^{-\frac{3}{2}}, \quad (71)$$

is dominated by the Nicholson asymptotic formulas:

$$J_\lambda(\lambda\beta_e)\cong(2/\lambda)^{\frac{1}{3}}\,\mathrm{Ai}(\xi), \quad (72a)$$

$$J_\lambda'(\lambda\beta_e)\cong-(2/\lambda)^{\frac{2}{3}}\,\mathrm{Ai}'(\xi), \quad (72b)$$

$$\xi=(\lambda/2)^{\frac{2}{3}}2(1-\beta_e). \quad (72')$$

Airy functions appearing in (72) may be defined by the integral

$$\mathrm{Ai}(x)=(1/\pi)\int_0^\infty d\tau\cos(\tau^3/3+x\tau), \quad (73)$$

and the expressions can be regarded as first terms of asymptotic expansion[15] with $(1-\beta_e)$ as a parameter of smallness. In the neighborhood of (71) Eq. (68) is then approximated by

$$d\mathcal{P}(\lambda)/d\Omega=(2/\pi)(e^2/R)\omega(\lambda/2) \times[(2/\lambda)^{\frac{1}{3}}\,\mathrm{Ai}'^2(\xi)+\tan^2\psi(\lambda/2)^{\frac{1}{3}}\,\mathrm{Ai}^2(\xi)]. \quad (74)$$

The effect of P_2 will now be studied in two limiting cases when the argument of the Airy function is much smaller and much larger than unity. Physically these correspond to the spectral ranges of subcritical and supercritical harmonics, respectively.

Case I. $\lambda\ll 2[2(1-\beta_e)]^{-3/2}$

With the aid of the expansions,

$$\mathrm{Ai}(x)\cong(1/2\pi)[3^{-1/6}\Gamma(\tfrac{1}{3})-3^{1/6}\Gamma(\tfrac{2}{3})x], \quad (75a)$$

$$\mathrm{Ai}'(x)\cong-(1/2\pi)[3^{1/6}\Gamma(\tfrac{2}{3})-(\tfrac{1}{2})3^{-1/6}\Gamma(\tfrac{1}{3})x^2], \quad (75b)$$

(74) may be written as

$$\begin{aligned}d\mathcal{P}(\lambda)/d\Omega=(e^2/R)\omega(2/\pi)\{&(1/4\pi^2)3^{\frac{1}{3}}\Gamma^2(\tfrac{2}{3})(\lambda/2)^{\frac{2}{3}}\\&+(1/4\pi^2)\tan^2\psi 3^{-\frac{1}{3}}\Gamma^2(\tfrac{1}{3})(\lambda/2)^{4/3}\\&-(1/2\pi)3^{-\frac{1}{2}}(\lambda/2)^2[2(1-\beta_e)]^2\\&-(1/\pi)3^{-\frac{1}{2}}(\lambda/2)^2\tan^2\psi[2(1-\beta_e)]\}. \quad (76)\end{aligned}$$

Expression (76) was calculated under the assumption that ψ is not too large. With this in mind,

$$2(1-\beta_e)\cong4\sin^2(\psi/2)+\cos\psi(1-\beta^2)+(\lambda/2n) \times\cos\psi(1-\beta^2\cos^2\psi)-(\lambda^2/16n^2)\cos\psi. \quad (77)$$

For the special case of radiation in the orbital plane, (76) and (77) may be combined to yield

$$\begin{aligned}d\mathcal{P}(\lambda)/d\Omega=(e^2/R)\omega(2/\pi)\{&(1/4\pi^2)3^{\frac{1}{3}}\Gamma^2(\tfrac{2}{3})(\lambda/2)^{\frac{2}{3}}\\&-(1/2\pi)3^{-\frac{1}{2}}(\lambda/2)^2(1-\beta^2)^2\\&+(1/16\pi)3^{-\frac{1}{2}}(\lambda/2)^2(1-\beta^2)(\lambda^2/n^2)\\&-(1/2\pi)3^{-\frac{1}{2}}(\lambda/2)^2(1-\beta^2)^2(\lambda/n)\}. \quad (78)\end{aligned}$$

The first two terms in the curly bracket of (78) are of classical, the last two of quantum origin. The latter are seen to represent small corrections to the leading energy independent classical term and are quite insignificant in practice. Expansion (78) being a limiting form of (74) valid mainly in the neighborhood of (71) was checked by expression (68) directly in terms of a Meissel series which constitutes a more appropriate representation for the subscritical range of the spectrum.

For other angles to which (76) is applicable the expressions are more involved but the conclusions the same. The case of very large angles of emission, $\beta_e\sim0$, requires special treatment. This may be carried out with the aid of the formula

$$J_\lambda(\lambda\beta_e)\cong(2\pi\lambda)^{-\frac{1}{2}}(\beta_e/2)^\lambda\exp\lambda. \quad (79)$$

Quantum effects are here even less impressive and the details will therefore be omitted.

Case II. $\lambda\gg 2[2(1-\beta_e)]^{-3/2}$

The asymptotic expressions,

$$\mathrm{Ai}(x)\cong\tfrac{1}{2}\pi^{-\frac{1}{2}}x^{-\frac{1}{4}}\exp[-\tfrac{2}{3}x^{\frac{3}{2}}], \quad (80a)$$

$$\mathrm{Ai}'(x)\cong-\tfrac{1}{2}\pi^{-\frac{1}{2}}x^{\frac{1}{4}}\exp[-\tfrac{2}{3}x^{\frac{3}{2}}], \quad (80b)$$

substituted in (74) yield

$$d\mathcal{P}(\lambda)/d\Omega=(e^2/R)\omega(\lambda/4\pi^2)\{[2(1-\beta_e)]^{\frac{1}{2}} +\tan^2\psi[2(1-\beta_e)]^{-\frac{1}{2}}\}\exp\{-\tfrac{2}{3}\lambda[2(1-\beta_e)]^{\frac{3}{2}}\}. \quad (81)$$

Expanding the exponential we obtain

$$d\mathcal{P}(\lambda)d\Omega=(d\mathcal{P}(\lambda)/d\Omega)^c\exp\{\tfrac{1}{16}(1-\beta^2+\psi^2)^{\frac{1}{2}}(\lambda^3/n^2) -(1-\beta^2+\psi^2)^{\frac{3}{2}}(\lambda^2/2n)\} \quad (82)$$

for angles close to the orbital plane. The approximate expressions that were employed in obtaining this formula are

$$P_2\cong1+\tfrac{1}{32}(\lambda/n)^2-\tfrac{1}{4}(\lambda/n)(1-\beta^2+\psi^2), \quad (83)$$

$$\beta_e\cong1-\tfrac{1}{2}(1+\lambda/2n)(1-\beta^2+\psi^2)+\tfrac{1}{32}(\lambda/n)^2. \quad (84)$$

It is clear from (81) that large angles need not be considered. The exponential factor for $\beta_e\ll1$ is powerful enough to suppress practically all radiation. The unfamiliar symbol in (82) is defined by

$$\begin{aligned}(d\mathcal{P}(\lambda)/d\Omega)^c=(e^2/R)\omega(\lambda/4\pi^2)&\\ \times\{(1-\beta^2+\psi^2)^{\frac{1}{2}}+\psi^2(1-\beta^2+\psi^2)^{-\frac{1}{2}}\}&\\ \times\exp\{-\tfrac{2}{3}\lambda(1-\beta^2+\psi^2)^{\frac{3}{2}}\},& \quad (85)\end{aligned}$$

and is equal to the classical expression for the radiated

[15] Expansions of this type have recently been obtained by F. W. J. Olver, Proc. Cambridge Phil. Soc. **48**, 414 (1952). The mathematical material used in this section will be found in Watson, *Bessel Functions* (The Macmillan Company, New York, 1945).

power in the supercritical range. Its multiplying factor in (82) represents the quantum effect.

Looking back to (54) we see that even for $\lambda^3/n^2 \sim 8$, when the Bessel function representation of Laguerre functions becomes somewhat doubtful, the quantum factor is still only $\sim \exp(10^{-4})$. This strongly suggests that the classical theory is valid for energies considerably higher than the range in which it has been here explicitly shown to hold. Before this suggestion is taken seriously, however, it is necessary, in view of (71), to show that the "safety factor" $(1-\beta^2+\psi^2)^{\frac{1}{2}}$ is not incidental to the limiting form of the particular asymptotic representation that has been employed.

This point may be checked by verifying that the Debye forms,

$$J_\lambda(\lambda\beta_e) \cong (2\pi\lambda)^{-\frac{1}{2}}(1-\beta_e^2)^{-\frac{1}{4}} \times \{\beta_e[1+(1-\beta_e^2)^{\frac{1}{2}}]^{-1} \exp(1-\beta_e^2)^{\frac{1}{2}}\}^\lambda, \quad (86a)$$

$$J_\lambda'(\lambda\beta_e) \cong (2\pi\lambda)^{-\frac{1}{2}}(1-\beta_e^2)^{-\frac{1}{4}}\{\beta_e^{-1}-\beta_e[1+(1-\beta_e)^{\frac{1}{2}}]^{-1}\} \times \{\beta_e[1+(1-\beta_e^2)^{\frac{1}{2}}]^{-1} \exp(1-\beta_e)^{\frac{1}{2}}\}^\lambda, \quad (86b)$$

which are applicable to the supercritical frequency range irrespectively of the electron energy (the Nicholson forms can be used only when the energy is very high), yield expression (82) in the relativistic limit. Substituting (86) into (62) we obtain

$$d\mathcal{P}(\lambda)/d\Omega = (e^2/R)\omega(\lambda/4\pi^2)\beta^3(1-\beta_e^2)^{-\frac{1}{2}} \times \{[\beta_e^{-1}-\beta_e(1+(1-\beta_e^2)^{\frac{1}{2}})^{-1}]^2+\beta^{-2}\tan^2\psi\} \times \{\beta_e[1+(1-\beta_e^2)^{\frac{1}{2}}]^{-1} \exp(1-\beta_e^2)^{\frac{1}{2}}\}^{2\lambda}, \quad (87)$$

a formula valid even if the velocity of the electron is not very high. It is easy to show with a few elementary manipulations that this more rigorous expression reduces to (82) in the limit $(1-\beta_e^2)^{\frac{1}{2}} \ll 1$.

The radiation formula in the intermediate frequency range cannot be expressed in terms of elementary functions. A cursory examination of the numerical tables[16] of Airy functions readily convinces one that the quantum effects in this region are even less significant than in Case II considered previously.

V. SUMMARY

In the domain of ultrarelativistic energies the difference between the velocity of light and that of the particle is decisive in the determination of its radiative properties. Here the classical results may cease to be valid, not only because the electron radiates so intensely that its final state falls into a region where quantum fluctuations count but also because quantum uncertainties may be large relative to this difference.

Quantum effects manifest themselves in the lack of precision in the definition of the orbit and in a shift of the frequency of the emitted radiation. The first effect, though exceedingly small relative to the total velocity of the electron, is sufficient at these energies to produce a considerable diminution in the radiated intensity; the second to give rise to an appreciable increase. If both, however, are properly taken into account, the net result is entirely nugatory.

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[16] Advancement of Sci., Math. Tables, *The Airy Integral* (Cambridge University Press, Cambridge, 1946). Part—, Vol. B.