slow electrons.[9] The present effect is in the right direction to explain part of this but the major part must still be attributed to source and backing-distortion. This is abundantly clear if we consider positrons where the finite wavelength effect depresses the number of slow positrons while the experiments show too many slow positrons even after screening corrections are made. Of course, for high energies the screening effects are much less important than those due to finite wavelength.

From this discussion it appears that the present results are not in disagreement with any known spectra which have been classified as allowed transitions (or as first forbidden with allowed shape). There are no known well-investigated cases of allowed shape spectra with large Z and reasonably high end point to provide a comparison of experiment and these calculations. However, it is entirely reasonable to expect that the effects discussed here can be detected with present experimental technique and they would clearly be of decisive importance for the classification of transitions involving large Z and W_0.

Turning now to the case of favorable parity change transitions, that is, those with spin change equal to $n+1$ (GT selection rules) the correction factor in these cases involves L_0. We have therefore examined the modification of the Kurie-Fermi plot for $Y^{90}(W_0\approx 5.50)$ which corresponds to about as high a Z and W_0 as is encountered in this type of transition. Moreover, reasonable thin sources were used in the measurements of this spectrum. In this case the correction involved here produces a linear plot over a somewhat greater energy range and does not change the conclusion that the transition is first forbidden. In the case of other known transitions with a unique matrix element similar conclusions should result.

Finally, it is fairly evident that the approximation which has hitherto been inherent in the beta-decay theory could not have led to any appreciable falsification of the endpoint energies as determined by extrapolation of the "straight" Kurie-Fermi plots.

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[9] For example, C. S. Wu and R. D. Albert, Phys. Rev. 75, 1107 (1949).

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Effect of a Short-Range Repulsion on Low Energy Singlet Nucleon-Nucleon Scattering*

M. H. Hull, Jr., and A. Herschman
Yale University, New Haven, Connecticut

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Modifications needed in order that low energy singlet scattering be described when a repulsive core is introduced in a phenomenological nucleon-nucleon interaction are investigated. It is found that if a square well is used to represent the attractive part of the interaction, a range of potential parameters allows a description of the data and also satisfies the charge independence hypothesis. When the Yukawa potential is investigated, the assumption of charge independence is more restrictive, and a single set of parameters is found.

I. INTRODUCTION

JASTROW[1] introduced a short range repulsive core into a phenomenological nucleon-nucleon interaction in order to allow a charge independent description of high energy scattering experiments. At low energies, the theoretical interpretation of scattering experiments had indicated charge independence of nuclear forces at an early date[2] without a repulsive core in the assumed interaction. The question arises as to what modification of the phenomenological interaction used to describe low energy scattering is necessary when a core is introduced. It is the purpose of this note to report on an investigation of this point.

The low energy singlet nucleon-nucleon scattering has been described by the f function of Breit, Condon and Present,[2] where

$$f=(C_0^2/\eta)\cot K_0-2\ln\eta+q_0/\eta$$

in the notation of BCP. The quantity K_0 is the phase shift due to the specifically nuclear interaction. For the limit of the charge e going to zero, one has[3]

$$\lim_{e\to 0}(f/a)=k\cot K_0,$$

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[1] R. Jastrow, Phys. Rev. 81, 165 (1951).

[2] Breit, Condon, and Present, Phys. Rev. 50, 825 (1936) (referred to as BCP); G. Breit and J. R. Stehn, Phys. Rev. 52, 396 (1937). For other early evidence not depending exclusively on scattering, see G. Breit and E. Feenberg, Phys. Rev. 50, 850 (1936) and references cited by them.

[3] G. Breit and M. H. Hull, Jr., Am. J. Phys. 21, 184 (1953).

where a is the nuclear Bohr radius and k^2 is proportional to the energy. The connection of this description with others current in the literature has been discussed in detail by Breit.[4] The function f can be expanded as a power series in the energy,[4,5] and the first two coefficients of the series have been used in the interpretation of low energy scattering data.[5]

In calculations of the phase shift, the supposition is made that the potential $V(r)$ representing the specifically nuclear interaction vanishes as $r\rightarrow\infty$ rapidly enough so that it may be neglected beyond a distance $r=r_0$. The f function may be obtained in the general case, therefore, by going to the limit $r_0\rightarrow\infty$ and taking the limit for the phase shift.[4] As given by Breit and Bouricius[5] and Hatcher, Arfken, and Breit,[5] the first two coefficients in the energy expansion of f about $E=0$ are equivalent, respectively, to the logarithmic derivative of the zero-energy wave function of the system at r_0, and the integral of the square of the zero-energy wave function from $r=0$ to r_0.

The present calculation is intended to exhibit the effect of the introduction of a hard core on the charge independence hypothesis (CIH). The discussion is facilitated by the employment of an auxiliary concept of the equivalence of potentials. Two potentials are said to be equivalent for a given type of scattering if they reproduce the values of the logarithmic derivative and of the "integral" at distances greater than the greatest distance at which a specific nuclear interaction takes place. This definition can be made for any energy; in the present work, however, it is always meant to apply to zero energy. Such potentials are "equivalent," therefore, in their ability to describe low energy scattering. Since the primary object is to exhibit the effect of the core rather than to give a definite answer to the question of validity of the CIH, the comparisons have been made by supposing, in a somewhat idealized fashion, that the experimental data agree with the CIH exactly for a definite pair of the range and depth parameters of a potential well of specified shape. As a means of summarizing the data in this manner, some of the older fits[6,7] were used. The introduction of the core requires a readjustment of the range and depth parameters in order to secure equivalence and this readjustment has to be made independently for the n-p and p-p potentials. A core of arbitrary radius may be expected to result, therefore, in the two adjustments for equivalence leading to different pairs of range and depth parameters. For an exceptional choice of the core radius, the two pairs might conceivably become equal. While the expectation expressed above is plausible on general grounds it does not give sufficient weight to the practical possibilities of differentiation between nearly equivalent well parameter pairs. In point of fact, it is found that, for a square well, a charge independent interaction when readjusted for equivalence remains practically charge independent for all core radii in the range of values investigated. For an initially charge independent Yukawa interaction, an adjustment for equivalence called for by the introduction of a core gives no charge independence for any reasonable core radius. In order to make it possible to have charge independence with a core it becomes necessary to suppose that experimental data correspond to interactions which for no core have relatively an effectively longer range for n-p scattering, as will be seen in more detail later.

The core is assumed to be "hard," as in Jastrow's work;[1] i.e., for r less than the core radius, the potential is infinite. The wave function vanishes at the core radius, therefore, and is zero for r less than the core radius.

TABLE I. Square well parameters for n-p and p-p interactions with repulsive core under conditions of equivalence.

r_0	p-p Interaction r_1	D_{p-p}	D^c_{p-p}	n-p Interaction r_1	D_{n-p}
1.00	0	10.50	11.33	0	11.33
0.9465	0.0505	13.44	14.27	0.0499	14.28
0.8935	0.1009	17.62	18.46	0.1001	18.49
0.8411	0.1510	23.82	24.67	0.1500	24.71
0.7890	0.2012	33.60	34.46	0.2000	34.48

II. SQUARE WELL CALCULATIONS

In this case, the distance r_0 at which the conditions of equivalence of interactions are applied is just the radius of the square well. The interaction without core was assumed to be that found in the work of BTE.[6] The exact values chosen are not of vital importance, since, as was pointed out in the introduction, an investigation of the effect of a core is the purpose of the work rather than a decision as to the validity of the CIH. In order to simplify the calculations, the Coulomb force was excluded from the attractive region of the p-p interaction. The effect of including it was then taken into account by means of the first-order perturbation formulas of BCP[2] and BTE.[6] In a sample case, the Schroedinger equation for the wave function, with the Coulomb potential included, was integrated numerically in order to check the accuracy of the perturbation calculation. Agreement between the results of the numerical integration and the perturbation calculation was within a half percent for the calculated well depth.

The results of the square well calculation are given in Table I in terms of the following notation: r_0 is the square well radius in units of e^2/mc^2, r_1 is the radius of the repulsive core in the same units, D is the square well depth in Mev, D^c is the depth of the square well in the p-p case when the effect of the Coulomb potential is taken into account by means of the perturbation calculation.

[4] G. Breit, Revs. Modern Phys. **23**, 238 (1951).

[5] G. Breit and W. G. Bouricius, Phys. Rev. **75**, 1029 (1949); Hatcher, Arfken, and Breit, Phys. Rev. **75**, 1389 (1949); G. Breit and R. D. Hatcher, Phys. Rev. **78**, 110 (1950).

[6] Breit, Thaxton, and Eisenbud, Phys. Rev. **55**, 1018 (1939) (referred to as BTE).

[7] Hoisington, Share, and Breit, Phys. Rev. **56**, 884 (1939).

Potentials with the parameters listed in Table I are equivalent, in the specific sense described in the introduction, to the potential without core represented by the parameters in the first row of the table. Comparison of the values of r_1 for the n-p and p-p cases and of $D_{n\text{-}p}$ with $D^c{}_{p\text{-}p}$ for a given well radius r_0 shows that the charge independence of the interaction holds throughout the range of parameters investigated within the precision of the calculations. The result obtained qualitatively by Breit[8] that the sum of core and square well radii under the conditions of equivalence is equal to the well radius without core is found to hold very closely in these calculations.

Note added in proof:—If the standard p-p square well case is calculated with the Coulomb potential included for $0<r<r_0$, the values of the logarithmic derivative and integral which result differ slightly from those used in the calculations for Table I. The parameters yielding equivalence for the case with core are then modified as follows: the values of r_1 are unchanged within the precision of the calculation, while the values of $D_{p-p}{}^c$ are 14.29, 18.51, 24.83, and 34.88 Mev for $r_0=0.947$, 0.894, 0.841, 0.789 e^2/mc^2, respectively. Comparison of these values with D_{n-p}, Table I, shows a trend with decreasing r_0 similar to the behavior of $\mathcal{C}_{p-p}$ and $\mathcal{C}_{n-p}$ for decreasing a in the Yukawa case (see Tables II or III for $a_s=0.42$ e^2/mc^2), although the magnitudes of the relative differences are much smaller in the square well case, and the conclusion concerning "practical" charge independence is not changed. Limiting cases, when $r_0=r_1$, were calculated for the square well interaction with the results that $r_0=r_1=0.496$ for the p-p interaction, and $r_0=r_1=0.494$ for the n-p case.

III. YUKAWA WELL CALCULATIONS

The wave functions for the Yukawa well were obtained by means of numerical integrations of the differential equations, with the Coulomb potential included from the beginning in the p-p case. In order to reduce the numerical work, the distance, r_0 at which the potential representing the nuclear interaction is assumed to become negligible was taken to be $3e^2/mc^2$ for the direct calculations. The effect of the tail of the Yukawa potential to $6e^2/mc^2$ was taken into account by means of the perturbation method for the logarithmic derivative given in BTE[6] [Eqs. (9) *et seq.* of that paper]. The potential beyond $6e^2/mc^2$ is not taken into account at all since it has no appreciable effect on the results.[9] As used here, the designation "Yukawa potential" signifies that much of the potential acting between $r=0$ and whichever of the values of r_0 is under consideration. The parameters for the interaction without core were those obtained by Hoisington, Share and Breit[7] in an analysis of p-p singlet S scattering: $\mathcal{C}_s=89.648mc^2$ for the depth parameter and $a_s=0.42e^2/mc^2$ for the range. The notation of Hatcher, Arfken and Breit[5] is employed, where the Yukawa interaction is represented by $\mathcal{C}e^{-r/a}/r/a$. The p-p and n-p parameters were assumed to be the same without core for the initial calculations. When it was found that, unlike the square well case, the introduction of a repulsive core in addition to the Yukawa interaction destroyed charge independence if it were assumed without core, a second n-p no core case was investigated. The parameters are $\mathcal{C}_s=81.663mc^2$ and $a_s=0.44e^2/mc^2$. The quantity $a_s{}^2\mathcal{C}_s=15.81$ was held constant when the parameters for the interaction without core were varied. The effect

TABLE II. Yukawa well parameters for n-p and p-p interactions with core under conditions of equivalence[a] for $r_0=3e^2/mc^2$.

	p-p interaction			n-p interaction					
a'	b'	$\mathcal{C}'$	$a'^2\mathcal{C}'$	b'	$\mathcal{C}'$	$a'^2\mathcal{C}'$	b'	$\mathcal{C}'$	$a'^2\mathcal{C}'$
0.44								81.663	15.81
0.42	0	89.648	15.81	0	89.648	15.81			
0.32	0.0801	271.9	27.84	0.0716	260.9	26.72	0.0866	285.0	29.18
0.25	0.150	826.7	51.67	0.135	773.2	48.32	0.153	851.4	53.21
0.22	0.182	1522	73.64	0.167	1385	67.05	0.184	1562	75.58
0.213	0.191	1793	81.32	0.173	1618	73.41	0.191	1828	82.94
0.21	0.193	1913	84.38	0.177	1734	76.45	0.195	1965	86.20
0.20	0.205	2477	99.06	0.188	2204	88.14	0.204	2484	99.34
0.195				0.191	2480	94.25	0.210	2826	107.5

TABLE III. Yukawa well parameters for interactions with core under conditions of equivalence[a] for $r_0=6e^2/mc^2$.

	p-p interaction			n-p interaction					
a'	b'	$\mathcal{C}'$	$a'^2\mathcal{C}'$	b'	$\mathcal{C}'$	$a'^2\mathcal{C}'$	b'	$\mathcal{C}'$	$a'^2\mathcal{C}'$
0.44							0	81.663	15.81
0.42	0	89.648	15.81	0	89.648	15.81			
0.32	0.0842	285	29.2	0.0744	270	27.6	0.0926	297	30.4
0.25	0.157	861	53.8	0.139	792	49.5	0.158	880	55.0
0.22	0.190	1585	76.7	0.171	1441	69.8	0.190	1601	77.5
0.21	0.201	2004	88.4	0.181	1787	78.8	0.200	2005	88.4
0.20	0.212	2605	104	0.192	2269	90.8	0.211	2592	104

[a] As defined in the introduction and applied independently to p-p and two n-p cases.

[8] G. Breit, Phys. Rev. **84**, 1053 (1951).
[9] Yovits, Smith, Hull, Bengston, and Breit, Phys. Rev. **85**, 540 (1952).

on the n-p parameters with core of changing the n-p no core case was calculated by means of another application of the perturbation method of BTE.[6] The information obtained in this calculation allows n-p no core parameters to be varied so that a charge independent potential with core can be found, as suggested in the introduction.

Results of the calculations are given in Table II for $r_0=3e^2/mc^2$ and in Table III for $r_0=6e^2/mc^2$. The procedure of making these approximations to $r_0=\infty$ is justified by the possibility of the limiting process discussed by Breit[4] and mentioned in Sec. I. The radius of the repulsive core is designated as "b" in the tables. The notation used is $a'=a(mc^2/e^2)$, $b'=b(mc^2/e^2)$, $\mathcal{C}'=\mathcal{C}/mc^2$. The parameters listed were determined under the conditions of equivalence as described in Sec. I.

Values of $a'^2\mathcal{C}'$ are included since these vary less rapidly than $\mathcal{C}'$ itself and were actually used in determining the parameters discussed below. Inspection of the tables shows that both with and without the inclusion of the tail of the Yukawa interaction between $r=3e^2/mc^2$ and $6e^2/mc^2$, no single triplet of values of $(a, \mathcal{C}, b)$ allows the n-p and p-p interactions to be simultaneously equivalent to the no core case where $\mathcal{C}=89.648mc^2$, $a=0.42e^2/mc^2$. When the n-p interaction parameters without core are changed to $(a_s', \mathcal{C}_s')=(0.44, 81.663)$ keeping $a_s'^2\mathcal{C}_s'=15.81$ and keeping $(a_s', \mathcal{C}_s')=(0.42, 89.648)$ for the p-p case, adjustments for equivalence can then yield the same (a, b) for n-p and p-p. The corresponding $\mathcal{C}$ differ, however, in general. Or else $(a, \mathcal{C})$ can be made the same but with different b in the two cases. The results of graphical and numerical interpolations for these cases are given in Table IV.

If linearity for small changes in the parameters is assumed, the triplet of values of parameters simultaneously satisfying the conditions of equivalence and the hypothesis of charge independence of the interaction can be found from the tabulated values of Tables II, III. The n-p no core parameters are allowed to vary, subject to holding $a_s^2\,\mathcal{C}_s$ constant. The sets of parameters which result are given in Table V.

TABLE IV. Values of $(a', b', \mathcal{C}')$ yielding equivalence when the no core interactions are $(a_s', \mathcal{C}_s')=(0.44, 81.663)$ for the n-p case and (0.42, 89.648) for the p-p case.

r_0	p-p interaction	n-p interaction
$3e^2/mc^2$	(*0.200*, *0.205*, 2475)	(*0.200*, *0.205*, 2485)
	(*0.199*, 0.207, *2800*)	(*0.199*, 0.206, *2800*)
$6e^2/mc^2$	(*0.214*, *0.206*, 1835)	(*0.214*, *0.206*, 1865)
	(*0.203*, 0.208, *2160*)	(*0.203*, 0.207, *2160*)

TABLE V. Interaction parameters $(a', b', \mathcal{C}')$ which are equivalent with and without core independently for n-p and p-p cases.

r_0	p-p interaction	n-p interaction
$3e^2/mc^2$	(0.42, 0, 89.648)	(0.439, 0, 81.726)
	(*0.201*, *0.204*, *2720*)	(*0.201*, *0.204*, *2720*)
$6e^2/mc^2$	(0.42, 0, 89.648)	(0.442, 0, 80.962)
	(*0.198*, *0.225*, *2840*)	(*0.198*, *0.225*, *2840*)

IV. DISCUSSION OF RESULTS

When a phenomenological nucleon-nucleon interaction with a repulsive core is required to be equivalent to an interaction without core which has been adjusted approximately to fit low energy scattering data, it appears that the range of the attractive part of the interaction is reduced and the strength is increased. This confirms the qualitative discussions of Jastrow[1] and especially of Breit.[8] When the attractive part of the interaction is represented as a square well, it further appears that if the n-p and p-p interactions are the same without core, the charge independence may be maintained for any value of the core when the conditions of equivalence are satisfied. When the Yukawa potential is used for the attractive part of the interaction, however, charge independence is destroyed by introduction of the core if it is assumed without core. If the p-p noncore parameters are fixed and the n-p noncore range allowed to vary with the strength fixed by requiring $a_s^2\mathcal{C}_s=15.81$, a set of four parameters a, b, $\mathcal{C}$, and a_s for the n-p noncore case can be found for which equivalence and charge independence are simultaneously obtained with a core. A different set would result for different p-p standard parameters and a different value of $a_s^2\mathcal{C}_s$, but in any case, the n-p "no core" range must be taken larger than the p-p "no core" range. A comparison of the recent discussions of low energy scattering by Yovits, Smith, Hull, Bengston, and Breit[9] for the p-p case and Salpeter[10] for the n-p case shows that a difference in the singlet ranges as determined from experimental data, with no core assumed, is possibly indicated. However, the experimental results seem to favor the n-p singlet range being *shorter* than the p-p range.

It may be pointed out that the core radius which comes out of these calculations is of the order of that assumed by Jastrow,[1] and also is in the range of values of core radii assumed in recent calculations,[11] where the core is used to represent relativistic effects or other uncalculated features of the pseudoscalar meson theory of the nucleon-nucleon interaction. The value of the core radius found here also is within the limit of $0.28e^2/mc^2$ set by Austern[12] in a discussion of the 100-Mev deuteron photoeffect.

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[11] M. Lévy, Phys. Rev. **86**, 806 (1952); **88**, 725 (1952); J. V. Lepore, Phys. Rev. **87**, 209 (1952); R. Jastrow, Phys. Rev. **87**, 209 (1952); Taketani, Machida, and O-Numa, Prog. Theoret. Phys. **7**, 45 (1952).
[12] N. Austern, Phys. Rev. **88**, 1207 (1952).