WAVE-LENGTH ENERGY DISTRIBUTION IN THE CONTINUOUS X-RAY SPECTRUM.

BY BERGEN DAVIS.

PROFESSOR DUANE and F. L. Hunt[1] have described some very interesting and important experiments on the energy radiated at various wave-lengths from a Coolidge X-ray tube.

The Coolidge X-ray tube was operated by a steady high voltage obtained from a large number of small storage cells. This voltage, however, was less than that required to produce the characteristic rays of tungsten. The radiation, after passing through a slit, was reflected from a cleavage surface of a calcite crystal. The wave-length was determined in the usual manner by the angle of reflection and the intensity of the radiation was measured by the ionization produced in the ionization chamber of the spectrometer. In this way an experimental relation was obtained between the energy and the wave-length of the radiation from a Coolidge X-ray tube.

The results of one experiment are shown by the full line in Fig. 3. This curve is reproduced from the curve published by Duane and Hunt. The results have not been corrected for variation of absorption with wave-length nor for the dependence of reflectivity of the crystal upon wave-length.

There is no radiation emitted at a wave-length less than λ_0. The curve rapidly rises to a maximum and then decreases with increasing wave-length. A point of great interest is that the shortest wave-length λ_0 corresponds to that required by the quantum relation.

$$P_0 e = \tfrac{1}{2} m v_0^2 = h\nu_0 = \frac{hc}{\lambda_0}.$$

The significance of these results is to be found in the fact that although the electrons of the cathode stream all have practically the same velocity, the radiation has a wide range of frequencies less than the frequency (ν_0) required by the Planck equation. Also as the voltage applied to the tube is varied the wave-length (λ_0) at which the curve starts is shifted, the curve still retaining the same form. Dr. A. W. Hull[2] has recently

[1] Proc. Amer. Phys. Soc., PHYS. REV., August, 1915.

[2] Proc. Amer. Phys. Soc., PHYS. REV., Jan., 1916.

shown that the wave-length (λ_0) of the end radiation fulfills the condition:

$$P_0 e = h\nu_0 = \frac{hc}{\lambda_0},$$

for all voltages applied to the X-ray tube, at least for all voltages up to 100,000 volts, which was the upper limit of the experiments.

It is proposed here to develop a theory of this continuous or "white" X-ray spectrum.

Two distinct hypotheses will be made as to the origin of the radiation.

A. A Pulse Hypothesis.

The radiation will be considered as consisting of the pulses arising from the accelerations of the electrons of the cathode stream upon their impacts with the atoms of the target.

B. A Quantum Hypothesis.

The radiation will be considered as arising from electrons of the atom, excited in some manner by the transfer of energy from the electrons of the cathode stream. The quantity of energy radiated will fulfill the condition:

$$\epsilon = h\nu.$$

The frequency also is controlled by the condition:

$$\tfrac{1}{2}mv^2 = h\nu.$$

In this hypothesis, only a small fraction of the electrons of the cathode stream make impacts that are effective in exciting radiation.

A. The Pulse Hypothesis.

The following fundamental assumptions are made:

(*a*) The "white" or continuous X-radiation are the electro-magnetic pulses produced by the accelerations of the electrons of the cathode stream upon their collisions with the atoms of the target. (The Stokes pulse theory of X-rays.)

(*b*) The energy radiated by an impinging electron is given by the equation:

$$E = \frac{2}{3}\frac{e^2}{c}\int a^2 dt,$$

where a is the acceleration of the electron, e is the charge and c is the velocity of propagation.

(*c*) The frequency (ν) of the emitted radiation fulfills the condition:

$$\tfrac{1}{2}mv_n^2 = h\nu,$$

where v_n is the component of the velocity of the electron normal to the atomic surface.

(*d*) The atoms of the target are considered to possess centers of force that strongly repel the colliding electrons. The repulsive force varies inversely as the cube of the distance from the center of force, and has a spherical distribution about this center.

(*e*) The electrons do not lose all of their kinetic energy at a single collision, but as they thread their way down through the surface of the target they lose their energy (velocity) by successive collisions with the atoms. These successive collisions continue until the translational energy of the electrons is exhausted.

A distinction is here made as to the nature of the continuous X-radiation and the characteristic radiation. The continuous radiation which is due to the accelerations of the impinging electrons is probably independent of the nature of the atoms of the target, except in so far as the repulsive force of the atoms for the electrons may depend on atomic properties. The characteristic radiation on the other hand depends directly on the atomic number of the element composing the target. This radiation arises from the electrons within the atom while the continuous radiation originates in the accelerations of the electrons of the cathode stream.

The difference between the continuous X-ray spectrum and the characteristic radiation is exhibited in a striking manner by the curves in Fig. 1, which Dr. A. W. Hull, of the Research Laboratory of the General Electric Company, has kindly allowed me to reproduce here. Curve 1 is the spectrum of the radiation from a tungsten target, and curve 2 is that from a molybdenum target. They were both taken at the same voltage of 43,000 volts. This voltage is less than that required to produce the characteristic radiation of tungsten, and is similar to that obtained for tungsten by Duane and Hunt (see Fig. 3). The voltage, however, is greater than that required to produce the characteristic rays of molybdenum. The curve 2 shows the characteristic (line) radiation superimposed on the continuous spectrum. The intensity of the characteristic radiation is great compared to that of the continuous spectrum.

The way in which the characteristic (line) spectrum is added to the continuous indicates that the two radiations may have different origins, and suggests the view advanced here, that the one arises from the electrons of the atoms and the other from the electrons of the cathode stream. These two curves (Fig. 1) were obtained under identical conditions.

When an electron impinges on the atomic surface (region of repulsive force), it is accelerated and a pulse is emitted. In two papers on a pulse theory of temperature radiation, Sir. J. J. Thomson[1] has described the kind of impact required by the laws of such temperature radiation. For the sake of brevity the results obtained by Professor Thomson will be applied to the present problem.

(*a*) It is shown that in order that the radiation may satisfy the second

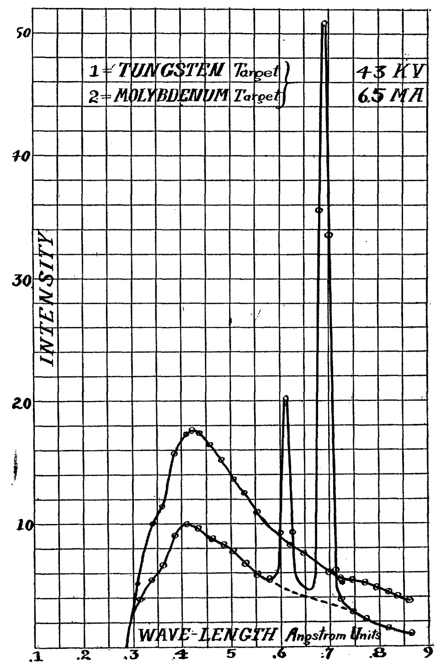

Fig. 1.

law of thermodynamics, the product of the kinetic energy of the colliding electron and the time of impact with the atom must be equal to a constant. This condition may be represented by:

$$\tfrac{1}{2}mv_n^2t = H = \text{a constant}. \qquad (1)$$

This is similar to Planck's quantum relation since a frequency is a reciprocal of a time.

(*b*) It is shown that the inverse cube law of force between the colliding electron and the atom gives approximately the above relation between the impact time and the kinetic energy of the electron.

(*c*) It is shown that approximately the rate of emission of energy during the impact of an electron is:

$$E = \frac{2}{3}\frac{e^2}{\pi e}\frac{v_n^2}{t}, \qquad (2)$$

[1] Phil. Mag., August, 1907, and July, 1910.

whose v_n is the normal component of velocity at beginning of the collision and t is the duration of the impact.

The radiation (2) may be expressed in terms of the normal velocity v_n by means of equation (1)

$$E = \frac{1}{3} \frac{m}{H} \frac{e^2}{\pi c} v_n^4. \tag{3}$$

The energy is proportional to the fourth power of the normal component of the velocity of the colliding electron.

It is assumed that there is some constant relation between the pulse length and the wave-length at which a greater part of the energy is radiated. That is, the constant H of equation (1) has such a value that the frequency of the radiation approximately fulfills the conditions of the quantum hypothesis:

$$\tfrac{1}{2} m v_n^2 = h\nu.$$

The energy radiated is:

$$E = \frac{4}{3} \frac{e^2}{m} \frac{h^2}{c\pi H} \nu^2.$$

or

$$E = k\nu^2. \tag{4}$$

The *energy radiated* is *proportional* to the *square* of the *frequency*.

B. Quantum Hypothesis.

There are serious objections to a pulse theory of the origin of X-radiation. It is doubtful if any pulse theory can satisfy all the results of experiment.

The diffraction effects obtained by means of the crystal grating indicates interference of many successive waves. There can be no true interference by the reflection of a single pulse from the successive molecular planes of the crystal. The pulse may however be resolved into its harmonic constituents and the interference may occur between the successive waves of each frequency. The crystal may perhaps possess the power of making such analysis of the pulse. But this would not lessen the difficulty. The resolution of a pulse into its harmonic constituents would in general give a series of harmonic terms extending to infinite frequency, whose coefficients might have definite magnitudes. The experiments however show that there is no measureable energy present at a frequency greater than the end frequency ν_0. In order to apply the pulse theory one must assume all the coefficients of the higher terms to be approximately zero. The investigation of Professor Thomson, however, previously referred to, shows that this is not the case.

It is quite possible that the mere acceleration of an electron may not necessarily cause radiation. The pulse hypothesis is of course founded on continuous mechanics. The ability of an electron to emit radiation may be in some way dependent on its position with reference to an atom. The success of the Bohr model of the atom strongly suggests that this may be so.

If a system of discontinuous mechanics is required, then of course a pulse hypothesis of radiation is not possible, and one must adopt some form of quantum theory to account for the radiation.

There are a number of physical phenomena that indicate that exchanges of energy probably take place in "quanta." In addition to temperature radiation one might mention the photo-electric effect, the excitation of characteristic X-radiation by impact of cathode rays, the secondary characteristic X-rays and other phenomena. All of these phenomena apparently involve exchanges of energy in definite quanta.

The following assumptions are made:

(*a*) If the impinging electron excites X-rays at all, it transfers all of its normal kinetic energy (that part of its energy due to the normal component of its velocity) into radiant energy.

(*b*) Only a small fraction of the collisions of the electrons result in the production of radiation. This small fraction of the electrons that is effective in producing radiation may be expressed as the probability p that any one electron makes an effective collision.

(*c*) The probability p that any one electron will produce X-radiation is taken proportional to the kinetic energy due to the normal component of its velocity

$$p = C\tfrac{1}{2}mv_n^2,$$

where C is a constant. Actual experiment of course shows that but a small proportion of the energy of the cathode stream appears as X-radiation. The greater part of the energy appears as heat in the target.

The energy emitted by the effective electron will be

$$\epsilon = h\nu.$$

The probability that any one electron in an encounter with an atom will emit radiation is p. The energy emitted by any one electron at an effective impact will be:

$$E = ph\nu.$$

This probability p is placed proportional to the kinetic energy due to the normal component of the velocity.

From which

$$E = C\tfrac{1}{2}mv_n^2h\nu.$$

But

$$\tfrac{1}{2}mv_n^2 = h\nu.$$

The probable energy emitted by one electron at a collision now becomes:

$$E = Ch^2\nu^2. \tag{5}$$

This expression is of the same form as equation (4).

The energy emitted is proportional to the square of the frequency.

The foregoing derivation of course, does not attempt to explain the process, or state the conditions that must obtain in order that an occasional electron shall be effective in producing radiation.

The Resultant Radiation from the Target.

A general expression for the radiated energy in terms of the frequency having been obtained, it remains to derive an equation for the resultant radiation produced by all of the electrons of the cathode stream upon their collisions with the atoms of the target. The electrons of the cathode stream all have the same original velocity ν_0 upon impact with the atoms at the surface of the target. The velocity at which they collide with the atoms progressively decreases.

All frequencies below the end frequency ν_0 may be emitted by the impact of the electrons against the atomic "surface." The forces acting are considered to be radial forces and hence only the normal component of the velocity will be affected, and this component only will determine the frequency. Thus electrons making impact at the pole p of an atom will emit the highest frequency, while those making impact at an angle will emit a lower frequency corresponding to that component of the velocity which is along the radius. This process will take place for all the impacts taking place with the atoms at various depths within the target. The total radiation at any frequency will be found by summing the radiations emitted at the impacts of the electrons that have a normal velocity component corresponding to that frequency as they make successive collisions with the atoms of the target.

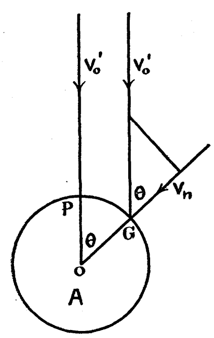

Fig. 2.

Let n' be the number of electrons that impinge on a zonal element of the surface of an atom at an angle (see Fig. 2). The radiation produced by these n' electrons will be by equation (4) or (5).

$$E = kn'\nu^2.$$

The frequency of the radiations may be expressed by:

$$\tfrac{1}{2}mv_n^2 = h\nu. \tag{6}$$

The electrons possess a velocity ν_0 at any depth within the target, and those that make impact at poles p of the atoms will emit radiation of frequency ν_0'.

$$\tfrac{1}{2}m(v')^2 = h\nu_0'. \tag{7}$$

Since

$$v_n = v_0' \cos \theta,$$

the frequency emitted at any angle θ may be expressed in terms of the polar frequency ν_0'.

$$\nu = \nu_0' \cos^2 \theta. \tag{8}$$

If I is the number of electrons that strike the atomic surfaces in an element of time (number of impacts), then

$$n' = I 2 \sin \theta \cos \theta d\theta. \tag{9}$$

The energy radiated by all those electrons having a velocity v_0', that make impact on a zonal element of surface of an atom will be

$$dw = kI(\nu_0')^2 \cos^4 \theta (2 \sin \theta \cos \theta d\theta).$$

By differentiation of equation (8) and substituting in above

$$dw = -kI\nu_0' \left(\frac{\nu}{\nu_0'}\right)^2 d\nu. \tag{10}$$

This expresses the energy lying between a frequency ν and $\nu + d\nu$ emitted by those electrons having a velocity v_0'.

This velocity v_0' is any velocity at any depth x within the surface of the target.

In order to derive the total radiation it is necessary to express the number of impacts at any velocity at any depth of penetration.

Let v_0 be the velocity with which N electrons enter the surface of the target. The number n_1 found in a layer dx deep will be

$$n_1 = N \frac{dx}{v_0}.$$

The number n present in any layer dx at any depth will be

$$n = n_1 \frac{v_0}{v_0'},$$

where v_0' is the velocity at any depth.

From these two expressions we obtain

$$n = N \frac{dx}{v_0'}.$$

The number of impacts will be proportional to the velocity and to the number having that velocity. The number of impacts in any region dx thick is

$$I = nv_0' = Ndx. \tag{11}$$

It is difficult to formulate an expression for the distribution of velocities of the electrons as they pass down through the surface of the target. One may imagine that the electrons will be differently affected, and that although they all enter the surface with the same velocity, yet at a depth within the surface, after a number of chance collisions there must be a wide range of velocities.

Widdington[1] has investigated the decrease of velocity of electrons on passing through thin films of various thickness. The decrease of velocity observed could be very well represented by:

$$v^4 = v_0^4 - ax.$$

Since $V = 0$ at a thickness corresponding to the range R, this may be written:

$$v^2 = v_0^2 \sqrt{1 - \frac{x}{R}}. \tag{12}$$

But this represents the law of velocity decrease for those few electrons only that passed directly through the film without deviation. The greater part of the entering electrons must have been scattered about, many issuing at various angles and velocities and some of them absorbed in the film itself.

The following consideration will show that Widdington's law cannot represent the velocity distribution completely. In the case of a solid target the electrons all strike the surface at the same velocity v_0. At some depth all of the electrons must have lost all of this initial velocity and have acquired the probability velocity distribution required by Maxwell's law for the temperature of the target. At intermediate depths there must be a gradual transition from the velocity v_0 to the Maxwell distribution. The true expression of the velocity distribution at various depths below the surface would be some equation which would gradually change from a velocity v_0 at the surface ($x = 0$) to an approach to the probability distribution as the depth x became greater and greater. It is manifestly a difficult matter to write such an equation empirically.

I have assumed as an *approximation* for the decrease of velocity for small depths of penetration a modification of Widdington's law of the following form:

$$(v_0')^2 = v_0^2 \left[1 - \left(\frac{x}{R}\right)^a \right]. \tag{13}$$

R is the maximum range of the electrons. The exponent a is a constant whose value may depend on the nature of the target.

[1] Proc. Royal Soc., April, 1912.

From equation (13),

$$\nu_0' = \nu_0 \left[1 - \left(\frac{x}{R}\right)^a \right],$$

$$d\nu_0' = -\nu_0 \frac{a}{R} x^{a-1} dx.$$

By substitution these two give

$$dx = -\frac{R}{a\nu_0}\left(1 - \frac{\nu_0'}{\nu_0}\right)^{\frac{1-a}{a}} d\nu_0'. \tag{13}$$

By means of equations (11) and (13), equation (10) becomes

$$w = kN\nu^2 d\nu \frac{R}{a\nu_0} \int_{\nu}^{\nu_0} \frac{d\nu_0'}{\nu_0'} \left(1 - \frac{\nu_0'}{\nu_0}\right)^{\frac{1-a}{a}}.$$

The indefinite integral of this cannot be obtained, but by expanding and taking four terms:

$$\begin{aligned} w = kN\frac{R}{a}\nu_0\left(\frac{\nu}{\nu_0}\right)^2 d\nu \Bigg\{ & \log\frac{\nu_0}{\nu} - \frac{1-a}{a}\left[1 - \frac{\nu}{\nu_0}\right] \\ & + \frac{(1-a)(1-2a)}{4a^2}\left[1 - \left(\frac{\nu}{\nu_0}\right)^2\right] \\ & - \frac{(1-a)(1-2a)(1-3a)}{18a^3}\left[1 - \left(\frac{\nu}{\nu_0}\right)^3\right] \\ & + \frac{(1-a)(1-2a)(1-3a)(1-4a)}{96a^4}\left[1 - \left(\frac{\nu}{\nu_0}\right)^4\right] \Bigg\}. \end{aligned} \tag{14}$$

The above expresses the energy, lying between a frequency interval ν and $\nu + d\nu$, radiated by N electrons upon their impact with the target of the X-ray tube. The frequency ν_0 is the end frequency or the highest frequency, and is that emitted by those electrons that make impact at the poles (or normal impact) with the first layer of atoms in the surface of the target.

It is more convenient for comparison with experimental results to express equation (14) in terms of the wave-lengths.

$$\begin{aligned} w = kN\frac{R}{a}\frac{1}{\lambda_0^3}\left(\frac{\lambda_0}{\lambda}\right)^4 d\lambda \Bigg\{ & \log\frac{\lambda}{\lambda_0} - \frac{1-a}{a}\left[1 - \frac{\lambda_0}{\lambda}\right] \\ & + \frac{(1-a)(1-2a)}{4a^2}\left[1 - \left(\frac{\lambda_0}{\lambda}\right)^2\right] \\ & - \frac{(1-a)(1-2a)(1-3a)}{18a^3}\left[1 - \left(\frac{\lambda_0}{\lambda}\right)^3\right] \\ & + \frac{(1-a)(1-2a)(1-3a)(1-4a)}{96a^4}\left[1 - \left(\frac{\lambda_0}{\lambda}\right)^4\right] \Bigg\}, \end{aligned} \tag{15}$$

Equation (15) is compared to the experimental results in Figs. 3, 4 and 5 where the broken line represents the plot of the equation. Fig. 3

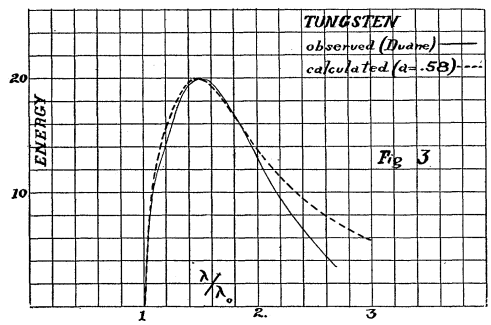

Fig. 3.

(the full line) represents the experimental results obtained by Duane and Hunt for a Coolidge tube with a tungsten target.

Figs. 4 and 5 represent the results for tungsten and molybdenum tar-

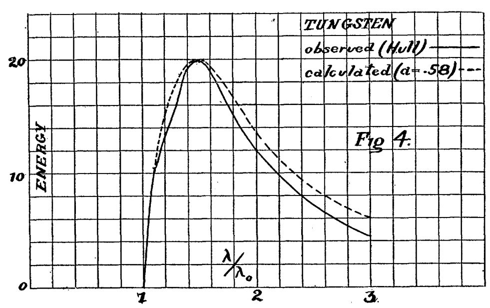

Fig. 4.

gets, kindly furnished me by Dr. A. W. Hull. In all of these curves the abscissæ represent the ratio of any wave-length to the shortest or end wave-length. I believe that none of these curves have been corrected

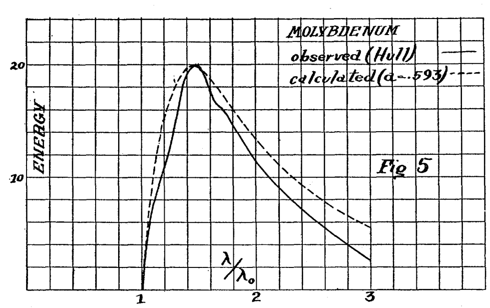

Fig. 5.

for the variation of absorption with the wave-length nor for the dependence of the reflectivity of the crystal upon wave-length.

In calculating equation (15) such a value of a was chosen as would make the maximum of the calculated curve coincide with that of the observed curve. It will be observed in Figs. 1, 4 and 5 that the maximum does not occur at the same ratio of λ/λ_0 for tungsten and molybdenum, but that this ratio is less for the element of smaller atomic number. The constant a also possesses the property of controlling the position of the maximum of the calculated curve. The agreement between theory and experiment is perhaps as good as might be expected, since the assumed law of loss of velocity of the electrons as they penetrate the target can only approximately represent the actual state of the case.

Relation Between the Energy Radiated at Constant Wave-Length and the Voltage Applied to the X-ray Tube.

Professor Duane and F. L. Hunt[1] also investigated the radiation emitted at a *constant wave-length* when the voltage applied to the X-ray tube was varied. The angle of reflection from the crystal was kept constant and the applied voltage was increased by steps. No radiation was observed at voltages less than that given by the relation

$$Pe = h\nu = \frac{hc}{\lambda},$$

but the radiation increases very rapidly with increase of voltage.

The theory here proposed (equations 14 and 15) also expresses a relation between the voltage and the radiation at constant wave-length. The variable in the energy equation in this case is ν_0 while ν remains constant. The constant frequency ν may be expressed by

$$Pe = h\nu,$$

while the variable frequency ν_0 is connected to the variable voltage P_0 by the relation

$$P_0e = h\nu_0.$$

From these we obtain

$$\frac{\nu}{\nu_0} = \frac{P}{P_0}.$$

If P/P_0 is substituted for ν/ν_0 in equation (14) and for ν_0^2 we write $(e/h)^2P_0^2$, the equation expresses the energy emitted at constant frequency as a function of the applied voltage. A plot of equation (14) so modified is shown in Fig. 6. The theory indicates that the energy in-

[1] Proc. Amer. Phys. Society, Phys. Rev., August, 1915.

creases rapidly with increasing voltage, but attains a maximum value. This maximum was not observed by Duane and Hunt since their experiments did not extend over a sufficient range of voltage.

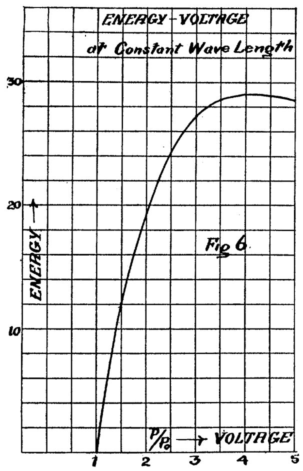

Fig. 6.

The Total Energy Radiated at an Applied Voltage P_0.

The total energy radiated at all wavelengths by an X-ray tube when operated at a voltage P_0 may be found by taking the total area under the curves (Figs. 3, 4, 5). This may be readily accomplished by the integration of equation (14) between the limits $\nu = 0$ and $\nu = \nu_0$.

For a given value of the constant a the terms within the bracket contribute only a numerical constant to the valuation of the integral.

Let J represent this numerical constant.

The integration of (14) gives

$$W = JkN\frac{R}{a}\nu_0^2. \tag{18}$$

Since

$$P_0 e = h\nu_0^2,$$

this becomes

$$W = \frac{k}{h^2}(JNe^2)\frac{R}{a}P_0^2. \tag{19}$$

The total energy radiated in the continuous spectrum (not including the characteristic) is proportional to the *square of the applied voltage*.

This result agrees with the experimental results of Beatty[1] who found that the total energy radiated was proportional to the atomic weight of the target and to the square of the voltage operating the X-ray tube.

In the development on the basis of the quantum hypothesis

$$k = Ch^2.$$

Let the constants JNe^2 be represented by K.

Equation (19) becomes

$$W = KC\frac{R}{a}P_0^2. \tag{20}$$

The range R, that is, the number of collisions of the electrons before losing their energy may be expected to depend on the atomic weight of the element composing the target.

[1] Proc. Roy. Soc., vol. 89, 1913.

Since the constant a varies but little with the atomic weight, one may place a/R proportional to the atomic weight of the target. Also the constant C expresses the probability of an electron making the kind of impact that is effective in producing radiation. This probability may be expected to depend in some way on the atomic weight. It will be assumed that C is proportional to the square of the atomic weight. Equation (20) becomes

$$W_1 = KAP_0^2. \tag{21}$$

where A represents the atomic weight of the element composing the target. This interpretation of the constants brings the theoretical equation in agreement with the results obtained by Beatty for the dependence of the energy upon the atomic weight.

The above result is obtained from the equation developed on the quantum hypothesis. The equation derived on the pulse hypothesis does not contain a natural constant that one may reasonably set proportional to the square of the atomic weight.

I wish to express my acknowledgment to my colleague, Professor A. P. Wills, for valuable assistance and discussion.

Phœnix Physical Laboratory,
Columbia University,
November, 1916.