because dx^a changes to 100 dx^a, we wish to regard the quantities dx^a as lengths, we still have not made a change of unit. The change of unit will cause g_{ab} to be replaced by $100^2\, g_{ab}$, and this will exactly compensate the 100^{-2} factor caused by the change of coordinates, leaving the components of the new metrical tensor numerically equal to those of the old. In this sense g_{ab} does behave like a set of pure numbers. But this particular effect is due to our having combined the change of units with a corresponding change of coordinates.[4] Actually the two sets of numbers g_{ab} that have equal numerical values belong to different metrical tensors and to different coordinate systems. If whenever we make a change of units we make this corresponding change of coordinates, we can think of dx^a as lengths and g_{ab} as pure numbers, ds being a length. But there is no need to make this coordinate change, and when we refrain from making it we see that the dx^a are pure numbers, the g_{ab} have the dimensions $(\text{length})^2$, and ds is a length.

[4] This is precisely the situation in Cartesian vector analysis. The transformation of coordinates given by (3) changes the components of the metrical tensor from δ_{ab} to $100^{-2}\delta_{ab}$. But the original metrical tensor δ_{ab} is changed to a new one $100^2\delta_{ab}$, and thus the final coefficients of the quadratic form for ds^2 are the same as the original.

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The Similarity Theory of Relativity and the Dirac-Schrödinger Theory of Electrons

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The similarity theory of relativity is shown to contain the recent Dirac-Schrödinger theory of electrons as a special case. The variational principle is built on the curvature scalar formed from the basic symmetric similarity tensor $S_{\sigma\tau}$, and leads to a unified field theory of the gravitational field, the electromagnetic field, a scalar field of the sort used by Jordan in his theory of the variable gravitational constant, a charged scalar particle field, and a vector particle field.

I. THE DIRAC-SCHRÖDINGER EQUATIONS

RECENTLY Dirac has proposed[1] a new classical theory of electrons in which the usual Maxwell equations for free space are modified by a simple nonlinear condition on the electromagnetic potential four-vector A_a, which fixes the gauge.

Let $a, b, \cdots$ take on the range 1, 2, 3, 4, with 4 referring to the time. Also, let the metrical tensor g_{ab} have the signature $+++-$ (instead of the $---+$ used by Dirac). Then in the earlier form of Dirac's theory the nonlinear gauge-fixing condition may be written

$$A_aA^a=-m^2c^4/e^2, \tag{1}$$

where m and e are, respectively, the mass and charge of an electron. Later Dirac modified this, but the present paper has to do with the theory based on (1).

The Maxwell equations for free space can be derived from the variational principle

$$\delta\int \mathfrak{L}(-g)^{\frac{1}{2}}\, dx^1dx^2dx^3dx^4=0, \tag{2}$$

where

$$\mathfrak{L}=-\tfrac{1}{4}F_{ab}F^{ab}=\tfrac{1}{4}F^a{}_bF^b{}_a, \tag{3}$$

$$F_{ab}=\partial A_a/\partial x^b-\partial A_b/\partial x^a, \tag{4}$$

and

$$F^a{}_b=g^{ac}F_{cb}. \tag{5}$$

In Dirac's theory the restriction (1) is taken care of by replacing (3) by

$$\mathfrak{L}=\tfrac{1}{4}F^a{}_bF^b{}_a-\tfrac{1}{2}\lambda(A_aA^a+m^2c^4/e^2), \tag{6}$$

where λ is an unknown function. This leads to field equations consisting of (1) together with

$$F^{ab}{}_{,b}=j^a, \tag{7}$$

where

$$j_a=\lambda A_a, \tag{8}$$

and, as a consequence of (7),

$$j^a{}_{,a}=0, \tag{9}$$

the comma denoting the covariant derivative with respect to g_{ab}. Thus a current four-vector j^a enters automatically, and Dirac showed that, without any further assumptions either as to the law of force or the structure of the electron, these equations correctly gave the nonvortical motion of beams of charged particles having m/e of electronic value.

The Dirac equations belonged to the special theory of relativity. They have been written here in general tensor form.

Schrödinger discovered[2] that these equations of Dirac

[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A209**, 291 (1951); Nature **168**, 906 (1951); Proc. Roy. Soc. (London) **A212**, 330 (1952).

[2] E. Schrödinger, Nature **169**, 538 (1952).

could be regarded as a limiting case of the Maxwell equations combined with the relativistic quantum-mechanical Gordon-Klein-Schrödinger wave equation for a scalar wave function representing charges. This wave equation can be written

$$g^{ab}\left(\partial_a-\frac{ie}{hc}A_a\right)\left(\partial_b-\frac{ie}{hc}A_b\right)\psi=\frac{m^2c^2}{h^2}\psi, \tag{10}$$

where ∂_a denotes the covariant derivative operator and h is Planck's constant divided by 2π.

From (10) one obtains the following expression for the current four-vector

$$j_a=\frac{-ie}{2hc}(\psi^*\psi_{,a}-\psi\psi^*_{,a})-\frac{e}{hc}A_a\psi\psi^*. \tag{11}$$

If, instead of the Dirac expression (8), one uses this value of j_a in (7), one has a theory of electric charges in an electromagnetic field in which the charges are represented by the quantum-mechanical scalar wave function ψ.

While Eqs. (7) and (11), are real, Eq. (10) is complex. Moreover, unlike the Dirac equations, the set (7), (10), (11) is gauge invariant and thus permits changes of gauge. Schrödinger discovered a particular choice of gauge that brings about a remarkable simplification. When this particular gauge is used, Eqs. (7), (10), (11) may be replaced by the real equations

$$F^{ab}{}_{,b}=j^a, \tag{7}$$

$$j_a=-(e/hc)A_a\varphi^2, \tag{12}$$

and

$$g^{ab}\varphi_{,ab}=\left(\frac{e^2}{h^2c^2}A_aA^a+\frac{m^2c^2}{h^2}\right)\varphi, \tag{13}$$

in which only real quantities occur.

Equations (7) and (12) correspond to Dirac's Eqs. (7) and (8), with $-(e/hc)\varphi^2$ corresponding to Dirac's λ. Further, Schrödinger pointed out that, if one assumes that the wave amplitude φ does not vary appreciably in a distance comparable to the Compton wavelength of an electron, the left-hand side of (13) may be neglected and the equation then reduces essentially to the Dirac gauge-fixing Eq. (1).

The set of Schrödinger Eqs. (7), (12), (13) can be obtained from the variational principle (2) with

$$\mathfrak{L}=\tfrac{1}{4}F^a{}_bF^b{}_a+g^{ab}\varphi_{,a}\varphi_{,b}+\frac{e^2}{h^2}\left(A^aA_a+\frac{m^2c^4}{e^2}\right)\varphi^2. \tag{14}$$

II. INADEQUACY OF THE PROJECTIVE THEORY

The fact that the Dirac Eqs. (1), (7), (8) are essentially five equations for the five field variables A_a, λ suggests a possible connection with the projective theory of relativity. Such a connection was found,[3] but in the projective theory there was no scalar field quantity directly corresponding to Dirac's field quantity λ.

The Schrödinger Eqs. (7), (12), (13) are also five equations for five field variables, namely, A_a and φ. And the possibility that these equations may belong to the projective theory of relativity is strengthened by the well-known fact[4] that the Gordon-Klein-Schrödinger scalar wave Eq. (10), as also the expression (11) for the current four-vector, may be obtained from a Lagrangian density

$$\mathfrak{L}=\gamma^{\alpha\beta}\psi_{,\alpha}\psi^*_{,\beta}, \tag{15}$$

where α, β have the range 0, 1, 2, 3, 4 (with 0 referring to the gauge variable), $\gamma_{\alpha\beta}$ is the fundamental projective tensor of zero index,[5] and ψ is a projective scalar of index different from zero.

In the projective theory of relativity one starts out with a symmetric projective tensor $G_{\alpha\beta}$ of index $2N$. This has fifteen components, and from these one can construct a symmetric affine tensor g_{ab}, a projective vector φ_α of index zero and having $\varphi_0=1$, and a projective scalar Φ of index N. The quantities g_{ab}, φ_a have usually been identified, respectively, with the gravitational and electromagnetic potentials, the φ_a acquiring an added gradient under a gauge transformation. Thus one could expect that the projective scalar Φ might be identified with the projective scalar ψ in (15), and that one could construct out of $G_{\alpha\beta}$ a unified field theory of the gravitational, electromagnetic, and charge-scalar fields.

This, however, is not possible in a projective theory of the sort under discussion. For in such a theory $G_{\alpha\beta}$ is not of zero index and so only the ratios of its components are significant, and not the components themselves. This has to do with the nature of a projective theory and corresponds to the use of homogeneous coordinates in projective geometry. It is taken care of in the formalism by just the device of the index and the corresponding gauge transformations. Consequently, of the fifteen components of $G_{\alpha\beta}$ only fourteen are independent and there is not a fifteenth quantity available to play the part of the projective scalar ψ in (15). In field equations built in terms of the projective curvature tensor formed from $G_{\alpha\beta}$, the projective scalar Φ and the projective vector φ_α enter only in the gauge invariant combination

$$\theta_a\equiv\varphi_a-\frac{1}{N}\frac{\partial\log\Phi}{\partial x^a}. \tag{16}$$

Thus the five quantities Φ, φ_a occur only in the four combinations θ_a.

If $G_{\alpha\beta}$ were of zero index it would have fifteen inde-

[3] B. Hoffmann, Phys. Rev. **87**, 703 (1952).

[4] O. Klein, Z. Physik. **37**, 895 (1926); **46**, 188 (1927).

[5] The gauge variable x^0 enters the components of a projective tensor only through a factor of the form $\exp(Kx^0)$ that is common to all the components. The constant K is called the index. Thus a projective tensor of zero index does not contain x^0. For further details of the projective formalism and references to the previous literature see the first paper listed in reference 7.

pendent components. But then the projective scalar Φ would be of zero index and so would not be suitable for the role of ψ in (15).

III. THE SIMILARITY THEORY

The principle of similarity discussed in the preceding paper[6] gives reasons for replacing the usual Riemannian geometry of the general theory of relativity by some form of general similarity geometry, so that the similarity transformations

$$g_{ab}\rightarrow\bar{g}_{ab}=k^2g_{ab}, \tag{17}$$

where k is independent of the coordinates of space-time, will be reflected in the underlying geometrical structure.

In previous papers[7] I developed the similarity geometry as a special case of the general conformal geometry. The latter is characterized by transformations of the type

$$g_{ab}\rightarrow\bar{g}_{ab}=\rho^2g_{ab}, \tag{18}$$

where ρ is any nonzero function of position. By specializing to the case where ρ is independent of the coordinates and is thus only a numerical parameter k, we obtain the similarity transformations (17).

There is an extensive literature dealing with the general conformal geometry. The classical conformal group in n dimensions has long been known to be related to the rotation group in $n+2$ dimensions, and the geometers have found that the appropriate mathematical tool for the study of the general conformal geometry is a conformal tensor calculus which employs entities $T_{\tau\cdots}{}^{\sigma\cdots}$, called conformal tensors, having $(n+2)^r$ components, where r is the number of indices. Thus in space-time, conformal tensors have 6^r components rather than the 4^r components of affine tensors or the 5^r components of projective tensors. The same is true of the similarity tensors in II.

There are many indications, in spinor theory, meson theory, and elsewhere, that entities having 6^r components are of significance for theoretical physics. Some are mentioned in the papers in reference 7. Schouten[8] has discussed this matter in detail from the point of view of conformal geometry, and his arguments equally support the similarity geometry used here, which may have a more plausible physical justification.

Weyl's theory, though conformal, did not employ the full resources of the general conformal geometry since it used tensors having only 4^r components. When one goes over to the general conformal geometry, the extra degrees of freedom release one from the necessity of bringing in the electromagnetic potentials as measures of a non-holonomy of length in parallel displacement. This removes a major physical difficulty in Weyl's theory. But there still remains the problem of finding an initial physical justification for the use of conformal geometry. The preceding paper gives physical reasons for the use of similarity transformations, but these reasons do not seem to extend beyond that. In the papers in reference 7 I was under the impression that one has to seek an affine metrical tensor g_{ab}. The preceding paper shows that this is not desirable, and the metrical tensor defined later in the present paper is one that undergoes similarity transformations. It is worth remarking that one can easily construct affine metrical tensors by simply dividing the metrical tensor used here by a suitable similarity scalar. This corresponds physically to fixing a particular unit of interval with reference to the physical quantity represented by the particular similarity scalar used in normalizing the metric. It thus corresponds to picking a "fundamental length" as the unit.

Let Greek letters σ, τ, $\cdots$ from the later part of the alphabet have the range 0, 1, 2, 3, 4, 5, where 0 refers to the gauge variable, 1, 2, 3 to space, 4 to time, and 5 to a further variable associated with the ∞^1 transformations of the type (17). The basic definitions are given in II, and we repeat here merely that the variables x^σ transform as

$$\begin{aligned} x^0&\rightarrow\bar{x}^0=x^0+N^{-1}\log k,\\ x^a&\rightarrow\bar{x}^a=\bar{x}^a(x^b),\\ x^5&\rightarrow\bar{x}^5=k^{-2}x^5, \end{aligned} \tag{19}$$

and that the field equations in II were built up from a symmetric second rank similarity tensor $S_{\sigma\tau}$ of index $2N$. Like all similarity tensors, $S_{\sigma\tau}$ is independent of x^5. It contains x^0 only in a multiplying factor $\exp(2Nx^0)$.

In II, for the sake of simplicity, the basic similarity tensor $S_{\sigma\tau}$ was subjected to the restrictions[9] that

$$S_{05}=0 \tag{20}$$

and that

$$S_{55}/S_{00} \text{ be independent of } x^a. \tag{21}$$

The field equations built on the curvature scalar formed from this restricted $S_{\sigma\tau}$ were found to give a unified field theory of the gravitational, electromagnetic, and vector meson fields.

In the present paper these restrictions on $S_{\sigma\tau}$ are removed, so that $S_{\sigma\tau}$ is here taken to be a general symmetric similarity tensor of index $2N$. The physical identifications of various combinations of the components of $S_{\sigma\tau}$ that are made in the present paper correspond to those made in II, but minor adjustments are made in the light of the paper in reference 6.

The curvature scalar Z formed from the general symmetric similarity tensor $S_{\sigma\tau}$ is computed in the appendix to this paper, the result being given in (99). In the course of the computation quantities g_{ab}, ψ_a, θ_a, φ, ω are defined, and Z is expressed in terms of them. Under coordinate transformations these quantities behave like

[6] B. Hoffmann, Phys. Rev. **89**, 49 (1953).

[7] B. Hoffmann, Phys. Rev. **72**, 458 (1947); **73**, 30 (1948); **73**, 1042 (1948). References to the previous literature given in these papers need not be repeated here. The second of these papers will here be referred to as II.

[8] J. A. Schouten, Revs. Modern Phys. **21**, 421 (1949).

[9] II, Eqs. (20) and (21). [Equation (22) in II contains a misprint. It should read $s_{\sigma\tau}=S_{\sigma\tau}/S_{00}$.]

ordinary affine tensors of ranks corresponding to their indices. Under similarity transformations they behave as follows:

$$g_{ab}\to k^2 g_{ab},\quad \psi_a\to k^2\psi_a,\quad \theta_a\to\theta_a,\qquad \varphi\to\varphi/k^2,\quad \omega\to\omega/k^2. \tag{22}$$

Since this is a four-dimensional theory, we consider a four-dimensional variational principle. The curvature scalar Z is independent of x^5, but contains x^0 in the initial factor Ψ^{-2}. So if we write

$$\bar{Z}=\Psi^2 Z, \tag{23}$$

$\bar{Z}$ will be independent of both x^5 and x^0.

The quantity $\bar{Z}$ contains the curvature scalar R formed from the g_{ab} in the term R/ω. In the general theory of relativity the Lagrangian density of the gravitational field is R. This suggests that the appropriate variational principle to use here is

$$\delta\int \omega\bar{Z}(-g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4=0. \tag{24}$$

If we use this variational principle, however, we encounter difficulties in assigning numerical values to certain quantities. To avoid these difficulties it is necessary to alter the coefficient of R and to alter or remove the term $20N^2\omega$. To alter the coefficient of R we may add a term $(\alpha-1)R(-g)^{\frac{1}{2}}$ to the integrand, where α is a constant; and to remove the term $20N^2\omega$ we can simply subtract a term $20N^2\omega(-g)^{\frac{1}{2}}$ from the integrand. Thus, as variational principle, we take

$$\delta\int \{\omega(\bar{Z}-20N^2)+(\alpha-1)R\}(-g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4=0, \tag{25}$$

where α is a numerical constant.

From (99) we see that (25) is the same as

$$\begin{aligned}\delta\int\Bigg[&\alpha R+\frac{2\omega}{(-g)^{\frac{1}{2}}}\frac{\partial}{\partial x^a}\left(\frac{1}{\omega}g^{ab}(-g)^{\frac{1}{2}}\frac{\partial\log\omega}{\partial x^b}\right)\\
&+18g^{ab}\frac{\partial\log\omega}{\partial x^a}\frac{\partial\log\omega}{\partial x^b}-\omega\psi^a{}_b\psi^b{}_a\\
&-\frac{1}{\omega}\left(\theta^a{}_b\theta^b{}_a+g^{bd}\psi_a\theta^a{}_b\frac{\partial\varphi}{\partial x^d}\right)\\
&+\frac{1}{2\omega^2}g^{ab}\frac{\partial\varphi}{\partial x^a}\frac{\partial\varphi}{\partial x^b}\\
&+\frac{10N}{\omega(-g)^{\frac{1}{2}}}\frac{\partial}{\partial x^a}\{\omega g^{ab}(-g)^{\frac{1}{2}}(\psi_b\varphi-\theta_b)\}\\
&+20N^2\{g^{ab}(\theta_a-\psi_a\varphi)(\theta_b-\psi_b\varphi)+\varphi^2/\omega\}\Bigg]\\
&\times(-g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4=0.\end{aligned} \tag{26}$$

Let the metric be so chosen that the centimeter is the unit of length. Let c be such that the second is the unit of time. And let the gravitational constant be chosen so that the unit of mass is the gram.

Consider first the case where g_{ab} alone is significant; that is, where ω is constant and θ_a, ψ_a, φ are all zero. The variational principle then reduces to

$$\delta\int R(-g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4=0,$$

which is that belonging to the general theory of relativity. This justifies the identification of g_{ab} with the gravitational-metrical tensor of Einstein's theory.

Next, keeping ω constant, but only θ_a and φ zero, we have

$$\delta\int(\alpha R-\omega\psi^a{}_b\psi^b{}_a)(-g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4=0. \tag{27}$$

This is the variational principle for the gravitational and electromagnetic[10] fields in the general theory of relativity with the electromagnetic potential four-vector given by

$$\psi_a=\frac{1}{c}\left(\frac{2\kappa\alpha}{\omega}\right)^{\frac{1}{2}}A_a, \tag{28}$$

where κ is the Einstein gravitational constant,

$$\kappa=8\pi\gamma/c^2, \tag{29}$$

and γ is the Newtonian gravitational constant.

When ω is allowed to be a function of x^a instead of being kept constant, we have further terms to consider in the variational principle. The significant point here is that, by (28), we have, in ω, a "variable gravitational constant" such as has been discussed by Jordan.[11] Some of the additional terms in the variational principle are analogous to those in Jordan's theory, so that his results may be applicable to the present theory.

Let us now go back to the case where ω is constant, and allow g_{ab}, ψ_a, and φ to enter, keeping θ_a zero. Then we have

$$\begin{aligned}\delta\int\Bigg\{&\alpha R-\omega\psi^a{}_b\psi^b{}_a+\frac{1}{2\omega^2}g^{ab}\frac{\partial\varphi}{\partial x^a}\frac{\partial\varphi}{\partial x^b}\\
&+20N^2(1/\omega+g^{ab}\psi_a\psi_b)\varphi^2\Bigg\}(-g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4=0,\end{aligned} \tag{30}$$

[10] In Eq. (79) the quantities ψ_{ab} are defined with a $\frac{1}{2}$ factor, so the $\frac{1}{4}$ factor in (3) is already included here.

[11] P. Jordan, Ann. Physik **I**, 15 (1947); Astron. Nachr. **276**, 13 (1948); *Schwerkraft und Weltall* (Vieweg, Braunschweig, 1952); G. Ludwig and C. Müller, Ann. Physik **2**, 76 (1948); Heckmann, Jordan, and Fricke, Astrophys. J. **28**, 113 (1951); G. Ludwig, *Fortschritte der Projectiven Relativitätstheorie* (Vieweg, Braunschweig, 1951); see also P. Bergmann, Ann. Math. **49**, 255 (1946).

where we have omitted the term

$$\frac{10N}{\omega(-g)^{\frac{1}{2}}}\frac{\partial}{\partial x^a}\{\omega g^{ab}(-g)^{\frac{1}{2}}(\psi_b\varphi-\theta_b)\}(-g)^{\frac{1}{2}},$$

which is a divergence when ω is constant and so contributes nothing to the field equations in the present case. This is precisely the Schrödinger variational principle, based on the Lagrangian density (14), extended to general relativity. Specifically, if Schrödinger's φ is denoted for the moment by $\bar{\varphi}$, we have

$$\varphi^2/2\alpha\omega^2=2\kappa h^2\bar{\varphi}^2/e^2, \tag{31}$$

$$20N^2\varphi^2/\alpha\omega=2\kappa m^2c^2\bar{\varphi}^2/e^2, \tag{32}$$

and

$$(20N^2/\alpha)\psi^a\psi_a\varphi^2=(2\kappa/c^2)A^aA_a\bar{\varphi}^2. \tag{33}$$

From these we have

$$\omega\psi^a\psi_a=(e^2/m^2c^4)A^aA_a,$$

or, by (28),

$$\alpha=e^2/2\kappa m^2c^2, \tag{34}$$

α being a pure number.

If m is the electronic mass and e the electronic charge, α is of the order 10^{41}.

Finally, if we take g_{ab}, θ_a to be variable, ω constant, and ψ_a, φ zero, we have

$$\delta\int\left\{\alpha R-\frac{1}{\omega}\theta^a{}_b\theta^b{}_a+20N^2\theta^a\theta_a\right\}\times(-g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4=0, \tag{35}$$

which corresponds to a combined gravitational and vector particle field, the latter having the same mass as the scalar field φ above.

When all quantities g_{ab}, θ_a, ψ_a, φ, ω are allowed to enter without restriction, various interaction terms appear linking these various fields.

We see, then, that the present theory contains the Dirac-Schrödinger equations as a special case. Since it has been built up in terms of tensors rather than spinors, there has been no possibility of its containing particles of spin $\frac{1}{2}$, so that electrons of the Dirac spinor type were excluded from the start. This raises a question as to the value of the mass m. If we take it to be the electronic mass, we have a unified field theory of the gravitational and electromagnetic fields containing scalar electrons of the sort described by the Dirac-Schrödinger equations; and alongside the scalar electron field there appears a vector particle field having the same mass but interacting differently with the Maxwellian part of the field. If we take m to be the mass of some meson, we have a unified theory of the gravitational, electromagnetic, vector meson, and charged scalar meson fields, the charge of the latter being taken equal to the electronic charge. And in either case we have in addition the further scalar field ω which belongs to the theory of Jordan.

APPENDIX

In this appendix we compute the value of the curvature scalar formed from the fundamental symmetric similarity tensor $S_{\sigma\tau}$. The aim of the computation is to express the curvature scalar in terms of certain combinations of the components $S_{\sigma\tau}$ that can be given immediate physical significance, and the mathematical definition of these combinations of the components goes hand in hand with the computation of the curvature scalar.

It is doubtful that the present computations could have been carried through without guidance from the analogous, though simpler, computations that have to be performed in the projective theory of relativity. The present computations are performed by the successive application of results known from the projective theory, and these will be stated shortly.

It is necessary to employ indices having various different ranges. The following conventions will therefore be employed consistently, and will be restated individually when first used:

Indices σ, τ, $\cdots$ from the middle of the Greek alphabet will have the range 0, 1, 2, 3, 4, 5.

Indices α, β, $\cdots$ from the beginning of the Greek alphabet will have the range 0, 1, 2, 3, 4 (as in projective relativity).

Indices p, q, $\cdots$ from the middle of the English alphabet will have the range 1, 2, 3, 4, 5.

Indices a, b, $\cdots$ from the beginning of the English alphabet will have the range 1, 2, 3, 4 (referring to space and time).

The index 4 will refer to time, and indices 1, 2, 3, to space. The index 0 will refer to the gauge, and the index 5 will correspond to the extra variable associated with the similarity transformations.

The components $S_{\sigma\tau}$ are independent of x^5 and contain x^0 only in the form of a common multiplying factor $\exp(2Nx^0)$, where N is a constant. Thus $S_{\sigma\tau}$ has the form $\exp(2Nx^0)f_{\sigma\tau}(x^1, x^2, x^3, x^4)$.

In the projective theory of relativity[12] one starts out from a symmetric projective tensor $G_{\alpha\beta}(\alpha, \beta=0, 1, 2, 3, 4)$ of the form $\exp(2Nx^0)f_{\alpha\beta}(x^1, x^2, x^3, x^4)$ and defines $\gamma_{\alpha\beta}$ by

$$\gamma_{\alpha\beta}=G_{\alpha\beta}/G_{00}, \tag{36}$$

so that $\gamma_{\alpha\beta}$ is independent of x^0 and

$$\gamma_{00}=1. \tag{37}$$

Then one writes

$$\varphi_\alpha=\gamma_{\alpha 0} \tag{38}$$

and

$$g_{\alpha\beta}=\gamma_{\alpha\beta}-\varphi_\alpha\varphi_\beta. \tag{39}$$

Since

$$g_{\alpha 0}\equiv 0, \tag{40}$$

the law of transformation shows that $g_{ab}(a, b=1, 2, 3, 4)$

[12] For references regarding the following details of the projective formalism, see II.

is an affine tensor; it is identified with the gravitational-metrical tensor of Einstein's general theory of relativity.

From (39) one has

$$\gamma_{\alpha\beta}=g_{\alpha\beta}+\varphi_\alpha\varphi_\beta. \tag{41}$$

Defining $\gamma^{\alpha\beta}$ by

$$\gamma^{\alpha\beta}\gamma_{\beta\gamma}=\delta^\alpha{}_\gamma, \tag{42}$$

one obtains, after a little computation,

$$\gamma^{ab}=g^{ab},\quad \gamma^{a0}=-g^{ab}\varphi_b,\quad \gamma^{00}=1+g^{ab}\varphi_a\varphi_b. \tag{43}$$

Also one finds that

$$\gamma=g, \tag{44}$$

where γ, g are, respectively, the determinants of $\gamma_{\alpha\beta}$ and g_{ab}.

The curvature scalar formed from $\gamma_{\alpha\beta}$ is denoted by B and turns out to have the form

$$B=R-\varphi^a{}_b\varphi^b{}_a, \tag{45}$$

where R is the curvature scalar formed from g_{ab},

$$\varphi^a{}_b=g^{ac}\varphi_{cb}, \tag{46}$$

and

$$\varphi_{cb}=\tfrac{1}{2}(\partial\varphi_c/\partial x^b-\partial\varphi_b/\partial x^c). \tag{47}$$

Finally, the curvature scalar formed from $G_{\alpha\beta}$ is denoted by $\bar{P}$ and turns out to have the form

$$\bar{P}=\Phi^{-2}\{B+2n\Phi^\alpha{}_{;\alpha}+n(n-1)\Phi^\alpha\Phi_\alpha\}, \tag{48}$$

where n is one less than the number of values that the indices α, β can take (and thus, in projective relativity $n=4$), the semicolon denotes the covariant derivative with respect to $\gamma_{\alpha\beta}$, indices are raised by means of $\gamma^{\alpha\beta}$, Φ is defined by

$$\Phi^2=G_{00}, \tag{49}$$

and Φ_α by

$$\Phi_\alpha=\partial\log\Phi/\partial x^\alpha. \tag{50}$$

We apply these various results, suitably modified, to the present problem of computing the curvature scalar formed from $S_{\sigma\tau}$.

We begin by writing

$$s_{\sigma\tau}=S_{\sigma\tau}/S_{00}, \tag{51}$$

so that $s_{\sigma\tau}$ is independent of x^0 (and, of course, of x^5 since $S_{\sigma\tau}$ is independent of x^5), and

$$s_{00}=1. \tag{52}$$

Denote the curvature scalar of $S_{\sigma\tau}$ by Z and that of $s_{\sigma\tau}$ by z. Then, from comparison with (36), (37), (48), (49), (50), we have (taking $n=5$ in (48))

$$Z=\Psi^{-2}\{z+10\Psi^\sigma{}_{/\sigma}+20\Psi^\sigma\Psi_\sigma\}, \tag{53}$$

where the solidus "/" denotes the covariant derivative with respect to $s_{\sigma\tau}$, indices are raised by $s^{\sigma\tau}$,

$$s^{\sigma\tau}s_{\tau\pi}=\delta^\sigma{}_\pi, \tag{54}$$

$$\Psi^2=S_{00}, \tag{55}$$

and

$$\Psi_\sigma=\partial\log\Psi/\partial x^\sigma. \tag{56}$$

To calculate z, the curvature scalar of $s_{\sigma\tau}$, we write

$$\varphi_\sigma=s_{\sigma0}, \tag{57}$$

and then define $w_{\sigma\tau}$ by

$$w_{\sigma\tau}=s_{\sigma\tau}-s_{\sigma0}s_{\tau0}=s_{\sigma\tau}-\varphi_\sigma\varphi_\tau. \tag{58}$$

Thus

$$w_{\sigma0}\equiv0. \tag{59}$$

By (58)

$$s_{\sigma\tau}=w_{\sigma\tau}+\varphi_\sigma\varphi_\tau. \tag{60}$$

By comparison with (38) to (43) we see that if p, q, $\cdots$ have the range 1, 2, 3, 4, 5,

$$s^{pq}=w^{pq},\quad s^{p0}=-w^{pq}\varphi_q,\quad s^{00}=1+w^{pq}\varphi_p\varphi_q, \tag{61}$$

where

$$w^{pq}w_{qr}=\delta^p{}_r. \tag{62}$$

Also, by comparison with (44), we have

$$w=s, \tag{63}$$

where w, s are, respectively, the determinants of w_{pq} and $s_{\sigma\tau}$. And by comparison with (45) we have

$$z=W-\varphi^p{}_q\varphi^q{}_p, \tag{64}$$

where W is the curvature scalar of w_{pq},

$$\varphi^p{}_q=w^{pr}\varphi_{rq}, \tag{65}$$

and

$$\varphi_{rq}=\tfrac{1}{2}(\partial\varphi_r/\partial x^q-\partial\varphi_q/\partial x^r). \tag{66}$$

Next we have to compute W, the curvature scalar of w_{pq}. To do this we follow again the pattern of equations (38) to (48). Thus we write

$$w_{55}=\omega^2 \tag{67}$$

and

$$\bar{w}_{pq}=w_{pq}/\omega^2 \tag{68}$$

so that

$$\bar{w}_{55}=1. \tag{69}$$

Then we write

$$\psi_p=\bar{w}_{p5} \tag{70}$$

and

$$\bar{g}_{pq}=\bar{w}_{pq}-\psi_p\psi_q, \tag{71}$$

so that

$$\bar{g}_{p5}=0. \tag{72}$$

Then

$$\bar{w}_{pq}=\bar{g}_{pq}+\psi_p\psi_q, \tag{73}$$

and, defining $\bar{w}^{pq}$ by

$$\bar{w}^{pq}\bar{w}_{qr}=\delta^p{}_r, \tag{74}$$

we have

$$\bar{w}^{ab}=\bar{g}^{ab}, \quad \bar{w}^{a5}=-\bar{g}^{ab}\psi_b, \quad \bar{w}^{55}=1+\bar{g}^{ab}\psi_a\psi_b \tag{75}$$

and

$$\bar{w}=\bar{g}, \tag{76}$$

where $\bar{w}$, $\bar{g}$ are, respectively, the determinants of $\bar{w}_{pq}$ and $\bar{g}_{ab}$.

The curvature scalar $\bar{W}$ formed from $\bar{w}_{pq}$ is then

$$\bar{W}=\bar{R}-\bar{\psi}^a{}_b\bar{\psi}^b{}_a, \tag{77}$$

where $\bar{R}$ is the curvature scalar formed from $\bar{g}_{ab}$,

$$\bar{\psi}^a{}_b=\bar{g}^{ac}\psi_{cb}, \tag{78}$$

and

$$\psi_{cb}=\tfrac{1}{2}(\partial\psi_c/\partial x^b-\partial\psi_b/\partial x^c). \tag{79}$$

To obtain the curvature scalar W formed from w_{pq} we now apply (48). While Φ contained x^0, ω is independent of the corresponding variable x^5. Thus we obtain

$$W=\omega^{-2}\left\{\bar{W}+\frac{8}{(-\bar{w})^{\frac{1}{2}}}\frac{\partial}{\partial x^a}[\bar{w}^{ab}(-\bar{w})^{\frac{1}{2}}\partial\log\omega/\partial x^b] +12\bar{w}^{ab}\frac{\partial\log\omega}{\partial x^a}\cdot\frac{\partial\log\omega}{\partial x^b}\right\}. \tag{80}$$

Finally, we must define the metrical tensor g_{ab}. We want this to transform according to (17). From (58) we see that under transformations of the form (19) $w_{\sigma\tau}$ behaves like $s_{\sigma\tau}$. Thus w_{ab} is unaltered, w_{a5} acquires a factor k^{-2}, and $\omega^2(\equiv w_{55})$ acquires a factor k^{-4}. Then, by (68), $\bar{w}_{ab}$ acquires k^4 and $\bar{w}_{a5}$ acquires k^2. So, by (71), $\bar{g}_{ab}$ acquires k^4. Thus we are led to define g_{ab} as

$$g_{ab}=\omega\bar{g}_{ab}, \tag{81}$$

since under a transformation (19) this g_{ab} acquires a factor k^2, in conformity with (17).

From (81),

$$g=\omega^4\bar{g}, \tag{82}$$

where g is the determinant of g_{ab}.

Let R be the curvature scalar formed from g_{ab}. Then, from (48),

$$\begin{aligned}R&=\omega^{-1}\left\{\bar{R}+\frac{6}{(-\bar{g})^{\frac{1}{2}}}\frac{\partial}{\partial x^a}\left[\bar{g}^{ab}(-\bar{g})^{\frac{1}{2}}\frac{\partial\log\omega}{\partial x^b}\right]+6\bar{g}^{ab}\frac{\partial\log\omega}{\partial x^a}\frac{\partial\log\omega}{\partial x^b}\right\}\\&=\omega^{-1}\bar{R}+\frac{6\omega}{(-g)^{\frac{1}{2}}}\frac{\partial}{\partial x^a}\left[\frac{1}{\omega}g^{ab}(-g)^{\frac{1}{2}}\frac{\partial\log\omega}{\partial x^b}\right]+6g^{ab}\frac{\partial\log\omega}{\partial x^a}\frac{\partial\log\omega}{\partial x^b}.\end{aligned} \tag{83}$$

Also, by (75), (76), (81), and (82), the last two terms in (80) can be written

$$\begin{aligned}&\omega^{-2}\left\{\frac{8}{(-\bar{w})^{\frac{1}{2}}}\frac{\partial}{\partial x^a}\left[\bar{w}^{ab}(-\bar{w})^{\frac{1}{2}}\frac{\partial\log\omega}{\partial x^b}\right]+12\bar{w}^{ab}\frac{\partial\log\omega}{\partial x^a}\frac{\partial\log\omega}{\partial x^b}\right\}\\&=\frac{8}{(-g)^{\frac{1}{2}}}\frac{\partial}{\partial x^a}\left[\frac{1}{\omega}g^{ab}(-g)^{\frac{1}{2}}\frac{\partial\log\omega}{\partial x^b}\right]+\frac{12}{\omega}g^{ab}\frac{\partial\log\omega}{\partial x^a}\frac{\partial\log\omega}{\partial x^b}.\end{aligned} \tag{84}$$

Combining (77), (78), (83), and (84), and writing

$$\psi^a{}_b=g^{ac}\psi_{cb}=\frac{1}{\omega}\bar{g}^{ac}\psi_{cb}=\frac{1}{\omega}\bar{\psi}^a{}_b, \tag{85}$$

we have

$$\begin{aligned}W&=\omega^{-1}R-\frac{6}{(-g)^{\frac{1}{2}}}\frac{\partial}{\partial x^a}\left[\frac{1}{\omega}g^{ab}(-g)^{\frac{1}{2}}\frac{\partial\log\omega}{\partial x^b}\right]+\frac{6}{\omega}g^{ab}\frac{\partial\log\omega}{\partial x^a}\frac{\partial\log\omega}{\partial x^b}\\&\quad+\frac{8}{(-g)^{\frac{1}{2}}}\frac{\partial}{\partial x^a}\left[\frac{1}{\omega}g^{ab}(-g)^{\frac{1}{2}}\frac{\partial\log\omega}{\partial x^b}\right]+\frac{12}{\omega}g^{ab}\frac{\partial\log\omega}{\partial x^a}\frac{\partial\log\omega}{\partial x^b}-\psi^a{}_b\psi^b{}_a\\&=\omega^{-1}R+\frac{2}{(-g)^{\frac{1}{2}}}\frac{\partial}{\partial x^a}\left[\frac{1}{\omega}g^{ab}(-g)^{\frac{1}{2}}\frac{\partial\log\omega}{\partial x^b}\right]+\frac{18}{\omega}g^{ab}\frac{\partial\log\omega}{\partial x^a}\frac{\partial\log\omega}{\partial x^b}-\psi^a{}_b\psi^b{}_a.\end{aligned} \tag{86}$$

To obtain the value of Z we must now compute the term $-\varphi^p{}_q\varphi^q{}_p$ in (64) and the terms $10\Psi^\sigma{}_{/\sigma}$ and $20\Psi^\sigma\Psi_\sigma$ in (53).

Since φ_p does not contain x^5, we have, from (66),

$$\varphi_{5p}=-\varphi_{p5}=\tfrac{1}{2}\partial\varphi_5/\partial x^p,$$

or, if we write

$$\varphi=\varphi_5, \tag{87}$$

$$\varphi_{5p}=-\varphi_{p5}=\tfrac{1}{2}\partial\varphi/\partial x^p. \tag{88}$$

Then, by (65), (66), (68), (75), and (76), we have

$$
\begin{aligned}
-\varphi^p{}_q\varphi^q{}_p &= -w^{pr}w^{qs}\varphi_{rq}\varphi_{sp} \\
&= -w^{ac}w^{bd}\varphi_{cb}\varphi_{da}-4w^{5c}w^{bd}\varphi_{cb}\varphi_{d5} \\
&\quad -2w^{5c}w^{5d}\varphi_{c5}\varphi_{d5}-2w^{55}w^{bd}\varphi_{5b}\varphi_{d5} \\
&= -\omega^{-4}\Big\{\bar{g}^{ac}\bar{g}^{bd}\Big[\varphi_{cb}\varphi_{da}+\psi_a\varphi_{cb}\frac{\partial\varphi}{\partial x^d} \\
&\quad +\tfrac{1}{2}\psi_a\psi_b\frac{\partial\varphi}{\partial x^c}\frac{\partial\varphi}{\partial x^d}\Big] \\
&\quad -\tfrac{1}{2}(1+\bar{g}^{ab}\psi_a\psi_b)\bar{g}^{cd}\frac{\partial\varphi}{\partial x^c}\frac{\partial\varphi}{\partial x^d}\Big\} \\
&= -\omega^{-2}\Big\{g^{ac}g^{bd}\Big[\varphi_{cb}\varphi_{da}+\psi_a\varphi_{cb}\frac{\partial\varphi}{\partial x^d}\Big] \\
&\quad -\frac{1}{2\omega}g^{ab}\frac{\partial\varphi}{\partial x^a}\frac{\partial\varphi}{\partial x^b}\Big\}. \qquad (89)
\end{aligned}
$$

In computing $\Psi^\sigma{}_{/\sigma}$ and $\Psi^\sigma\Psi_\sigma$ we shall need the values of s^{ab}, s^{a0}, and s^{00}. From (61), (68), (75), (76), and (87) we have

$$
s^{ab}=w^{ab}=\omega^{-2}\bar{w}^{ab}=\omega^{-2}\bar{g}^{ab}=\omega^{-1}g^{ab}, \qquad (90)
$$

$$
\begin{aligned}
s^{a0} &= -w^{ap}\varphi_p=-w^{ab}\varphi_b-w^{a5}\varphi_5 \\
&= -\omega^{-1}g^{ab}\varphi_b-\omega^{-2}\bar{w}^{a5}\varphi \\
&= -\omega^{-1}g^{ab}\varphi_b+\omega^{-1}\psi_b\varphi \\
&= -\omega^{-1}g^{ab}(\varphi_b-\psi_b\varphi), \qquad (91)
\end{aligned}
$$

and

$$
\begin{aligned}
s^{00} &= 1+w^{pq}\varphi_p\varphi_q=1+\omega^{-2}\{\bar{g}^{ab}\varphi_a\varphi_b-2\bar{g}^{ab}\psi_b\varphi_a\varphi \\
&\quad +(1+\bar{g}^{ab}\psi_a\psi_b)\varphi^2\} \\
&= 1+\omega^{-1}\{g^{ab}\varphi_a\varphi_b-2g^{ab}\psi_b\varphi_a\varphi \\
&\quad +(\omega^{-1}+g^{ab}\psi_a\psi_b)\varphi^2\} \\
&= (1+\varphi^2/\omega^2)+\omega^{-1}g^{ab}(\varphi_a-\psi_a\varphi)(\varphi_b-\psi_b\varphi). \qquad (92)
\end{aligned}
$$

Also

$$
s=w=\omega^8\bar{w}=\omega^8\bar{g}=\omega^4 g. \qquad (93)
$$

We have now

$$
\begin{aligned}
\Psi^\sigma{}_{/\sigma} &= \frac{1}{(-s)^{\frac{1}{2}}}\frac{\partial}{\partial x^\sigma}\Big[s^{\sigma\tau}(-s)^{\frac{1}{2}}\frac{\partial\log\Psi}{\partial x^\tau}\Big] \\
&= \frac{1}{\omega^2(-g)^{\frac{1}{2}}}\frac{\partial}{\partial x^a}\Big[\omega^2(-g)^{\frac{1}{2}}\Big(s^{ab}\frac{\partial\log\Psi}{\partial x^b}+Ns^{a0}\Big)\Big] \\
&= \frac{1}{\omega^2(-g)^{\frac{1}{2}}} \\
&\quad \times\frac{\partial}{\partial x^a}\Big[\omega(-g)^{\frac{1}{2}}g^{ab}\Big(\frac{\partial\log\Psi}{\partial x^b}-N(\varphi_b-\psi_b\varphi)\Big)\Big], \qquad (94)
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi^\sigma\Psi_\sigma &= s^{\sigma\tau}\frac{\partial\log\Psi}{\partial x^\sigma}\frac{\partial\log\Psi}{\partial x^\tau}=s^{\alpha\beta}\frac{\partial\log\Psi}{\partial x^\alpha}\frac{\partial\log\Psi}{\partial x^\beta} \\
&= s^{ab}\frac{\partial\log\Psi}{\partial x^a}\frac{\partial\log\Psi}{\partial x^b}+2Ns^{a0}\frac{\partial\log\Psi}{\partial x^a}+N^2s^{00} \\
&= \omega^{-1}g^{ab}\frac{\partial\log\Psi}{\partial x^a}\frac{\partial\log\Psi}{\partial x^b}-2N\omega^{-1}g^{ab}(\varphi_b-\psi_b\varphi) \\
&\quad \times\frac{\partial\log\Psi}{\partial x^a}+N^2(1+\varphi^2/\omega^2) \\
&\quad +N^2\omega^{-1}g^{ab}(\varphi_a-\psi_a\varphi)(\varphi_b-\psi_b\varphi) \\
&= N^2(1+\varphi^2/\omega^2)+N^2\omega^{-1}g^{ab} \\
&\quad \times\Big(\varphi_a-\frac{1}{N}\frac{\partial\log\Psi}{\partial x^a}-\psi_a\varphi\Big) \\
&\quad \times\Big(\varphi_b-\frac{1}{N}\frac{\partial\log\Psi}{\partial x^b}-\psi_b\varphi\Big). \qquad (95)
\end{aligned}
$$

We note that, except for the initial factor Ψ^{-2}, the quantities φ_a and Ψ enter only in the combinations φ_{ab} and $\varphi_a-(1/N)(\partial\log\Psi/\partial x^a)$. So we write

$$
\theta_a=\varphi_a-\frac{1}{N}\frac{\partial\log\Psi}{\partial x^a}, \qquad (96)
$$

$$
\theta_{ab}=\tfrac{1}{2}(\partial\theta_a/\partial x^b-\partial\theta_b/\partial x^a)=\varphi_{ab}, \qquad (97)
$$

and

$$
\theta^a{}_b=g^{ac}\theta_{cb}. \qquad (98)
$$

Then, collecting the partial results (64), (86), (89), (94), (95), and substituting in (53), we have

$$
\begin{aligned}
Z=\Psi^{-2}\Big\{ &\frac{1}{\omega}R+\frac{2}{(-g)^{\frac{1}{2}}}\frac{\partial}{\partial x^a}\Big[\frac{1}{\omega}g^{ab}(-g)^{\frac{1}{2}}\frac{\partial\log\omega}{\partial x^b}\Big] \\
&+\frac{18}{\omega}g^{ab}\frac{\partial\log\omega}{\partial x^a}\frac{\partial\log\omega}{\partial x^b}-\psi^a{}_b\psi^b{}_a \\
&-\frac{1}{\omega^2}\Big[\theta^a{}_b\theta^b{}_a+g^{bd}\psi_a\theta^a{}_b\frac{\partial\varphi}{\partial x^d}-\frac{1}{2\omega}g^{ab}\frac{\partial\varphi}{\partial x^a}\frac{\partial\varphi}{\partial x^b}\Big] \\
&+\frac{10N}{\omega^2(-g)^{\frac{1}{2}}}\frac{\partial}{\partial x^a}[\omega g^{ab}(-g)^{\frac{1}{2}}(\psi_b\varphi-\theta_b)] \\
&+20N^2(1+\varphi^2/\omega^2) \\
&+\frac{20N^2}{\omega}g^{ab}(\theta_a-\psi_a\varphi)(\theta_b-\psi_b\varphi)\Big\}. \qquad (99)
\end{aligned}
$$