plication factor. Assuming that the average energy of an electron incident on top of the lead is 100 Mev, the critical energy for air, and that all electrons under the lead having energies greater than 1 Mev will pass through the chamber wall and be detected, then the curves show that the peak multiplication factor at that point is seven. Thus, on the average, 2.3 air-shower electrons incident on top of the A lead in an area equal to that of the ion chamber will produce a burst of 16 electrons in the chamber. Since the area of the chamber is 356 cm^2, showers of density greater than 65 electrons per square meter will be detected. Cocconi[12] has found the number of showers per second with densities greater than D electrons per square meter at Echo Lake to be $1.68D^{-1.46}$. This rate will be increased by the factor 1.25 at Climax because of the difference in altitude. This gives 17 showers per hour that should be detected at the maximum of the transition. The ID rate, the extensive showers detected by the chamber and the shower counters, is about two per hour with no A lead and twelve per hour at the maximum, which is in fair agreement with the rough calculations above. The shower counters are 96 percent efficient for the density of 65 particles per square meter.

It is thus evident that the observed chamber transition of an additional 200 bursts per hour cannot be due to low density extensive air showers. Wei[13] has obtained rates of occurrence of narrow air showers and their absorption properties, concluding that they are mixtures of electrons and a hard component, very likely mesons. He finds[14] 160 narrow showers per hour of two particles and 87 per hour of three particles. These rates are comparable to the observed increase in ion chamber rate of about 200 per hour. Thus it is possible we were detecting the electron component of narrow air showers. The average energy of such electrons, and thus the transition effect to be expected, is not known.

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[12] G. Cocconi and V. Tongiorgi, Phys. Rev. **75**, 1058 (1949).

[13] J. P. N. Wei, Phys. Rev. **79**, 670 (1950).

[14] J. P. N. Wei, thesis, Yale University, 1950.

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A Note on Isotropy in Nuclear Gamma-Radiation*

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The question of the possibility of competing radiations exhibiting different angular distributions (in the sense that one radiation is isotropic and another anisotropic) is investigated. Except for the singular cases $\frac{3}{2}\rightarrow\frac{3}{2}$ and $5\rightarrow4$ by quadrupole emission, our conclusion is as follows: if one radiation of multipole order 2^L is observed to be isotropic, all competing gamma-radiation will be isotropic if $L\geq j-\frac{1}{2}$, where j is the angular momentum of the emitting state. The possibility of occurrence of singular cases, for which the conclusion would not apply, has been investigated for all radiations for which $L\leq3$. In the case $\frac{3}{2}\rightarrow\frac{3}{2}$, $L=2$ the radiation is isotropic and the applicability of this to the problem of the excited state of Li^7 is briefly discussed.

I. INTRODUCTION

We consider a compound nuclear state formed by capture of some particle by a target nucleus and the subsequent decay of the state by two (or more) competing processes. In what follows we restrict our attention to the case in which the competing radiations are γ-rays. It will be clear that our conclusions apply equally well for the emission of α-particles (or, in general, spinless particles). The problem to be considered may be stated as follows: if one of the gamma-radiations is observed to be isotropic, under what conditions must the competing radiations also exhibit isotropic angular distributions? Conversely, under what circumstances may the competing radiations exhibit an anisotropic distribution with respect to the incident beam or some other physically defined direction? It is understood, of course, that the transmutation leading to the γ-emitting state involves p (or higher angular momentum) waves for the relative motion of incident particles and target nucleus. This problem arises, for example, in the proton bombardment of B^{11} with the subsequent emission of an isotropic (within experimental error) 16-Mev γ-ray to the C^{12} ground state presumably competing with a double cascade of 12- and 4-Mev radiations.[1]

* This paper is based on work performed for the AEC at the Oak Ridge National Laboratory.

[1] Kern, Moak, Good, and Robinson, Phys. Rev. **83**, 211(A) (1951); G. B. Arfken and L. C. Biedenharn, *ibid.*, 238(A).

It is obvious that from a compound state of angular momentum j with all $2j+1$ degenerate substates equally populated and randomly phased (e.g., capture of unpolarized s-particles), all radiation must be isotropic. If the converse were valid, i.e., if isotropy of one radiation implied equal populations and random phases for the emitting state, it would follow that all other radiation would have to be isotropic. However, while this would be a sufficient condition for isotropy, it is not a necessary one, since isotropy follows even with unequal populations. Thus, gamma-radiation, observed with polarization insensitive detectors, would have the same angular distribution for emission from the substates $\pm m$. Then, if the population of the substate m is a_m, a sufficient condition for isotropy is

$$2a_0=a_1+a_{-1}=\cdots=a_j+a_{-j} \quad (j \text{ integer}), \qquad (1)$$

or, generally, a_m+a_{-m} independent of m. Of course, uniform population is a special case. As is shown below, Eq. (1) is a necessary condition for isotropy except in a very few special cases which are explicitly exhibited in the following. Therefore, the question in hand is to determine whether the observation of a single isotropic 2^L pole radiation implies Eq. (1).

II. ISOTROPY THEOREM

Following Falkoff and Uhlenbeck[2] we write the angular distribution of pure 2^L pole gamma-radiation from the state j, m with population a_m to a state j', m' in the form,

$$I(\vartheta)\sim\sum_{mM} a_m(C_{mM}{}^{jLj'})^2F_L{}^M(\vartheta). \qquad (2)$$

Here $C_{mM}{}^{jLj'}$ is the vector addition coefficient, and M is the magnetic quantum number of the radiation. The $F_L{}^M(\vartheta)$ are the classical angular distribution functions for 2^L pole electromagnetic radiation and for our purposes it is most convenient to represent them in the form[3]

$$F_L{}^M(\vartheta)=\sum_{P=\pm1}[D_{MP}{}^{(L)}(0\vartheta0)]^2, \qquad (3)$$

where the $D_{MP}{}^{(L)}$ are the elements of the irreducible rotation matrix of order L. It will be noted that, by the assumption of random phases, we have eliminated cross terms in Eq. (2). If the state j is excited by a process that defines only a single direction in space, this assumption is always justified.

Using the properties of the rotation matrices $D^{(L)}$, we can transform $F_L{}^M$ as follows:[3]

$$\begin{aligned}F_L{}^M(\vartheta)&=(-)^{M+1}\sum_P D_{MP}{}^{(L)}(0\vartheta0)D_{-M-P}{}^{(L)}(0\vartheta0)\\&=(-)^{M+1}\sum_P\sum_{\tau=0}^{2L} C_{P-P}{}^{LL\tau}C_{M-M}{}^{LL\tau}D_{00}{}^{(\tau)}(0\vartheta0)\\&=(-)^{L+1}\left(\frac{16\pi}{2L+1}\right)^{\frac{1}{2}}\\&\quad\times\sum_{\nu=0}^{L} C_{1-1}{}^{LL2\nu}C_{M0}{}^{L2\nu L}Y_{2\nu}{}^0(\cos\vartheta), \qquad (4)\end{aligned}$$

since $C_{1-1}{}^{LL\tau}+C_{-11}{}^{LL\tau}$ vanishes when $2L-\tau$ is odd. We now introduce a notation for the sum over M in Eq. (2),

$$f_{Ljj'}{}^m\equiv\sum_M(C_{mM}{}^{jLj'})^2F_L{}^M(\vartheta), \qquad (5)$$

which is the quantum-mechanical angular distribution function for radiation emitted from the substate m. From the definition it is clear that the functions f are even in m, since $F_L{}^M$ is even in M. We utilize this symmetry by summing in Eq. (2) over non-negative m only and, after introducing the new populations b_m defined by

$$\begin{aligned}b_m&=a_m+a_{-m}, \quad m\neq0,\\ b_0&=a_0,\end{aligned} \qquad (6)$$

where the second of Eq. (6) applies only for j integer, Eq. (2) takes the form,

$$I(\vartheta)\sim\sum_{m\geq0}^{j} b_mf_{Ljj'}{}^m(\vartheta). \qquad (7)$$

By hypothesis $I(\vartheta)$ is independent of ϑ for one pair of values of L, j'. If the $f_{Ljj'}{}^m$ are a linearly independent set of functions in m for each L, j, j', it would follow that all radiation from j is isotropic. This is because a known solution of Eq. (7) is b_m a constant independent of $m(b_m=2b_0)$. If the $f_{Ljj'}{}^m$ are linearly independent, the solution for the b_m is unique. Therefore, we investigate the conditions under which the f^m-functions form a linearly independent set. Substituting Eq. (4) into Eq. (5), we obtain

$$\begin{aligned}f_{Ljj'}{}^m(\vartheta)&=(-)^{L+1}\left(\frac{16\pi}{2L+1}\right)^{\frac{1}{2}}\\&\quad\times\sum_{\nu=0}^{L}\sum_M(C_{mM}{}^{jLj'})^2C_{1-1}{}^{LL2\nu}C_{M0}{}^{L2\nu L}Y_{2\nu}{}^0. \qquad (8)\end{aligned}$$

Using the Racah coefficients,[4] the summation over M can be performed. The relation required is

$$\begin{aligned}C_{M0}{}^{L2\nu L}C_{Mm}{}^{Lj\,j'}=\sum_{j_1}(2L+1)^{\frac{1}{2}}(2j_1+1)^{\frac{1}{2}}\\ \times C_{0m}{}^{2\nu jj_1}C_{Mm}{}^{Lj_1j'}W(L2\nu j'j;Lj_1). \qquad (9)\end{aligned}$$

From the properties of the vector addition coefficients we have

$$\begin{aligned}\sum_M(C_{mM}{}^{jLj'})^2C_{M0}{}^{L2\nu L}=(2j'+1)(2L+1)^{\frac{1}{2}}(2j+1)^{-\frac{1}{2}}\\ \times C_{0m}{}^{2\nu jj}W(L2\nu j'j;Lj). \qquad (10)\end{aligned}$$

Then the final form for $f_{Ljj'}{}^m(\vartheta)$ is[4]

$$\begin{aligned}f_{Ljj'}{}^m(\vartheta)&=(-)^{L+1+j-m}(16\pi)^{\frac{1}{2}}(2j'+1)\\&\quad\times\sum_{\nu=0}^{L}C_{m-m}{}^{jj2\nu}C_{1-1}{}^{LL2\nu}(4\nu+1)^{-\frac{1}{2}}\\&\quad\times W(L2\nu j'j;Lj)Y_{2\nu}{}^0(\cos\vartheta). \qquad (11)\end{aligned}$$

We now establish a necessary condition for the linear independence of the f^m set. The Racah coefficients in

[2] D. L. Falkoff and G. E. Uhlenbeck, Phys. Rev. **79**, 323 (1950).
[3] For example, G. Goertzel, Phys. Rev. **70**, 897 (1946).
[4] G. Racah, Phys. Rev. **62**, 438 (1942); Phys. Rev. **63**, 367 (1943).

Eq. (11) vanish unless the "triangular conditions" are simultaneously satisfied for all three sets $(LL2\nu)$, $(Lj'j)$, and $(jj2\nu)$. That is, it must be possible to form a triangle from each of these sets of three numbers. This condition is just the angular momentum conservation for $(Lj'j)$. For the other pair we deduce that the maximum value of ν is the lesser of the two numbers j and L, if j is an integer, or the lesser of $j-\frac{1}{2}$ and L if j is a half-integer. Since the $Y_{2\nu}{}^0$ are linearly independent, it is necessary that

$$L \geqslant j-\tfrac{1}{2} \tag{12}$$

for the $f_{Ljj'}{}^m$ to be linearly independent. This covers both cases: j integer or half-integer.

If the condition expressed by Eq. (12) is fulfilled, the linear independence of the f^m is assured, provided that the determinant

$$|d_m{}^\nu| \equiv |C_{m-m}{}^{jj2\nu}C_{1-1}{}^{LL2\nu}W(L2\nu j'j;Lj)| \neq 0. \tag{13}$$

Since the factors depending only on ν factor out of the determinant, we must investigate (a) the possibility of $\det C_{m-m}{}^{jj2\nu}$ vanishing, (b) the possibility of roots of $C_{1-1}{}^{LL2\nu}$, and (c) the possible roots of $W(L2\nu j'j;Lj)$.

(a) We cannot immediately conclude that $\det C_{m-m}{}^{jj2\nu}\neq 0$ from the unitary property of the vector addition coefficients, since we have restricted m to non-negative values and also because 2ν is always an even integer. This reduces the determinant in question to dimension $j+1$ (j integer) or $j+\frac{1}{2}$ (j half-integer). Nevertheless, we can show that $\det C_{m-m}{}^{jj2\nu}\neq 0$ by defining a matrix $B_m{}^\nu$ whose rows are $C_{m-m}{}^{jj2\nu}$ except for $m=0$, and for $m=0$ we set $B_0{}^\nu = 2^{-\frac{1}{2}}C_{00}{}^{jj2\nu}$. Then, we have

$$(\tilde{B}B)_{\lambda\nu} = \sum_\mu \tilde{B}_{\lambda\mu}B_{\mu\nu} = \sum_{\mu\neq 0} C_{\mu-\mu}{}^{jj2\nu}C_{\mu-\mu}{}^{jj2\lambda} + \tfrac{1}{2}C_{00}{}^{jj2\nu}C_{00}{}^{jj2\lambda}$$
$$= \tfrac{1}{2}\sum_{\mu=-j}^{j} C_{\mu-\mu}{}^{jj2\nu}C_{\mu-\mu}{}^{jj2\lambda} = \tfrac{1}{2}\delta_{\lambda\nu}. \tag{14}$$

Therefore $\det B_m{}^\nu \neq 0$ and $\det C_{m-m}{}^{jj2\nu}\neq 0$.

(b) The vector addition coefficient $C_{1-1}{}^{LL2\nu}$ can be expressed in terms of $C_{00}{}^{LL2\nu}$ by using

$$Y_L{}^M Y_L{}^{-M} = \tfrac{1}{2}(2L+1)\pi^{-\frac{1}{2}} \times \sum_{\nu=0}^{L}(4\nu+1)^{-\frac{1}{2}}C_{00}{}^{LL2\nu}C_{M-M}{}^{LL2\nu}Y_{2\nu}{}^0.$$

Using the properties of the spherical harmonics, we obtain

$$C_{1-1}{}^{LL2\nu} = C_{00}{}^{LL2\nu}[\nu(2\nu+1)-L(L+1)]/L(L+1). \tag{15}$$

Since $C_{00}{}^{LL2\nu}\neq 0$ for any L, ν, the roots in question are those of the square bracket in Eq. (15). With $L\geqslant 1$ the lowest root is $L=14$, $\nu=10$; and, although our isotropy theorem does not hold in this case, 2^{14} pole radiation is hardly of physical interest. (For spinless particles the only change in Eq. (11) is the replacement of $C_{1-1}{}^{LL2\nu}$ by $C_{00}{}^{LL2\nu}$.)

(c) Investigation of the Racah coefficient $W(L2\nu j'j;Lj)$ shows that (with the triangular conditions satisfied) this coefficient vanishes for

$$j=j'=\tfrac{3}{2},\quad L=2,\quad \nu=1, \tag{16a}$$

and for

$$j=5,\quad j'=4,\quad L=2,\quad \nu=1. \tag{16b}$$

We have investigated all cases for $L=1$, 2, and 3. Thus, while there may be similar anomalous cases for radiation higher than octupole (the investigation of which would be exceedingly laborious), it is felt that from a practical point of view these cases would not be of great interest.

We therefore conclude that for the emission of octupole or lower multipole radiation a necessary and sufficient condition for the impossibility of isotropic and anisotropic radiations originating from the same level is $L\geqslant j-\frac{1}{2}$, where L refers to the isotropic radiation, and that neither Eq. (16a) nor Eq. (16b) applies.

In the $B^{11}(p,\gamma)C^{12}$ reaction preliminary results[1] show isotropy for the cross-over and definite anisotropy for the 12-Mev radiation which is known to precede the 4-Mev γ-ray. The condition (12) is fulfilled. Therefore, accepting the experimental evidence at face value, we must conclude that the 16- and 12-Mev radiations originate from different levels.

For the quadrupole emission from $j=\frac{3}{2}$ to $j'=\frac{3}{2}$ the angular distribution of the radiation will be isotropic.[5] This result has a possible application to the interpretation of the $Li^7(p,p')Li^{7*}(\gamma)Li^7$ and $B^{10}(n,\alpha)Li^{7*}(\gamma)Li^7$ reactions. The observation of isotropic γ's in these cases does not compel one to conclude that $j=\frac{1}{2}$ for the relevant state of Li^{7*}. As a logical possibility $j=\frac{3}{2}$ should be considered. However, other evidence may be cited which makes the interpretation $j=\frac{3}{2}$ somewhat difficult.[6] Nevertheless, it may be of interest in connection with other reactions that isotropic radiation emitted from odd mass nuclei does not necessarily imply $j=\frac{1}{2}$ for the emitting state.

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[5] This conclusion is implicitly contained in the results of S. Devons, Proc. Phys. Soc. (London) **62A**, 580 (1949) and also in the results of reference 2. For Eq. (16b) only the coefficient of $Y_2{}^0$ is absent and the angular distribution is not isotropic.

[6] D. R. Inglis, Phys. Rev. **82**, 181 (1951); L. G. Elliott and R. E. Bell, Phys. Rev. **76**, 178 (1949).