On the Relativistic Pseudoscalar Meson Theory of the Deuteron

MAURICE M. LÉVY*
Institute for Advanced Study, Princeton, New Jersey
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The existence of stationary states of the neutron-proton system is discussed on the basis of a two-particle Dirac equation, including a retarded pseudoscalar interaction. This interaction (in which the possibility of creation of nucleons pairs is disregarded) is treated according to two different models: in the first one, the negative energy states of each nucleon are supposed to be filled, and the interaction is limited to transitions between positive energy states only; in the second model, which is the generalization of the "one-particle" theory, the negative energy states are considered as empty. It is shown that, in both cases, no stable stationary state exists with such an interaction.

I. INTRODUCTION

THE calculation of the interaction between two nucleons in the pseudoscalar meson theory leads, as is well known, to a static approximation which is too strongly singular at small distances to allow for the existence of stationary states of the system. It has long been hoped that a relativistic calculation of this interaction might lower its singularity at the origin, and a tentative approach to this problem will be made in this paper. Within the limits of the approximations which will be made below, the answer is, however, negative.

There are, in general, two different ways to treat relativistically a bound system of two particles interacting through a field. In the first method, which has been mostly used in the past, the interaction is calculated independently of the equations of motion of the two particles ("adiabatic" approximation); this interaction is then introduced into a wave equation, and the motion of the particles is worked out as a second step. In the second method, the calculation of the interaction to a certain order in the coupling constant is not separated from the derivation of the equations of motion, valid to the same order. A "non-adiabatic" treatment of this form has been used by Tamm[1] and Dancoff[2] for the study of the scalar interaction between two nucleons. Only the first method will be discussed in this paper (see, however, our concluding remarks).

The relativistic energy density of interaction of a system of two nucleons, calculated by means of the method of Møller[3] to the second order in the coupling constant, can be written, in the pseudoscalar case, as:

$$W(1,2)=-\bar{\psi}(2)\gamma_5^{(2)}\psi(2)\bar{\Delta}(x_1-x_2)\bar{\psi}(1)\gamma_5^{(1)}\psi(1), \quad (1),$$

where the indices 1, 2 refer to the two particles. $\psi(1)$ $\psi(2)$ are the wave functions of the nucleons, $\bar{\Delta}$ is the well-known four dimensional Green function as defined by Schwinger.[4] The interaction (1) includes nondiagonal elements which connect two particle states with others, in which pairs of nucleons are present.[5] Following Van Hove,[6] we shall make, in the following, two essential approximations:

(a) The nondiagonal elements of $W(1,2)$ will be neglected.

(b) In the momentum representation of the function $\bar{\Delta}$, the fourth component of the momentum of each nucleon will be identified with its corresponding free energy (a system of units in which $\hbar=c=1$ is used throughout this paper). This will enable us to express the energy of interaction in a three-dimensional form and, therefore, to use the conventional definition of stationary states, in which the times of the two-particles are set equal.

With these two assumptions, the matrix elements of the energy of interaction between a neutron and a proton, corresponding to a pseudoscalar meson field (with pseudoscalar and pseudovector couplings), can be written, in the momentum representation, as

$$V^{(2)}_{m_1,m_2}=-\tfrac{1}{2}f_3^2\sum_{n_1,n_2}\chi^{(1)*}_{n_1}\rho^{(1)}_2\chi^{(1)}_{m_1}\mathbf{T}\chi^{(2)*}_{n_2}\rho^{(2)}_2\chi^{(2)}_{m_2}$$
$$\times\left[\frac{1}{\epsilon_K^2-(E_{n_1}-E_{m_1})^2}+\frac{1}{\epsilon_K^2-(E_{n_2}-E_{m_2})^2}\right] \quad (2)$$

with $f_3=f_1+(2M/\mu)f_2$, $\epsilon_K^2=K^2+\mu^2$; $\mathbf{T}$ is an arbitrary isotopic spin operator. In the interaction $V^{(2)}$, a second term has been neglected [Eq. (20)] of Van Hove's paper), which can be removed from the second-order interaction by means of a canonical transformation given by Dyson.[7] This term gives no contribution to real processes which occur with conservation of the energy. However, for virtual processes, in which the energy is not conserved in the intermediate states, this term can contribute to the interaction. We note, however, that this term vanishes for a pure pseudoscalar coupling ($f_2=0$, $f_3=f_1$); in order to avoid any am-

* On leave of absence from the Centre National de la Recherche Scientifique, Paris, France.

[1] I. Tamm, J. Phys. (U.S.S.R.) **9**, 449 (1945).

[2] S. M. Dancoff, Phys. Rev. **78**, 382 (1950).

[3] C. Møller, Z. Physik **70**, 786 (1931); Ann. Physik **14**, 531 (1932).

[4] J. Schwinger, Phys. Rev. **75**, 651 (1949) (Eq. A.21).

[5] Equation (1) can be generalized as to include higher order effects, by introducing a more general function $\bar{\Delta}'(x_1-x_2)$, satisfying an inhomogeneous linear integral equation.

[6] L. Van Hove, Phys. Rev. **75**, 1519 (1949).

[7] F. J. Dyson, Phys. Rev. **72**, 929 (1948).

TABLE I. Classification of the 16 components of the fourier coefficient $\varphi(p)$ of the wave function according to the spin direction and the sign of the corresponding free energy of particles (1) and (2).

		$E^{(1)}=+E_p$		$E^{(1)}=-E_p$	
		$\sigma_3^{(1)}=+1$	$\sigma_3^{(1)}=-1$	$\sigma_3^{(1)}=+1$	$\sigma_3^{(1)}=-1$
$E^{(11)}=+E_p$	$\sigma_3^{(11)}=+1$	$\varphi_{11}^{(1)}$	$\varphi_{11}^{(2)}$	$\varphi_{12}^{(1)}$	$\varphi_{12}^{(2)}$
	$\sigma_3^{(11)}=-1$	$\varphi_{11}^{(3)}$	$\varphi_{11}^{(4)}$	$\varphi_{12}^{(3)}$	$\varphi_{12}^{(4)}$
$E^{(11)}=-E_p$	$\sigma_3^{(11)}=+1$	$\varphi_{21}^{(1)}$	$\varphi_{21}^{(1)}$	$\varphi_{22}^{(1)}$	$\varphi_{22}^{(2)}$
	$\sigma_3^{(11)}=-1$	$\varphi_{21}^{(3)}$	$\varphi_{21}^{(4)}$	$\varphi_{22}^{(3)}$	$\varphi_{22}^{(4)}$

biguity, we shall therefore restrict ourselves to this case.[8]

Now, in order to calculate the energy levels of the system, our next step is to insert the interaction (2) into a wave equation. In the framework of the "adiabatic" approximation, the only possibility we have is to introduce $V^{(2)}$ into a two-particle Dirac equation,

$$(H_1+H_2-E)\psi(\mathbf{x}_1,\mathbf{x}_2)+V^{(2)}\psi(\mathbf{x}_1,\mathbf{x}_2)=0, \qquad (3)$$

where $H_i=\boldsymbol{\gamma}^{(i)}\cdot\mathbf{grad}^{(i)}+M$, and E is the total energy of the system. We consider, here, the equation as given in a phenomenological way, and resulting more or less from an intuitive generalization of the ordinary notion of potential. There exists, however, field-theoretical arguments which justify, within the approximations (a) and (b), the inclusion of $V^{(2)}$ into Eq. (3) (see footnote 16).

With an equation like (3), a delicate question arises, as to the way to treat the transitions to negative energy states of the nucleons, which are included in interaction (2). In this connection we note that the very use of Eq. (3), and also the approximations (a) and (b) which have been made for $V^{(2)}$, exclude the possibility of taking into account the creation of virtual nucleon pairs. There are, therefore, only two ways to use Eq. (3): model (A)—to consider all negative energy states as filled, and, consequently, to limit the interaction to transitions between positive energy states only; and model (B)—to consider all negative energy states as empty, and, therefore, allow the interaction to connect free states of all energies: this is the generalization to two particles of the "one-particle" theory.

This last model, which is only possible because pair creation is excluded, is rather unattractive. The slightest departue from the assumptions (a) and (b) would invalidate the discussion of this model, whereas it will still be possible to use model (A) as a basis of comparison with more elaborate calculations (to ascertain, for example, the importance of pair creation, etc. . . .).

In the following, the existence and the stability of stationary states of the neutron-proton system will be discussed separately in the framework of models (A) and (B). For this purpose, Eq. (3) will be expressed in the momentum representation, and the behavior at high momenta of the fourier coefficients of the wave function will be investigated. It will be shown, however, that both models do not allow for stationary states of the system. There are two essential reasons for this negative result. First, the part of the interaction which behaves as r^{-3} in the nonrelativistic limit (and which, in the relativistic domain, is less singular), is, however, still too strongly singular for small separations of the nucleons to allow for the existence of stationary states. The second reason is that there also occurs, in the interaction, terms which tend to a δ-function in the nonrelativistic approximation, but which cannot be eliminated covariantly, because they depend on momenta in the relativistic domain (see Sec. III). These terms dominate in the whole range of momentum transfers, and reverse the sign of the interaction. However, even if the isotopic factor is chosen in such a way as to make the complete interaction attractive, the singularity of these pseudo-"contact" terms is too strong to allow for stationary states of the system. In contrast with the nonrelativistic calculations, the singlet states are not favored with respect to the triplet states.

II. GENERAL EQUATIONS

In the barycentric system, the sixteen-components wave function $\psi(\mathbf{x})$ ($\mathbf{x}=\mathbf{x}_1-\mathbf{x}_2$) is expanded by means of its fourier coefficients:

$$\psi(\mathbf{x})=\sum\varphi(\mathbf{p})u_{\mathrm{I}}(\mathbf{p})u_{\mathrm{II}}(-\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}, \qquad (4)$$

where u_{I} and u_{II} are the amplitudes of the Dirac spinors corresponding to particles (1) and (2). The amplitude $\varphi(\mathbf{p})$ has 16-components which are classified in four sets of four-component amplitudes φ_{ij} $(i,j=1,2)$ according to the sign of the corresponding free energy of each nucleon. The "even" components correspond to $i=j$, the "odd" components, to $i\neq j$. In model (A), only the first set φ_{11} will be different from zero; in model (B), the four sets will give contributions. The four components of each partial amplitude φ_{ij} are numbered according to the different spin orientations of the two nucleons. The complete classification of the components of $\varphi(\mathbf{p})$ is summarized in Table I.

With these notations, the amplitude φ_{ij} obeys the following equation:

$$[E-E_i(p)-E_j(p)]\varphi_{ij}(\mathbf{p})$$
$$=\lambda\sum_{k,l}\int_{(\mathbf{p}')}(\mathbf{p}',k|\rho_2^{(1)}|\mathbf{p},i)\mathbf{T}(\mathbf{p}',l|\rho_2^{(11)}|\mathbf{p},j)$$
$$\times\Delta_{kl}{}^{i,j}(\mathbf{p},\mathbf{p}')\varphi_{kl}(\mathbf{p}')d\mathbf{p}' \qquad (5)$$

where

$$\Delta_{k,l}{}^{ij}(\mathbf{p},\mathbf{p}')=\tfrac{1}{2}\Big[\frac{1}{\omega^2-[E_i(p)-E_k(p')]^2}$$
$$+\frac{1}{\omega^2-[E_j(p)-E_l(p')]^2}\Big], \qquad (6)$$

[8] Since the special contribution of the pseudovector coupling is more singular than $V^{(2)}$, and negative conclusion relating to the latter will also probably be valid for the former.

and $\omega=[(\mathbf{p}-\mathbf{p}')^2+\mu^2]^{\frac{1}{2}}$, $E_1(p)=+E_p$, $E_2(p)=-E_p$, $E_p=(p^2+M^2)^{\frac{1}{2}}$, and $\lambda=(2\pi)^{-3}f_1^2$. The matrix elements of ρ_2 are easily calculated. One has, for example,

$$(\mathbf{p}', k|\rho_2^{(\mathrm{I})}|\mathbf{p}, i)=u_k^*(\mathbf{p}')\rho_2^{\mathrm{I}}u_i(\mathbf{p})$$
$$=\left[\frac{[M+E_i(p)][M+E_k(p')]}{4E_i(p)E_k(p')}\right]^{\frac{1}{2}}\boldsymbol{\sigma}_{\mathrm{I}}\cdot(\mathbf{P}_i-\mathbf{P}_k'), \quad (7)$$

where $\mathbf{P}_i=\{1/[M+E_i(p)]\}\mathbf{p}$, and where $\boldsymbol{\sigma}_{\mathrm{I}}$ (and $\boldsymbol{\sigma}_{\mathrm{II}}$) are two commuting Pauli spin vector-matrices, relating to particles (1) and (2).

III. DISCUSSION OF MODEL (A)

We shall consider separately the singlet and the triplet states of our system, the corresponding eigenvalues of the isotopic spin operator being $T^{(1)}$ and $T^{(3)}$.

Case (1) Singlet States

We have $\varphi_{11}^{(1)}=\varphi_{11}^{(4)}=0$, $\varphi_{11}^{(2)}=-\varphi_{11}^{(3)}=B(\mathbf{p})$. The equation for B can easily be brought to the following form:

$$(E-2E_p)B(\mathbf{p})$$
$$=\lambda T^{(1)}\int\frac{1}{4E_pE_{p'}}\left[1-\frac{\mu^2}{\omega^2-(E_p-E_{p'})^2}\right]B(\mathbf{p}')d\mathbf{p}'. \quad (8)$$

The first term of the kernel goes over, in the nonrelativistic limit, to $1/4M^2$, that is, in coordinate space, to $(1/4M^2)\delta(\mathbf{x})$. It corresponds, therefore, to the "contact-term" of the static pseudoscalar interaction, the existence of which was first emphasized by Belinfante.[9] Its contribution vanishes for states of angular momentum different from zero. The second term of the kernel tends, in the nonrelativistic limit, to the ordinary Yukawa potential.

For the ^1S state, which is the only one to be physically interesting, we first look at the analytical behavior of the kernel of Eq. (8) for high values of the momenta. The first term is actually preponderent in the whole range of variation of p and p'. The second one is of the order of μ^2/M^2 and tends rapidly to zero for high momenta. Neglecting it, we get, apart from a normalization coefficient: $B(\mathbf{p})=[E_p(E-2E_p)]^{-1}$, and the value of E is determined by the equation,

$$\frac{1}{\lambda T^{(1)}}=\int\frac{d\mathbf{p}'}{4E_{p'}^2[E-2E_{p'}]}. \quad (9)$$

The integral of the right-hand side diverges logarithmically for high momenta (a nonrelativistic treatment of the "contact" term would have led to a linear divergence). The system, therefore, does not have any stable ^1S state.

We have to note, however, that, in momentum space, such an analytical examination as has just been made can be misleading. If the isotopic factor $T^{(1)}$ is chosen in such a way as to make the ordinary Yukawa potential attractive, the first term of the kernel of Eq. (8) represents a singular repulsive interaction, less singular, however, in the relativistic region, than a δ-function.[10] On the other hand, it can be shown, by means of the standard coordinate space method, that an interaction consisting of an ordinary attractive potential $V(r)$ plus a repulsive δ-function will allow for the existence of bound states. In this case, however, we shall show in the Appendix that a straightforward transformation of the problem into the momentum representation [which would lead to an equation analogous to (8)], is entirely misleading: this difficulty is due to the fact that the fourier transformation implies regularity conditions for the wave function which are too restrictive from the physical point of view.

However, in the case of a repulsive δ-function, a limiting process can be defined, by which the contribution of the singular potential is removed from the momentum space wave equation. The physical justification to this procedure is that, in fact, a genuine contact interaction can always be eliminated covariantly at the outset, by addition of a direct coupling term in the initial hamiltonian.[11] This is not the case in our problem, because the singular term of the kernel of Eq. (8) depends on momenta, and goes over to a δ-function only in the nonrelativistic limit. We have reached, therefore, the following conclusions:

(a) If Eq. (8) is considered as given, in the momentum representation, we can rightly say that it has no solution corresponding to a finite value of the coupling constant.

(b) Every limiting process which can be defined for suppressing the singular part of the kernel of Eq. (8) will have the aspect of an arbitrary cutoff at large momenta, with no physical justification whatever.

(c) This limiting process would, moreover, be misleading, because it could also be used in order to remove an attractive singular interaction, whereas, in this case, we know that no stable stationary states are possible. As will be seen in the study of the triplet states, an

[9] F. Belinfante, *Theory of Heavy Quanta* (Leiden thesis, 1939), p. 88.

[10] In order to have an idea of the singularity of this interaction, the passage to the "velocity dependent" operators of Wheeler [Phys. Rev. **50**, 643 (1936)] can be made, with the following result:

$$\int\frac{1}{4E_pE_{p'}}B(\mathbf{p}')d\mathbf{p}'\sim\int\frac{1}{rr'}K_1(Mr)K_1(Mr')b(\mathbf{x}')d\mathbf{x}',$$

where $b(\mathbf{x})$ is the fourier transform of $B(\mathbf{p})$. The right-hand side behaves like $1/r^2$ for small distances, provided that the integral on $\mathbf{x}'$ converges.

[11] An interaction of the form $(f_1/2M)^2\boldsymbol{\sigma}^{(\mathrm{I})}\cdot\boldsymbol{\sigma}^{\mathrm{II}}\delta(\mathbf{r})$ can be eliminated covariantly by writing the hamiltonian of the interacting nucleon and pseudoscalar meson fields in the following way:

$$H=H(\text{free nucleon})+H(\text{free meson})+H(\text{interaction})+H_{\mathrm{dir}},$$

where H_{dir} is given by

$$H_{\mathrm{dir}}=-(f_1^2/4M^2)\int\psi^*(1)\psi^*(2)\boldsymbol{\sigma}^{(\mathrm{I})}\cdot\boldsymbol{\sigma}^{(\mathrm{II})}\psi(1)\psi(2)d\mathbf{x}_1d\mathbf{x}_2$$

where $\psi(1)$ and $\psi(2)$ are the spinor-wave functions of the nucleons. In the calculations of Van Hove (see reference 6), there occurred other real "contact" terms which were cancelled in this way.

attractive singular interaction does occur, which we have no right to suppress.

These conclusions will be illustrated more precisely in the Appendix, where the superposition of a repulsive δ-function and a regular attractive potential is examined in momentum space.[12]

As a concluding remark, we note that the logarithmic divergence of the right-hand side of (9) is not due, as in the problem investigated by Ma,[13] to the cancellation of the quadratic terms in the denominators of the interaction (2), which become bilinear functions of p and p'. If this were true, the occurence of the divergence would be independent of the value of the angular momentum; furthermore, it would occur as well for the scalar case, and in Eq. (8), for the second term of the kernel. It is true that we lose two powers of p at high momenta, one on the left-hand side of the equation, which becomes linear in p, the other in the denominator of the interaction term. But, on the other hand, we gain two other powers of p, from the introduction of the factor $(4E_pE_{p'})^{-1}$. It can be seen that the divergence actually comes from the peculiar angular dependence of the matrix elements of ρ_2.

Case (2) Triplet States

To simplify the discussion, we are obliged, in this case, to separate the angular variables. Using the same notations as Kemmer,[14] we introduce the three symbolic vectors

$$\begin{aligned}(2j+1)^{\frac{1}{2}}\mathbf{X}_j{}^m&=\{[(j+m)_2]^{\frac{1}{2}}Y_j{}^{m-1};\\ &\quad -[(j+m+1)(j-m+1)]^{\frac{1}{2}}Y_j{}^m;\\ &\quad [(j-m)_2]^{\frac{1}{2}}Y_j{}^{m+1}\},\\ \mathbf{Y}_j{}^m&=\{-[(j+m)(j-m+1)]^{\frac{1}{2}}Y_j{}^{m-1};\\ &\quad -2mY_j{}^m;[(j+m+1)(j-m)]^{\frac{1}{2}}Y_j{}^{m+1}\},\\ (2j+1)^{\frac{1}{2}}\mathbf{Z}_j{}^m&=\{[(j-m)_2]^{\frac{1}{2}}Y_j{}^{m-1};\\ &\quad [(j+m)(j-m)]^{\frac{1}{2}}Y_j{}^m;\\ &\quad [(j+m)_2]^{\frac{1}{2}}Y_j{}^{m+1}\},\end{aligned} \tag{10}$$

where the notation $(n)_2$ means $n(n+1)$. The amplitude $\varphi_{11}(\mathbf{p})$ can now be considered also as a vector, the components of which are $\varphi_{11}{}^{(1)}$, $\varphi_{11}{}^{(2)}=\varphi_{11}{}^{(3)}$, $\varphi_{11}{}^{(4)}$. In polar coordinates, it is expressed as

$$\phi_{11}=\mathbf{X}_{j-1}{}^m A^{(1)}(p)+\mathbf{Z}_{j+1}{}^m A^{(2)}(p). \tag{11}$$

We use a reduced system of units: $\epsilon=E/2M$, $\rho=p/M$, $k^2=\mu^2/2M^2$, $\chi=1-k^2$, and, moreover, we put

$$A^{(1)}+A^{(2)}=R,\quad jA^{(1)}-(j+1)A^{(2)}=S. \tag{12}$$

The radial equations are then simply

$$\begin{aligned}&[\epsilon-(1+\rho^2)^{\frac{1}{2}}]R(\rho)\\ &=2\pi\lambda T^{(3)}\int_0^\infty\{[\mathcal{K}_j(\rho,\rho')-j\mathcal{L}_j{}^{(1)}(\rho,\rho')]R(\rho')\\ &\qquad+\mathcal{L}_j{}^{(2)}(\rho,\rho')S(\rho')\}\rho'^2d\rho',\end{aligned} \tag{13a}$$

$$\begin{aligned}&[\epsilon-(1+\rho^2)^{\frac{1}{2}}]S(\rho)\\ &=-2\pi\lambda T^{(3)}\int_0^\infty\{[\mathcal{K}_j(\rho,\rho')-(j+1)\mathcal{L}_j{}^{(1)}(\rho,\rho')]S(\rho')\\ &\qquad+j(j+1)\mathcal{L}_j{}^{(2)}(\rho,\rho')R(\rho')\}\rho'^2d\rho',\end{aligned} \tag{13b}$$

with

$$\begin{aligned}\mathcal{K}_j&=(P^2+P'^2)\Delta_{j-1}-2PP'\Delta_j,\\ \mathcal{L}_j{}^{(1)}&=(P^2+P'^2)(\Delta_{j-1}-\Delta_{j+1})/(2j+1),\\ \mathcal{L}_j{}^{(2)}&=(P^2-P'^2)(\Delta_{j-1}-\Delta_{j+1})/(2j+1),\end{aligned} \tag{14}$$

where we have put $P=|\mathbf{P}_1|$ and

$$\Delta_j(\rho,\rho')=\frac{[1+(1+\rho^2)^{\frac{1}{2}}][1+(1+\rho'^2)^{\frac{1}{2}}]}{4[(1+\rho^2)(1+\rho'^2)]^{\frac{1}{2}}}\frac{1}{\rho\rho'}Q_j(U_1). \tag{15}$$

Here, $U_1=(1/\rho\rho')[(1+\rho^2)^{\frac{1}{2}}(1+\rho'^2)^{\frac{1}{2}}-\chi]$, and Q_j is the Legendre function of the second kind. In particular, one has:

$$Q_0(U_1)=\tfrac{1}{2}\,\mathrm{Log}[(U_1+1)/(U_1-1)]. \tag{16}$$

For $j=1$ (ground state of the deuteron), we can write explicitly:

$$\begin{aligned}\mathcal{K}_1&=\frac{1}{4[(1+\rho^2)(1+\rho'^2)]^{\frac{1}{2}}}\left[1-\frac{k^2}{\rho\rho'}Q_0(U_1)\right],\\ \mathcal{L}_1{}^{(1)}&=\frac{1}{4[(1+\rho^2)(1+\rho'^2)]^{\frac{1}{2}}}\cdot\frac{[(1+\rho^2)(1+\rho'^2)]^{\frac{1}{2}}-1}{2\rho\rho'}\\ &\qquad\times[U_1-(U_1{}^2-1)Q_0(U_1)],\\ \mathcal{L}_1{}^{(2)}&=\frac{1}{4[(1+\rho^2)(1+\rho'^2)]^{\frac{1}{2}}}\cdot\frac{1}{2\rho\rho'}\left[\frac{\rho^2}{1+(1+\rho^2)^{\frac{1}{2}}}\right.\\ &\qquad\left.-\frac{\rho'^2}{1+(1+\rho'^2)^{\frac{1}{2}}}\right][U_1-(U_1{}^2-1)Q_0(U_1)].\end{aligned} \tag{17}$$

U_1 tends to $k^2/\rho\rho'$ for small values of ρ and ρ', and to 1 for high momenta. $\mathcal{K}_1$ includes, in the nonrelativistic limit, the contact interaction and the Yukawa potential. $\mathcal{L}_1{}^{(1)}$ and $\mathcal{L}_1{}^{(2)}$ include terms which have, for low momenta, the characteristic r^{-3} and r^{-2} behavior.[15] For

[12] We gratefully acknowledge interesting discussions with Dr. Oppenheimer, in connection with this point.

[13] S. T. Ma, Phys. Rev. **82**, 275 (1951).

[14] N. Kemmer, Helv. Phys. Acta **10**, 47 (1937).

[15] It should be noted that it is rather ambiguous to speak, in the momentum representation, of a singularity equivalent to r^{-3}. The r^{-3} interaction, being too strongly singular at the origin, cannot be fourier transformed, unless an appropriate convergence factor is used. In this case, the "fourier transform" of r^{-3} is defined as a constant, and, therefore, cannot be distinguished from the fourier transform of a three-dimensional δ-function.

high momenta, the expression corresponding to the first term in the bracket of $\mathfrak{L}_1^{(1)}$ has the same structure as the first term in $\mathfrak{K}_1$, but differs by a factor of one-half. Therefore, these two singular terms cancel each other in the kernel of Eq. (13b), but not in (13a). More precisely, we can neglect the terms in $(k^2/\rho\rho')Q_0(U_1)$ and $(U_1^2-1)Q_0(U_1)$, which are very small compared to unity, and write:

$$R(\rho)\sim a/\rho^2+\mathcal{O}(1/\rho^3), \quad S(\rho)\sim b/\rho^2+\mathcal{O}(1/\rho^3). \quad (18)$$

Defining also:

$$c_n\sim T^{(3)}\int^{\infty} d\rho'/\rho'^n, \quad (19)$$

(c_1 is logarithmically divergent, c_2 and c_3 are convergent), we obtain the following set of equations:

$$\begin{aligned}(1+\tfrac{1}{4}\pi\lambda c_1)a+\tfrac{1}{4}\pi\lambda c_2 b=0,\\ (1+\tfrac{1}{2}\pi\lambda c_3)b-\tfrac{1}{2}\pi\lambda c_2 a=0,\end{aligned} \quad (20)$$

together with the condition $a\neq 0$. The Eq. (20) imply therefore that $\lambda\sim 1/c_1$, which means that the system tends to collapse for small separations of the nucleons. The logarithmic divergence represented by c_1 is not removed if the "contact" term is dropped, because of a term in $\mathfrak{L}_1^{(1)}$, which has the same structure for high momenta, and which, in the nonrelativistic limit, includes the $1/r^3$ interaction. (This term would lead to a linear divergence, if treated nonrelativistically.) As was noted before, this term, being attractive, cannot be removed by means of a limiting process analogous to the one which is defined in the Appendix.

IV. DISCUSSION OF MODEL (B)

In this model, the whole set of amplitudes φ_{ij} has to be considered. The kernel of Eq. (5), defined by (6), takes three distinct values according to the possibilities: $i=k$, $j=l$; $i\neq k$, $j\neq l$; and $i=k$, $j\neq l$ (which is equivalent to $i\neq k$, $j=l$). The consequence is that the triplet and the singlet states cannot, in general, be treated separately. However, we are only interested, for the moment, in the behavior of the equations for $p\gg M$. In this extreme relativistic case, the triplet and singlet systems can be separated; furthermore, inside each of these corresponding reduced sets of equations, the contributions of the "even" and "odd" parts of the amplitude φ can also be studied separately, which is very convenient for the discussion of the results.

Case (1) Singlet States

The amplitude $\varphi(\mathbf{p})$ can be considered, in this case as an antisymmetrical tensor. The non-vanishing components can be written as: $\varphi_{11}^{(2)}=-\varphi_{11}^{(3)}=\frac{1}{2}(A+B)$, $\varphi_{22}^{(2)}=-\varphi_{22}^{(3)}=\frac{1}{2}(A-B)$; $\varphi_{12}^{(1)}=-\varphi_{21}^{(1)}=C_1$, $\varphi_{12}^{(3)}=-\varphi_{21}^{(2)}=\varphi_{12}^{(2)}=-\varphi_{21}^{(3)}=C_2$, $\varphi_{12}^{(4)}=-\varphi_{21}^{(4)}=C_3$.

(a) Equations for the "Even" Components

These equations are very similar to (8). We set the following definition:

$$D_{\pm}(\mathbf{p},\mathbf{p}')=\frac{M^2}{2}\left[\frac{1}{\omega^2-(E_p-E_{p'})^2}\pm\frac{1}{\omega^2-(E_p+E_{p'})^2}\right], \quad (21)$$

and we have the equations:

$$\begin{aligned}EA-2E_pB&=\lambda T^{(1)}\int\frac{\mu^2}{M^2}\cdot\frac{1}{2E_pE_{p'}}D_-(\mathbf{p},\mathbf{p}')A(\mathbf{p}')d\mathbf{p}',\\ EB-2E_pA&=-\lambda T^{(1)}\int\frac{1}{2E_pE_{p'}}\\ &\quad\times\left[1-\frac{\mu^2}{M^2}D_+(\mathbf{p},\mathbf{p}')\right]B(\mathbf{p}')d\mathbf{p}'.\end{aligned} \quad (22)$$

In the nonrelativistic limit, D_+ goes over to the ordinary Yukawa potential and D_- goes to zero. Neglecting terms of the order of μ^2/M^2, we have, therefore, the approximate solutions: (apart from a normalization coefficient): $A=[(E^2-4E_p^2)]^{-1}$, $B=(E/2E_p)A$. For large momenta, B is very small compared to A, and we see that, for the latter, the "contact" terms arising from the contributions of the two even parts φ_{11} and φ_{22} have cancelled each other. There exist a finite eigenvalue for the energy-level which is roughly given by:

$$\frac{1}{\lambda T^{(1)}}=-\frac{E}{2}\int\frac{d\mathbf{p}'}{E_{p'}^2[E^2-4E_{p'}^2]}. \quad (23)$$

(b) Equations for the "Odd" Components

In this group of equations, there appear component of angular moment $j-1$ and $j+1$. Denoting the three functions C_1, C_2, C_3 by a single symbolic vector $\mathbf{C}$, we can write, in polar coordinates, the angular dependence of these functions as:

$$\mathbf{C}=\mathbf{X}_{j-1}{}^mH_1(p)+\mathbf{Z}_{j+1}{}^mH_2(p). \quad (24)$$

For $j=0$ (^1S state), only the second term of the right-hand side is different from zero, and the corresponding equation, with the reduced units of Sec. III, case (2), is the following:

$$\epsilon H_2=2\pi\lambda T^{(1)}\int_0^{\infty}[1-(U_1-1)Q_0(U_1)-(U_2-1)Q_0(U_2)]\times\frac{H_2(\rho')\rho'^2d\rho'}{4[(1+\rho^2)(1+\rho'^2)]^{\frac{1}{2}}}, \quad (25)$$

where we have put: $U_2=(1/\rho\rho')\{[(1+\rho^2)(1+\rho'^2)]^{\frac{1}{2}}+\chi\}$. It is found that, in this case, the contribution to the "contact" terms arising from the odd components of the amplitude add together, and the integral of the right-hand side is linearly divergent. The reason for this

even stronger singularity is that the left-hand side of Eq. (25) is independent of the momentum, instead of being linear in p as in the case of the even components.

Case (2) Triplet States

We shall in this case, only summarize the results, which are entirely analogous to what we have seen before.

(a) There appear, in the equations, two kinds of singular terms which, in the nonrelativistic limit, include the contact interaction and the r^{-3} potential. In the extremely relativistic limit ($p \gg M$), these interaction terms produce logarithmic divergences for the even components and linear divergences for the odd components.

(b) The preceding singularities cancel each other completely in the contributions from the even components and, on the contrary, add together in the contributions from the odd ones.

At first sight, it seems, therefore, that there exists a possibility to allow for the existence of stationary states of the system: the odd components φ_{12} and φ_{21} of the amplitude $\varphi(\mathbf{p})$ must be set equal to zero. However, in the formalism of the "one-particle" model, this possibility is only apparent. Indeed, we have seen that the separation of the equations into separate systems for the odd and even components can only be made in the extreme relativistic limit. If one sets, therefore, $\varphi_{12}=\varphi_{21}=0$, the equations corresponding to intermediate values of the momenta will imply that the whole amplitude $\varphi(\mathbf{p})$ has to be equated to zero.[16] Consequently, there does not exist, in model (B), any stable stationary states of the neutron-proton system.

V. SUMMARY AND CONCLUSION

The results of the preceding investigation can be summarized as follows: In the pseudoscalar meson theory, a retarded interaction between two nucleons, valid to the second order in the coupling constant, has been considered. In this interaction, two essential approximations have been made: (a) The creation of virtual nucleons pairs is excluded; (b) in the momentum representation of the interaction, the fourth component of the momenta is identified with the free energy of the corresponding nucleon.

This interaction has been introduced into a two-particle Dirac equation, and the transitions between free states of the nucleous have been treated in two ways, according to whether the negative energy states are supposed to be filled or empty. In both cases, it has been shown that the interaction, although less singular than the static pseudoscalar potential, does still not allow for stable stationary states of the deuteron. On the other hand, it is well known that this interaction leads, in the Born approximation, to an unacceptable description of the neutron-proton scattering at high energies. The evidence obtained so far seems, therefore, to indicate that this interaction [with approximations (a) and (b)] is completely unsuited to the relativistic treatment of the neutron-proton system.

This conclusion can be substantially modified if a different approach to the problem, with less restrictive approximations than (a) and (b), is used. This can be done, for example, by means of the method which has been used by Tamm[1] and Dancoff[2] for the scalar meson interaction, or by means of the covariant equation which has recently been proposed by Bethe and Salpeter.[17] In the Tamm-Dancoff method, it has been found by Dancoff himself (unpublished) and by the author, that, in the pseudoscalar theory, the second-order interaction will probably allow for stationary states of the deuteron. However, one does not know whether such states will be acceptable (especially in view of the scattering experiments) or whether the corresponding value of the coupling constant (fitted to give the correct value of the ground-state binding energy) is sufficiently small to make a second order calculation consistent. These problems are being investigated, and the results will be reported later.

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APPENDIX. THE DEFINITION OF A REPULSIVE CONTACT INTERACTION IN THE MOMENTUM REPRESENTATION

We consider, in coordinate space, the Schrödinger equation for a system of two particles interacting through a potential represented, as a function of the relative distance, in Fig. 1. This potential consists of an ordinary attractive interaction $V(r)$, plus a singular repulsive potential $G_a(r)$, with a very short-range a and

[16] This conclusion would be different if, instead of Eqs. (5), we had a system of eight equations only, with an amplitude including only the eight "even" components. Such a system of equations can be justified, to some extent, in the following way:

Starting from the covariant integral equation for two particles which has been proposed by Bethe and Salpeter (see reference 17), one makes the approximation (*b*) of Sec. I, in the interaction. With this approximation, it is possible to derive an equation for an amplitude corresponding to equal times for the two particles (that is, the amplitude for the two-particle part of the state which is considered). This equation is apparently identical in form to Eq. (5), but is only equivalent to a system of eight equations, the amplitude itself including only the "even" components. Therefore, this special equation would allow for the existence of stationary states of the system. However, the approximation (*b*) of Sec. I appears, in the equation of Bethe and Salpeter, as artificial and superfluous. It seems more promising, in this case, to have the equation for the amplitude corresponding to different times of the particles solved as it stands, the times being set equal only afterwards, in the solution itself. Such an investigation is being carried on, and the results will be reported later.

[17] H. A. Bethe and E. Salpeter, Phys. Rev. **82**, 309 (1951), and private communication. See also, Y. Nambu, Prog. Theor. Phys. **5**, 614 (1950), and M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).

a strength proportional to a^{-3}. One has therefore:

$$\lim_{a\to 0} G_a(r)\sim\delta(\mathbf{r}). \tag{A1}$$

The fourier transform of $V(r)$ will be denoted by $W(p)$; we list below some fourier transforms of $G_a(r)$ for different shapes of the coordinate potential:

$$\begin{aligned}&\text{Well:} && U_a^w(p)=(3U_0/a^3p^3)(\sin ap-ap\cos ap),\\ &\text{Gaussian:} && U_a^g(p)=U_0\exp(-a^2p^2),\\ &\text{Exponential:} && U_a^e(p)=U_0/(1+a^2p^2)^2.\end{aligned} \tag{A2}$$

All these functions tend to U_0 for $a\to 0$, which we take as the fourier transform of $U_0\delta(r)$.

In coordinate space, we denote by $\bar{\psi}(r)$ the wave function of the system, in the absence of the singular interaction $G_a(r)$. We suppose that it has been normalized in such a way as $\bar{\psi}(0)=1$. In the presence of $G_a(r)$, the corresponding wave function $\psi_a(r)$ has, when $a\to 0$, the following limit:

$$\psi_0(r)=\lim_{a\to 0}\psi_a(r)=\begin{cases}\bar{\psi}(r) & \text{for } r>0,\\ 0 & \text{for } r=0.\end{cases} \tag{A3}$$

$\psi_0(r)$ has therefore a singularity at the origin.

In momentum space, we denote by $\bar{\Phi}$, Φ_a and $\Phi_0=\lim_{a\to 0}\Phi_a$, the fourier transforms of $\bar{\psi}$, ψ_a and ψ_0 respectively. A completely "naive" translation of (A3) could be written:

$$\Phi_0=\bar{\Phi}(p), \quad \int\Phi_0(\mathbf{p})d\mathbf{p}=0, \tag{A4}$$

because these conditions would be fulfilled only for very unusual forms of $V(r)$. Instead, we write, in momentum space, the Schrödinger equations satisfied by $\bar{\Phi}$ and Φ_a:

$$(p^2-Mw)\bar{\Phi}(\mathbf{p})+(2\pi)^{-3}M\int W(\mathbf{p}-\mathbf{p}')\times\bar{\Phi}(\mathbf{p}')d\mathbf{p}'=0, \tag{A5}$$

$$(p^2-Mw)\Phi_a(\mathbf{p})+(2\pi)^{-3}M\int[U_a(\mathbf{p}-\mathbf{p}')+W(\mathbf{p}-\mathbf{p}')]\Phi_a(\mathbf{p}')d\mathbf{p}'=0, \tag{A6}$$

(w being the binding energy), and take the limit of the last equation as follows:

$$(p^2-Mw)\Phi_0(\mathbf{p})+(2\pi)^{-3}M\int W(\mathbf{p}-\mathbf{p}')\Phi_0(\mathbf{p}')d\mathbf{p}' = -\lim_{a\to 0}(2\pi)^{-3}M\int U_a(\mathbf{p}-\mathbf{p}')\Phi_a(\mathbf{p}')d\mathbf{p}'=0. \tag{A7}$$

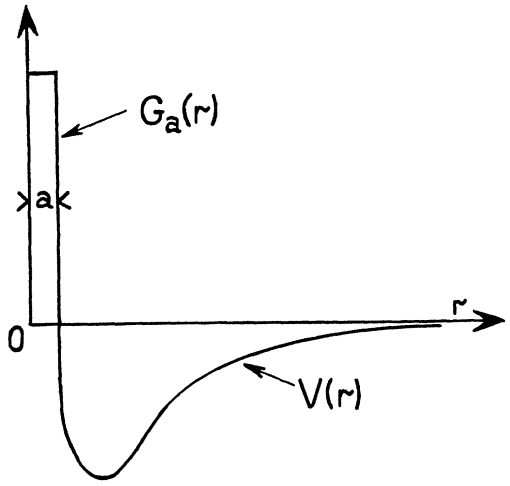

FIG. 1. Coordinate space potential as a function of the relative distance.

This shows that the first condition (A4) is correct, but that the second one must be replaced by the second part of (A7). We see also that the effect of the singular interaction is completely eliminated from the momentum representation by the limiting process of (A7). This limiting process, if we look at the analytical form of the functions given in (A2), appears merely as a more or less strong cutoff which makes to vanish otherwise divergent integrals.

In order to understand what the situation is, with respect to Eq. (8) of the text, we assume for a while that the limit of the second term of (A6) can be taken as the product of the limits of its constituents. We obtain an equation:

$$(p^2-Mw)\Phi_0(\mathbf{p})+(2\pi)^{-3}M\int[U_0+W(\mathbf{p}-\mathbf{p}')]\Phi_0(\mathbf{p}')d\mathbf{p}'=0 \tag{A8}$$

which is analogous to Eq. (8) of the text. (It coincides with it in the nonrelativistic limit, if $U_3=p^2/4M^2$, and if $V(r)$ is the ordinary Yukawa potential.) An analytical examination of this equation analogous to our procedure of Sec. III, shows, as it must be, that it has no solution for a value of U_0 different from zero. The momentum representation is therefore, in this case, misleading, because (A8) is the equation which we would have written if we had started directly with $\delta(\mathbf{r})$, instead of defining it at the limit of $G_a(r)$. On the other hand, the limiting process of (A7) is also, in a sense, misleading, because, if $G_a(r)$ were attractive, Eq. (A7) would still have a solution, because the limit of the right-hand side is independent of the sign of $U_a(p)$.