A Remark on the Conventional Perturbation Theory

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It is shown that Dyson's S-matrix theory can also be attained by the perturbation method used in the conventional field theory formalism.

I. INTRODUCTION

ONE can show explicitly in a step-by-step way that the conventional perturbation theory is just Dyson's S-matrix theory[1] and that the Feynman[2]-Dyson rules are the rules which have been used conventionally. Recently, Yang and Feldman[3] and Källen[4] have succeeded in formulating the S-matrix theory directly in the Heisenberg representation; our arguments have then the same advantages as theirs because we start from the Schrödinger representation.

II. RÉSUMÉ OF THE CONVENTIONAL PERTURBATION METHOD

The Schrödinger representation will be used; the hamiltonian density $H(\mathbf{x})$ and the hamiltonian

$$\bar{H}=\int d^3xH(\mathbf{x})$$

are all time-independent. The expression $\Phi(t)$ denotes the Schrödinger time-dependent functional; $\{\Phi_n{}^0(q)\}$ is a complete orthonormal set of solutions of the stationary Schrödinger eigenvalue problem $\bar{H}_0\Phi_n{}^0(q)=E_n\Phi_n{}^0(q)$. The index n indicates the set of quantum numbers which characterize the state $\Phi_n{}^0(q)$. Further, we shall always use the subscripts 0 and 1 to refer, respectively, to free fields and interaction terms.

Instead of putting the initial time $t_0=0$ as is usually done, we shall take $t_0=-\infty$. Certain precautions are then to be taken concerning the indeterminacy of $\exp(-i\hbar^{-1}E_nt)$ at $t=\pm\infty$. However, they are conventional; on some occasions one multiplies the integrand by a factor $\exp(-\epsilon|t|)$ with $\epsilon>0$ and one then takes the limit $\epsilon\rightarrow+0$ after the integration, while in other cases one may simply imagine that $\pm\infty$ is $\pm T$, because the value of T will not appear in the final results or $\exp(\pm i/\hbar E_nT)$ will merely be a constant phase factor.

The following Eqs. (1) and (2) are well known.[5] Let

$$\Phi(t)=\sum\nolimits_n A_n(t)\exp(-i/\hbar E_nt)\Phi_n{}^0(q).$$

From the Schrödinger equation

$$i\hbar\partial\Phi(t)/\partial t=(\bar{H}_0+\bar{H}_1)\Phi(t),$$

we have

$$i\hbar\dot{A}_f(t)=\sum\nolimits_i A_i(t)\exp(i/\hbar(E_f-E_i)t)\bar{H}_{fi}{}^1, \tag{1}$$

where

$$\bar{H}_{fi}{}^1=(\Phi_f{}^0,\bar{H}_1\Phi_i{}^0).$$

Let us define

$$\bar{H}_{fi}{}^1=\lim_{\epsilon\rightarrow1-0}(\epsilon\bar{H}_{fi}{}^1)=\bar{H}_{fi}{}^1$$

and

$$A_i(t)=\lim_{\epsilon\rightarrow1-0}[A_i{}^{(0)}(t)+\epsilon A_i{}^{(1)}(t)+\epsilon^2A_i{}^{(2)}(t)+\cdots]$$
$$=A_i{}^{(0)}(t)+A_i{}^{(1)}(t)+A_i{}^{(2)}(t)+\cdots.$$

We obtain as a solution of (1):

$$A_f{}^{(0)}(t)=A_f{}^{(0)}(-\infty)=\text{const.},$$

$$A_f{}^{(k)}(t)=(1/i\hbar)^k\sum\nolimits_i\bigg\{\sum\nolimits_{\phi(k-1)}\sum\nolimits_{\phi(k-2)}\cdots\sum\nolimits_{\phi(1)}\int_{-\infty}^{t}d\tau_k\bar{H}^1{}_{f\phi(k-1)}\exp[i\hbar^{-1}(E_f-E_{\phi(k-1)})\tau_k]$$
$$\times\int_{-\infty}^{\tau_k}d\tau_{k-1}\bar{H}^1{}_{\phi(k-1)\phi(k-2)}\exp[i\hbar^{-1}(E_{\phi(k-1)}-E_{\phi(k-2)})\tau_{k-1}]\cdots$$
$$\times\int_{-\infty}^{\tau_2}d\tau_1\bar{H}^1{}_{\phi(1)i}\exp[i\hbar^{-1}(E_{\phi(1)}-E_i)\tau_1]\bigg\}A_i{}^{(0)}(-\infty), \tag{2}$$

where the superscripts $k=1,2,\cdots$ indicate the order of approximation. One may also write

$$A_m(t)=\sum\nolimits_n\bar{S}_{mn}{}'(t)A_n{}^{(0)}(-\infty), \tag{3}$$

which at the same time serves to define the matrix

$$\bar{S}'(t)=\{\bar{S}_{mn}{}'(t)\}.$$

We note that $\bar{S}'(-\infty)=1$, and since in the S-matrix theory only stationary problems are considered, $\bar{S}'(\infty)$ (with a slight change as will be mentioned later) can be

[1] F. J. Dyson, Phys. Rev. **75**, 486 (1949); **75**, 1736 (1949).
[2] R. P. Feynman, Phys. Rev. **76**, 749 and 769 (1949).
[3] C. N. Yang and D. Feldman, Phys. Rev. **79**, 972 (1950).
[4] G. Källen, Arkiv f. Fysik **19**, 187 (1950).
[5] W. Pauli, Handbuch der Physik Bd. **24** (2te Auflage), p. 158.

shown to be equivalent to Dyson's S-matrix $U(\infty)$. Since $\bar{H}_1=\int d^3xH_1(\mathbf{x})$ is time-independent and

$$\int_{-\infty}^{\infty}d\tau_k\exp[ih^{-1}(E_f-E_{\phi(k-1)})\tau_k]\int_{-\infty}^{\tau_k}d\tau_{k-1}\exp[ih^{-1}(E_{\phi(k-1)}-E_{\phi(k-2)})\tau_{k-1}]\cdots\int_{-\infty}^{\tau_2}d\tau_1\exp[ih^{-1}(E_{\phi(1)}-E_i)\tau_1]$$

$$=\frac{\int_{-\infty}^{\infty}d\tau_k\exp[ih^{-1}(E_f-E_i)\tau_k]}{(ih^{-1})^{k-1}(E_{\phi(k-1)}-E_i)(E_{\phi(k-2)}-E_i)\cdots(E_{\phi(1)}-E_i)}=\frac{2\pi}{(ih^{-1})^{k-1}}\cdot\frac{h\delta(E_f-E_i)}{(E_{\phi(k-1)}-E_i)\cdots(E_{\phi(1)}-E_i)}$$

where, in doing the integration, we have made use of the device

$$\lim_{\epsilon\to+0}[\exp(-\epsilon|t|)].$$

In essence, this just corresponds to throwing away the transients in the customary procedure. We observe that the so-called matrix element of kth order in the conventional perturbation theory, multiplied by the δ-function between the energies of the initial and final state, can be also written as follows:

$$ME^{(k)}\delta(E_f-E_i)=\sum\nolimits_{\phi(k-1)}\sum\nolimits_{\phi(k-2)}\cdots\sum\nolimits_{\phi(1)}\frac{\bar{H}^1{}_{f\phi(k-1)}\bar{H}^1{}_{\phi(k-1)\phi(k-2)}\cdots\bar{H}^1{}_{\phi(1)i}\delta(E_f-E_i)}{(E_i-E_{\phi(k-1)})\cdots(E_i-E_{\phi(1)})}$$

$$=(1/(ih)^{k-1}2\pi h)\sum\nolimits_{\phi(k-1)}\cdots\sum\nolimits_{\phi(1)}\int_{-\infty}^{\infty}d\tau_k\bar{H}^1{}_{f\phi(k-1)}\exp[ih^{-1}(E_f-E_{\phi(k-1)})\tau_k]\cdots$$

$$\times\int_{-\infty}^{\tau_2}d\tau_1P^1{}_{\phi(1)i}\exp[ih^{-1}(E_{\phi(1)}-E_i)\tau_1].\quad(4)$$

Our program is first to show that the customary method of calculation leads exactly to Feynman-Dyson's rules, and then to point out that the S-matrix attained in our way is in full agreement with Dyson's.

III. EQUIVALENCE OF THE CONVENTIONAL METHOD OF CALCULATION AND FEYNMAN-DYSON'S RULES

We will restrict our discussion to quantum electrodynamics. We shall use Heaviside and natural ($\hbar=c=1$) units, and a repeated Greek index is to be summed over 1 to 4, while a repeated Latin index is to be summed only over 1 to 3. Since we follow the conventional formulation of field theory, the groundwork is well known.[6] The total hamiltonian density may be written in the form

$$H(\mathbf{x})=H_0(\mathbf{x})+H_1(\mathbf{x})$$

where, with $\bar{\psi}=\psi^*\beta$, we have

$$H_1(\mathbf{x})=ie\bar{\psi}(\mathbf{x})\gamma_\mu\psi(\mathbf{x})A_\mu(\mathbf{x});\quad(\mathbf{x}=(x,y,z)).^7\quad(5)$$

The quantization of the electron field is the ordinary one according to Pauli's exclusion principle.[7] For the electromagnetic field we shall follow a treatment by Gupta,[8] where an indefinite metric has been introduced for the scalar photon part. Let us introduce the following fourier expansions:

$$\psi(\mathbf{x})=(2\pi)^{-\frac{3}{2}}\int d^3p\sum_{s=1}^{4}f(\mathbf{p},s)u(\mathbf{p},s)\exp(i\mathbf{p}\cdot\mathbf{x}),$$

$$\psi^*(\mathbf{x})=(2\pi)^{-\frac{3}{2}}\int d^3p\sum_{s=1}^{4}f^*(\mathbf{p},s)u^*(\mathbf{p},s)\exp(-i\mathbf{p}\cdot\mathbf{x}),$$

$$A_i(\mathbf{x})=(2\pi)^{-\frac{3}{2}}\int d^3q[2|\mathbf{q}|]^{-\frac{1}{2}}\times\{C_i(\mathbf{q})\exp(i\mathbf{q}\cdot\mathbf{x})+C_i^+(\mathbf{q})\exp(-i\mathbf{q}\cdot\mathbf{x})\},$$

$$A_0(\mathbf{x})=i^{-1}A_4(\mathbf{x})=(2\pi)^{-\frac{3}{2}}\int d^3q[2|\mathbf{q}|]^{-\frac{1}{2}}\times\{C\ (\mathbf{q})\exp(i\mathbf{q}\cdot\mathbf{x})+C_0^+(\mathbf{q})\exp(-i\mathbf{q}\cdot\mathbf{x})\},$$

where $u^*(p,s)$ and $u(p,s)$ are the Dirac plane wave spinors, and an asterisk will always be used to denote a hermitian conjugate, while a dagger denotes an adjoint. $f^*(p,s)$ and $f(p,s)$ are, respectively, the creation and annihilation operators for the electron field, $f^*(p,s)f(p,s)$ represents the number of electrons of momentum $\mathbf{p}$ at the spin state s (due to our fourier expansions this number is to be understood, here as well as for the electromagnetic field, as the number per unit volume in wave number space), and $[f(p,s),f^*(p',s')]_+=\delta_{pp'}\,\delta_{ss'}$ etc. For the photon field, C_μ and C_μ^+ are still annihilation and creation operators ($\mu=1$, 2, 3, and 4), while $C_i^+C_i$ ($i=1$, 2, or 3) represents the number of photons associated with the ith component; the number of scalar photons is now given by $-C_0^+C_0$. The com-

[6] G. Wentzel, *Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1949).

[7] Following Dyson (reference 1) we have deliberately not chosen the charge conjugate expression.

[8] S. N. Gupta, Proc. Phys. Soc. (London) **63**, 681 (1950); also K. Bleuler, Helv. Phys. Acta **23**, 567 (1950). Note: Our intention is only to obtain $\langle P(A_\mu(x)A_\nu(y))\rangle_0=\frac{1}{2}\delta_{\mu\nu}D_F(x-y)$ in a simple and more satisfactory way; that this can be equally well accomplished for our purpose without introducing any indefinite metric, see S. T. Ma, Phys. Rev. **80**, 729 (1950) and other literature quoted by him.

mutation relations are

$$[C_1(q), C_1^+(q)]_- = [C_2(q), C_2^+(q)]_- = [C_3(q), C_3^+(q)]_- = 1,$$
$$[C_0(q), C_0^+(q)]_- = -1,$$

with all other commutators being zero. As far as the perturbation calculation is concerned, no complications are involved.[8]

Before proceeding, we would like to make the following remarks:

$$\bar{H}^1_{\Phi(j)\Phi(j-1)} \exp[i(E_j - E_{j-1})t]$$
$$= (\Phi_j^0, \bar{H}_1 \exp[i(E_j - E_{j-1})t]\Phi_{j-1}^0) = (\Phi_j^0, \bar{H}_1(t)\Phi_{j-1}^0),$$

where $\bar{H}_1(t) = \int d^3x H_1(\mathbf{x}, t)$ and $H_1(\mathbf{x}, t)$ is obtained from $H_1(x)$ by the substitution that replaces every $\exp(i\mathbf{k}\cdot\mathbf{x})$ by $\exp(ik_\mu x_\mu)$ and every $\exp(-i\mathbf{k}\cdot\mathbf{x})$ by $\exp(-ik_\mu x_\mu)$, with $x_\mu = (\mathbf{x}, it)$, $k_\mu = (\mathbf{k}, ik_0)$ and $\mathbf{k}$ standing either for $\mathbf{p}$ or for $\mathbf{q}$. It is clear that this is true. For instance, if Φ_{j-1}^0 goes to Φ_j^0 by the transition in which a positive energy electron (p_1, $s_1 = 1$ or 2) absorbs a transverse photon (q, $i = 1$ or 2, say) and goes to a new positive energy state (p_2, $s_2 = 1$ or 2), then

$$E_j - E_{j-1} = (p_2^2 + K^2)^{\frac{1}{2}} - \{(p_1^2 + K^2)^{\frac{1}{2}} + |q|\}.$$

We also have

$$(\Phi_j^0, \bar{H}_1 \exp[i(E_j - E_{j-1})t]\Phi_{j-1}^0)$$
$$= \Big(\Phi_j^0, (2\pi)^{-9/2} ie \int d^3x f^*(\mathbf{p}_2, s_2) f(\mathbf{p}_1, s_1) \times \bar{u}(\mathbf{p}_2, s_2)\gamma_i u(\mathbf{p}_1, s_1) C_i(\mathbf{q})[2|\mathbf{q}|]^{-\frac{1}{2}} \times \exp[-i\mathbf{p}_2\cdot\mathbf{x} + i(p_2^2 + K^2)^{\frac{1}{2}} t + i\mathbf{p}_1\cdot\mathbf{x} - i(p_1^2 + K^2)^{\frac{1}{2}} t + i\mathbf{q}\cdot\mathbf{x} - i|q|t]\Phi_{j-1}^0\Big)$$
$$= \Big(\Phi_j^0, \int d^3x ie\bar{\psi}(\mathbf{x}, t)\gamma_\mu\psi(\mathbf{x}, t)A_\mu(\mathbf{x}, t)\Phi_{j-1}^0\Big),$$

because the other terms in the corresponding fourier expansions of $\bar{\psi}(\mathbf{x})$, $\psi(\mathbf{x})$ and $A_\mu(\mathbf{x})$ contribute nothing by the orthogonality of $\{\Phi_n^0(q)\}$. K is the mass of electron and i is either 1 or 2 and not to be summed.

$$\delta_+(a) = (2\pi)^{-1}\int_0^\infty dt \exp(-iat) = 1/2\delta(a) + 1/2\pi ia$$

$$\times \int_{-\infty}^{\infty} dk_0 \exp(\pm ik_0 t)\delta_+(k_\mu^2 + K^2) = \begin{cases} \exp(-i|E|t)/2|E| & \text{if } t > 0 \\ \exp(+i|E|t)/2|E| & \text{if } t < 0 \end{cases} \quad (6_1)$$

$$\times \int_{-\infty}^{\infty} dk_0 \exp(\pm ik_0 t)\delta_+(k_\mu^2 + K^2)k_0 = \begin{cases} \mp(1/2)\exp(-i|E|t) & \text{if } t > 0 \\ \pm(1/2)\exp(+i|E|t) & \text{if } t < 0, \end{cases} \quad (6_2)$$

where $E = +(|\mathbf{k}|^2 + K^2)^{\frac{1}{2}}$. The derivation of the above equations is trivial.

In the conventional perturbation calculation we observe that in order to accomplish the transition from the initial state (i) to the final state (f), we must have in our matrix element (ME) enough operators to annihilate i and enough operators to create f. Following Dyson, we shall call them free operators. The rest of the operators in ME just cause the so-called virtual effects; in order to yield a nonzero contribution to ME and so to the physical problem under consideration, these operators should occur in pairs, each of which refers to a virtual particle. Customarily, one calculates ME by listing schemes which take account of all the possible ways belonging to the same order of approximation to reach f from i *via* virtual states. Because of momentum conservation ensured by the δ-function contained in $\int d^3x H_1(\mathbf{x})$, some momentum sums reduce to a single term and the calculation becomes easier. We shall keep these δ-functions explicitly, not carrying out the sums over momenta in which they occur. Virtual particles are those that did not exist in the initial state and therefore must have been created first in order to be annihilated (this just corresponds to Dyson's order "P").[1] This statement involves no ambiguity concerning the photon field, because we have followed Gupta's method of quantization.[8,9] However, we can first annihilate one of the negative energy electrons which are always available and fill up the hole again, but then the other way around is prohibited (exclusion principle); in this case the intermediate holes will be thought of as virtual particles. By Eq. (6_1) the contribution to ME due to a virtual photon operator pair (summed over all intermediate states of the virtual photon) can be collected as follows:

$$A_\mu(x)A_\nu(y) \to \delta_{\mu\nu}\int d^3q[(2\pi)^{-\frac{3}{2}}(2|q|)^{-\frac{1}{2}}]^2 \times \exp(iq_\mu x_\mu)\exp(-iq_\mu y_\mu)$$
$$= \delta_{\mu\nu}(2\pi)^{-3}\int d^3q \exp[i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})](2|\mathbf{q}|)^{-1} \times \exp[-i|\mathbf{q}|(t_x - t_y)]$$
$$= \delta_{\mu\nu}(2\pi)^{-3}\int d^3q \exp[i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})] \times \int_{-\infty}^{\infty} dq_0 \exp[-iq_0(t_x - t_y)]\delta_+(q_\mu^2)$$
$$= \delta_{\mu\nu}(2\pi)^{-3}\int d^4q \exp[iq_\mu(x_\mu - y_\mu)]\delta_+(q_\mu^2)$$
$$= 1/2\delta_{\mu\nu}D_F(x-y), \quad (7)$$

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[9] See a discussion by F. J. Belinfante, Phys. Rev. **76**, 226 (1949).

where $t_x > t_y$, because the virtual photon must be created first in order to be annihilated; and $\rightarrow$ means that, so far as the contribution to ME is concerned, a pair $A_\mu(x)A_\nu(y)$ referring to the same virtual photon may be replaced by the right-hand side, which is just Dyson's $\langle P(A_\mu(x)A_\nu(y))\rangle_0{}^1$. It should be clear that this is correct and is what one obtains according to the conventional rules, except that one has usually used up the momentum δ-functions earlier and we now keep them there. The contribution to ME from a free photon operator is obvious. In the final result no photon operators will remain in ME. The contribution to ME due to a virtual electron operator pair (summed over all intermediate states of the virtual electron) according to the conventional method of calculation can be rewritten in a similar manner; however, because of the negative energy electrons we need to distinguish two cases:

(a) A Virtual Electron of Positive Energy ($s=1$ and 2) First Created and Then Annihilated

The $\bar{\psi}$ and ψ in ME either are free operators or they just cause virtual effects. The contribution to ME from a free electron operator is simply:

$$(2\pi)^{-\frac{3}{2}}u(p,s)\exp(ip_\mu x_\mu)$$

and

$$(2\pi)^{-\frac{3}{2}}\bar{u}(p,s)\exp(-ip_\mu x_\mu),$$

where $s=1$, 2, 3, or 4. Since the virtual electron of positive energy must be created first in order to be annihilated, the corresponding operator pair must occur in ME in the order $\cdots\psi(x)\cdots\bar{\psi}(y)\cdots$ with $t_x>t_y$; according to the fourier transform and the meaning of $f(p,s)$ and $f^*(p',s')$, only the combination $\cdots f(p,s)\cdots \times f^*(p,s)\cdots$ with the same p and s (here $s=1$ and 2) can give a nonzero contribution. Further, we observe that in the final result all the creation and annihilation operators will be used up and none will remain in ME. Hence the contribution due to the virtual pair can be first indicated as follows:

$$\cdots(2\pi)^{-3}\int d^3p\sum_{s=1}^{2}\exp[ip_\mu(x_\mu-y_\mu)]$$
$$\times\{(\bar{\psi}(x)\gamma_\mu u(p,s))\cdots(\bar{u}(p,s)\gamma_\sigma\psi(y))\}\cdots;$$

in this way all the operators $\bar{\psi}$ and ψ will be replaced by the ordinary Dirac plane wave spinors

$$\bar{u}(p',s')=u^*(p',s')\beta$$

and $u(p',s')$ multiplied by $\exp(\pm ip_\mu' x_\mu')$, respectively, and with the corresponding momentum integrals and $\sum_{s'}$ summed over 1 and 2 or over 3 and 4 before them. Let P and Q be two arbitrary matrix operators, and introduce the ordinary projection operator

$$\Lambda^\pm(k)=\tfrac{1}{2}[1\pm(\boldsymbol{\alpha}\cdot\mathbf{k}+K\beta)/|E|],$$

where K is mass of electron and $E=+(|\mathbf{k}|^2+K^2)^{\frac{1}{2}}$. We have the well-known relations:

$$\sum_{s=1}^{2}(\bar{u}(p',s')Pu(p,s))(\bar{u}(p,s)Qu(p'',s''))$$
$$=\sum_{s=1}^{2}(\bar{u}_\alpha(p',s')P_{\alpha\gamma}u_\gamma(p,s)\bar{u}_\delta(p,s)Q_{\delta\epsilon}u_\epsilon(p'',s''))$$
$$=\bar{u}_\alpha(p',s')P_{\alpha\gamma}(\Lambda^+(p)\beta)_{\gamma\delta}Q_{\delta\epsilon}u_\epsilon(p'',s'')$$

and

$$\sum_{s=3}^{4}(\bar{u}(p,s)Pu(p',s'))(\bar{u}(p'',s'')Qu(p,s))$$
$$=\sum_{s=1}^{4}(\bar{u}(p,s)Pu(p',s'))(\bar{u}(p'',s'')Q\Lambda^-(p)u(p,s))$$
$$=\sum_{s=3}^{4}(\bar{u}_\alpha(p,s)P_{\alpha\gamma}u_\gamma(p',s')\bar{u}_\delta(p'',s'')Q_{\delta\epsilon}u_\epsilon(p,s))$$
$$=P_{\alpha\gamma}u_\gamma(p',s')\bar{u}_\delta(p'',s'')Q_{\delta\epsilon}(\Lambda^-(p)\beta)_{\epsilon\alpha}.$$

Since, finally, there are no field operators in ME, we can, instead of following the customary procedure, evaluate ME according to the above equations by making the following replacement for each associated pair $\psi_\alpha(x)\bar{\psi}_\beta(y)$ when $t_x>t_y$:

$$\psi_\alpha(x)\bar{\psi}_\beta(y)\rightarrow(2\pi)^{-3}\int d^3p\exp[ip_\mu(x_\mu-y_\mu)](\Lambda^+(p)\beta)_{\alpha\beta}$$
$$=(2\pi)^{-3}\int d^3p\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})]\exp[-i|E|(t_x-t_y)]$$
$$\times\tfrac{1}{2}(\beta+(\boldsymbol{\alpha}\cdot\mathbf{p}+K\beta)\beta/|E|)_{\alpha\beta}$$
$$=(2\pi)^{-3}\int d^3p\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})]\int_{-\infty}^{\infty}dp_0$$
$$\times\exp[-ip_0(t_x-t_y)]\delta_+(p_\mu{}^2+K^2)$$
$$\times[p_0\beta+\boldsymbol{\alpha}\cdot\mathbf{p}\beta+K]_{\alpha\beta}$$
$$=-(2\pi)^{-3}\int d^4p\exp[ip_\mu(x_\mu-y_\mu)]\delta_+(p_\mu{}^2+K^2)$$
$$\times(ip_\mu\gamma_\mu-K)_{\alpha\beta},\quad(8_1)$$

where we have made use of Eqs. (6_1) and (6_2).

(b) A Virtual Hole First Created, Then Annihilated

The corresponding operator pair in ME must now occur in the order $\cdots\bar{\psi}(x)\cdots\psi(y)\cdots$ with $t_x>t_y$, and only the combination $\cdots f^*(p,s)\cdots f(p,s)\cdots$ with the same p and s (here $s=3$ and 4) will yield a nonzero contribution. Following the same procedure as in (*a*), it is immediately seen that this contribution to ME is equivalent to the following substitution:

$$\bar{\psi}_\alpha(x)\psi_\beta(y)\rightarrow(2\pi)^{-3}\int d^3p\exp[-ip_\mu(x_\mu-y_\mu)](\Lambda^-(p)\beta)_{\beta\alpha}$$
$$=+(2\pi)^{-3}\int d^4p\exp[-ip_\mu(x_\mu-y_\mu)]$$
$$\times\delta_+(p_\mu{}^2+K^2)(ip_\mu\gamma_\mu-K)_{\beta\alpha}.\quad(8_2)$$

By (8_1) and (8_2), and remembering Dyson's definition and interpretation of the order "P", we observe that Dyson's rule:[3] "replacing each ordered pair $P(\bar{\psi}_\alpha(x)\psi_\beta(y))$ by $\langle P(\bar{\psi}_\alpha(x)\psi_\beta(y))\rangle_0=1/2\eta(x,y)S_{F\beta\alpha}(x-y)$" is just what has been done in the conventional method of calculation.

In our discussion above we have left out the sign factor θ_k resulting from

$$f_k\Phi^0{}_{\text{electron}}(n_1,\cdots,n_k,\cdots)=\theta_k n_k\Phi^0{}_{\text{electron}}(n_1,\cdots,1-n_k,\cdots)$$

and

$$f_k{}^*\Phi^0{}_{\text{electron}}(n_1,\cdots,n_k,\cdots)=\theta_k(1-n_k)\Phi^0{}_{\text{electron}}(n_1,\cdots,1-n_k,\cdots),$$

where $\theta_k=(-1)^{\nu_k}$,

$$\nu_k=\sum_{j=1}^{k=1} n_j,$$

and $k\equiv(p,s)$. However, we notice that if f_k and $f_k{}^*$ referring to the same virtual particle were adjacent to each other, the resulting sign would be $\theta_k{}^2=+1$, while the sign factor resulting from bringing each associated operator pair together and then arranging the order of the pairs in a suitable way for the purpose of calculation can be obtained obviously in the same way as has been amply discussed by Dyson.[1] By the closure theorem we have

$$\sum_j(\Phi_{j+1}{}^0,\bar{H}_1(t)\Phi_j{}^0)(\Phi_j{}^0,\bar{H}_1(t')\Phi_{j-1}{}^0)=(\Phi_{j+1}{}^0,\bar{H}_1(t)\bar{H}_1(t')\Phi_{j-1}{}^0).$$

Hence, if we define the order "P" according to Dyson, we obtain immediately

$$\begin{aligned}\sum_{\phi(k-1)}\sum_{\phi(k-2)}\cdots\sum_{\phi(1)}&\int_{-\infty}^{\infty}d\tau_k\bar{H}^1{}_{f\phi(k-)}(\tau_k)\\ &\times\int_{-\infty}^{\tau_k}d\tau_{k-1}\bar{H}^1{}_{\phi(k-1)\phi(k-2)}(\tau_{k-1})\cdots\\ &\times\int_{-\infty}^{\tau_2}d\tau_1\bar{H}^1{}_{\phi(1)i}(\tau_1)\\ =\bigg(\Phi_f{}^0,&\int_{-\infty}^{\infty}d\tau_k\bar{H}_1(\tau_k)\int_{\infty}^{\tau_k}d\tau_{k-1}\bar{H}_1(\tau_{k-1})\cdots\\ &\times\int_{-\infty}^{\tau_2}d\tau_1\bar{H}_1(\tau_1)\Phi_i{}^0\bigg)\\ =\bigg(\Phi_f{}^0,&\frac{1}{k!}\int_{-\infty}^{\infty}d^4x_1\int_{-\infty}^{\infty}d^4x_2\cdots\int_{-\infty}^{\infty}d^4x_k\\ &\times P(H_1(x_1),H_1(x_2),\cdots,H_1(x_k))\Phi_i{}^0\bigg)\end{aligned}\tag{9}$$

and

$$\begin{aligned}ME^{(k)}&\delta(E_f-E_i)\\ &=(2\pi)^{-1}i^{k-1}\bigg(\Phi_f{}^0,\frac{(-1)^{k-1}}{k!}\int d^4x_1\cdots\int d^4x_k\\ &\qquad\times P(H_1(x_1),\cdots,H_1(x_k))\Phi_i{}^0\bigg)\end{aligned}\tag{10}$$

which is an immediate consequence of (4) and (9).

It has been shown by Dyson, purely on the basis of the S-matrix, that the self-energy effects can be formally cancelled in all orders by a constant, independent of the physical situation in which the effects occur. It is clear that his argument can be directly applied to (10), so that the idea of mass renormalization is also formally justified on the basis of the conventional formulation of field theory. According to (2) and (3), we have

$$\begin{aligned}\bar{S}_{fi}{}'(\infty)=\bigg(\Phi_f{}^0,\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1\cdots\int d^4x_n\\ \times P(H_1(x_1),\cdots,H_1(x_n))\Phi_i{}^0\bigg).\end{aligned}\tag{11}$$

Since the self-energy effects merely produce a renormalization of the electron mass which, though infinite, has no observable consequences, we may now cancel it formally in the following way by writing

$$\begin{aligned}H(\mathbf{x})&=H_0(\mathbf{x})+\delta m\bar{\psi}(\mathbf{x})\psi(\mathbf{x})+H_1(\mathbf{x})-\delta m\bar{\psi}(\mathbf{x})\psi(\mathbf{x})\\ &=H_0(x)+\delta m\bar{\psi}(\mathbf{x})\psi(\mathbf{x})+H^I(\mathbf{x}),\end{aligned}$$

and substituting $H^I(x)$, in place of $H_1(x)$, in (10) and (11), while understanding the mass of the electron to be $m+\delta m$ instead of m. It is interesting to note that all this can be based also on the conventional field theory formalism. From (11) we then obtain

$$\begin{aligned}\bar{S}_{fi}(\infty)&=\bigg(\Phi_f{}^0,\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1\cdots\int d^4x_n\\ &\qquad\times P(H^I(x_1),\cdots,H^I(x_n))\Phi_i{}^0\bigg)\\ &=(\Phi_f{}^0,s(\infty)\Phi_i{}^0),\quad(x=(\mathbf{x},it)),\end{aligned}\tag{12}$$

where $S(\infty)$ is just Dyson's $U(\infty)$.

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