Relativistic Corrections in High Energy n-p Scattering*

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A method, due to Breit, of generalizing phenomenological nuclear interactions to give a relativistically invariant cross section in the first Born approximation, has been extended to apply to a broad class of nonrelativistic interactions; namely, those containing isotopic spin operators and any kind of Pauli spin and space dependence. A procedure is given for obtaining an equivalent Pauli operator for the Dirac operators so obtained, which reduces the calculation of the relativistic cross section (in the first Born approximation), formally, to a nonrelativistic calculation. The cross section is then obtained in a form in which the relativistic corrections are explicitly split off from the nonrelativistic terms.

The method is applied to the symmetric-exchange interaction with tensor forces. The form of the relativistic interaction is not uniquely determined by that of the nonrelativistic one, but a number of the more conservative possibilities are investigated. The corrections to the 90-Mev n-p scattering thus obtained are analyzed in relation to the data. The possibility of exploiting these corrections to adjust the theory to experiment is discussed.

I. INTRODUCTION AND DISCUSSION OF RESULTS

THE symmetric exchange postulate for the nucleon-nucleon interaction was seriously discredited by the 90-Mev neutron–proton scattering experiments of Segrè and his co-workers.[1] It was shown by Christian and Hart[2] that the angular distribution of the 90-Mev cross section is incompatible with the relatively large proportion of space exchange in the symmetric exchange factor $\frac{1}{3}(\sigma_1\cdot\sigma_2)(\tau_1\cdot\tau_2)$ if the scattering is treated on a nonrelativistic basis. On the other hand, the symmetric exchange is an attractive theoretical concept because it accounts for the saturation of nuclear binding energies and embodies the charge symmetry of nuclear forces (which is at least approximately demonstrable at low energies). This is probably not the case[3] with the half-exchange, half-ordinary force found by Christian and Hart to give the best fit of the 90-Mev scattering results.

The true explanation of nuclear saturation is of course not necessarily to be found entirely in an exchange effect in the two-nucleon interaction. It may originate in some part from many-body forces, for example, or the nucleon–nucleon interaction may not be expressible by a quantum-mechanical potential operator at all. Nevertheless, one is not yet compelled to abandon the two-body force as the phenomenological interaction. Specifically, one might suppose that the nonrelativistic interaction is something similar to the symmetric exchange interaction, correct at nonrelativistic energies, and explaining saturation. The 90-Mev scattering would then deviate from that predicted by this interaction through effects that become important only at appreciably relativistic energies. (Reconciling the symmetric exchange theory with the deviations from charge equivalence shown by the 32-Mev p–p scattering[4] might, of course, require additional postulates).

That such a point of view is not necessarily incompatible with at least the meson theory of nuclear forces is shown by the field theory calculations of Watson and Lepore.[5] *A priori*, the main objection to attributing the deviations from the scattering results to relativistic corrections alone is that the deviations are too large[5a] compared with the nominal figure of 5 percent at 90 Mev obtained by assuming the magnitude of such corrections to be v^2/c^2 times the nonrelativistic cross section. In contrast with this, Watson and Lepore find relativistic contributions of the order of magnitude of the nonrelativistic cross section itself. There seems to be nothing, therefore, in current theory that rules out the possibility of relativistic corrections much greater than the nominal v^2/c^2.

In the present paper the relativistic corrections obtainable with symmetric exchange are investigated from a phenomenological point of view. The cross section in the Born approximation is generalized to a relativistically covariant form. The interaction potential is a Dirac operator, whose matrix elements are required to fit those of the nonrelativistic interaction at low energies. (Two phenomenologically adjusted nonrelativistic interactions are used, a square well, and a Yukawa well). The low energy fitting requirement does not yield a unique Dirac interaction operator. The corrections to the cross section obtained by utilizing several of the more conservative possibilities allowed by this non-uniqueness have been calculated. Each appears as a function that can be added to the differential cross section with a more or less arbitrary multiplicative coefficient. We say "more or less arbitrary" for the

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[1] Hadley, Kelley, Leith, Segrè, Wiegand, and York, Phys. Rev. **75**, 351 (1949).
[2] R. S. Christian and E. W. Hart, Phys. Rev. **77**, 441 (1950).
[3] E. Gerjuoy, Phys. Rev. **77**, 568 (1950).
[4] H. P. Noyes, Phys. Rev. **78**, 90 (1950).
[5] K. M. Watson and J. V. Lepore, Phys. Rev. **76**, 1157 (1949).
[5a] 46 percent of the theoretical value with respect to total cross section, using the square well with Rarita-Schwinger constants, or 21 percent with a Yukawa well (Born approximation triplet, exact singlet). See Table II, and Section VII of this paper.

following reason. Each such coefficient appears in the Dirac interaction potential as the multiplier of a term whose matrix elements are of order v^2/c^2 compared to the leading term (the latter has matrix elements of zero order, which reduce to those of the nonrelativistic interaction in the limit $v/c=0$). Thus, if the only fitting requirement is agreement of the relativistic interaction with the nonrelativistic one in the *zero*-energy limit, these coefficients are entirely arbitrary.

Actually, of course, since the relativistic interaction must fit the experimental data at all energies, not only in the zero-energy limit, the following difficulty arises. If the nonrelativistic approximation is to be considered to be correct for the low and moderate-energy phenomena to which it has been adjusted (and this would be the simplest and hence most desirable situation), the v^2/c^2 terms must not be allowed to become appreciable at these energies. This sets an upper limit to the values of the multiplicative coefficients under discussion. As is shown in detail in the last section, if we are to be able to adjust the 90-Mev cross section by means of these coefficients, without exceeding this upper limit, then since the discrepancies are so much greater than v^2/c^2 in their ratio to the nonrelativistic cross section, a very favorable combination of properties is needed in the purely relativistic contributions to the cross section; namely, (1) a fair degree of coincidence of their values as functions of angle with the values of the discrepancies and (2) a substantial amount of constructive interference at 90 Mev in the amplitudes that combine to give these contributions, so that they will have an actual numerical ratio greater than their formal ratio v^2/c^2 to the nonrelativistic terms at this energy, but not at the low and moderate energies.

In the last section of this paper, the numerical results obtained are analyzed to see how close they come to satisfying these requirements. Requirement (1) is satisfied surprisingly well for the Yukawa interaction, at least, but requirement (2) is so far from being satisfied that it is impossible to have any confidence in the justifiability of handling either interaction along the lines desired. This means that, if the program originally proposed, that of utilizing relativistic terms to adjust the symmetric theory to experiment at 90 Mev, is to be carried out, the form of the nonrelativistic interaction cannot be treated as given, but must instead undergo some further adjustment along with the relativistic coefficients even at low and moderate energies. (We are assuming here that the more radical possibilities in generalizing the nonrelativistic interaction either lead to the same result or are to be rejected, at least tentatively, as excessively speculative). Although such a method is not as simple as could be desired, nevertheless the favorable qualitative behavior of the relativistic corrections [satisfaction of requirement (1)] obtained with the Yukawa interaction indicates that agreement with experiment might be obtainable with a relativistic interaction containing only a very few parameters in addition to those contained in the conventional nonrelativistic interactions. Further research on this subject is now in progress.

An important difference between the case of large coefficients and that of small ones may be noted. If requirements (1) and (2) are met, so that small coefficients can be used, the relativistic contributions to scattering would be expected to level off with increasing energy, as the disappearance of the fortuitous constructive interference counteracts the increase in these terms due to the increase of the ratio v/c, on which they depend more or less monotonically in the near-relativistic region. If these requirements are not met, and large coefficients are required, a very rapid increase of the peculiarly relativistic contributions would be expected, going as v^2/c^2 in the near-relativistic region; thus these terms might rapidly come to predominate, making the scattering at moderate relativistic energies entirely different from that given by the nonrelativistic interaction.

II. GENERAL PROCEDURE

Entirely apart from detailed considerations involving meson theory, there are two very general conditions on the relativistic generalization of a given nucleon–nucleon interaction. These are: (a) Relativistic covariance of physical quantities obtained from it, and (b) reduction of these quantities, in the zero velocity limit, to the corresponding quantities obtained from the given nonrelativistic interaction. Breit[6] has devised a simple procedure, satisfying these conditions, for generalizing the nucleon–nucleon interaction with respect to those matrix elements that appear in the Born approximation cross section. In the same paper he presents explicit generalizations, along these lines, of certain types of nonrelativistic interactions. We adopt his method here, adapting it to the case of symmetric forces and Pauli spin operators of general type.

In order to obtain a covariant cross section, certain terms of purely kinetic origin (i.e., common to all interactions) must be revised. But the essential quantity with which we are concerned is the dynamic one, namely the transition matrix element of the interaction operator of the neutron–proton system. Assume a nonrelativistic interaction V depending on the relative position vector $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ of the two particles, and in addition on the Pauli and isotopic spins. The initial and final momenta of the particles are $\mathbf{p}_{1i}$, $\mathbf{p}_{2i}$ and $\mathbf{p}_{1f}$, $\mathbf{p}_{2f}$, respectively. After elimination of the motion of the center of mass, the transition matrix element is

$$\mathfrak{M}=\langle \mathbf{p}_f, S_f, T_f | V | \mathbf{p}_i, S_i, T_i \rangle, \tag{2.1}$$

in which

$$\mathbf{p}_i=\tfrac{1}{2}(\mathbf{p}_{1i}-\mathbf{p}_{2i}), \quad \mathbf{p}_f=\tfrac{1}{2}(\mathbf{p}_{1f}-\mathbf{p}_{2f}); \tag{2.2}$$

S and T are the Pauli and isotopic spin quantum numbers of the two-nucleon system.

[6] G. Breit, Phys. Rev. **53**, 153 (1938).

Requirement (a) above, when applied to the Born approximation cross section, was shown by Breit[6] to be equivalent to the requirement that under Lorentz transformations of the state vectors (including densities) and the relativistic potential operator, the relativistic generalization of $\mathfrak{M}$ must be invariant. In the paragraphs that follow, we present the general procedure for satisfying conditions (a) and (b), starting with a nonrelativistic interaction containing very general space and Pauli and isotopic spin dependence.

Since isotopic spin state vectors and operators are relativistically invariant, it is sufficient to consider the relativistic generalization of operators of the form

$$\text{(Pauli spin operator)}\times\text{(space operator)}. \quad (2.3)$$

The most general interaction of the type described above will be a sum of operators of the form (2.3), with isotopic spin operators as coefficients. The relativistic interaction is thus obtainable by replacing (2.3) by its relativistic counterpart in each term.

Let U denote an operator of the form (2.3), i.e., not containing isotopic spin. We seek a relativistic generalization, U_{rel}, of U, satisfying conditions (a) and (b). The difficulties standing in the way of a formulation of a relativistically invariant interaction in configuration space can be avoided by using a momentum representation of U. The factor (space operator) in (2.3) is then replaced by the equivalent operator in terms of the momenta. This operator factor, when the interaction is diagonal in the space variable $\mathbf{r}$, can be written in the explicit form

$$U_p=\int\int |\mathbf{p}_f\rangle d\mathbf{p}_f \mathfrak{U}_p(\mathbf{p}_{if}) d\mathbf{p}_i \langle \mathbf{p}_i|, \quad (2.4)$$

where

$$\mathbf{p}_{if}\equiv\mathbf{p}_i-\mathbf{p}_f; \quad (2.5)$$

$\mathfrak{U}_p(\mathbf{p}_{if})$ is a c-quantity (in general a tensor with c-numbers as components) equal to the matrix elements of U_p between the state vectors $\langle \mathbf{p}_f|$ and $|\mathbf{p}_i\rangle$:

$$\mathfrak{U}_p(\mathbf{p}_{if})=\langle \mathbf{p}_f| U_p |\mathbf{p}_i\rangle. \quad (2.6)$$

At this point, for later convenience, we introduce the notation

$$\{f(\mathbf{p})\}_{\text{op}}\equiv\int |\mathbf{p}_f\rangle d\mathbf{p}_f f(\mathbf{p}_{if}) d\mathbf{p}_i \langle \mathbf{p}_i|; \quad (2.7)$$

in words, $\{f(\mathbf{p})\}_{\text{op}}$ is the operator whose matrix element between $\langle \mathbf{p}_f|$ and $|\mathbf{p}_i\rangle$ is $f(\mathbf{p}_{if})$. $f(\mathbf{p}_{if})$ may be a scalar or tensor, or tensor component.

The spin factor of (2.3), to be denoted by U_σ, will be left in the customary representation. The factored form of U is now

$$U=U_pU_\sigma. \quad (2.8)$$

The transition to the relativistic domain will be made by constructing a mapping which defines the correspondence between nonrelativistic and relativistic state vectors, and by characterizing the relativistic interactions whose matrix elements between these relativistic states satisfy conditions (a) and (b).

III. RELATIVISTIC NOTATION

We shall use, throughout, the very advantageous scheme of expressing the Dirac operators in terms of dichotomic variables.[7] If we write $|\rho_-'\rangle$ for the eigenvector of ρ_3 having the eigenvalue -1, we obtain a basic set of one-particle solutions of the Dirac equation in the form

$$|E, \mathbf{p}, (\sigma)\rangle=C\left(1-\frac{c\boldsymbol{\alpha}\cdot\mathbf{p}}{E+\mu c^2}\right)|\rho_-'\rangle|\sigma\rangle|\mathbf{p}\rangle, \quad (3.1)$$

where $|\sigma\rangle$ is any σ-spin state vector, $|\mathbf{p}\rangle$ is a momentum eigenvector normalized to one particle per unit volume. C is a normalization constant:

$$C=[1+c^2p^2/(E+\mu c^2)]^{-\frac{1}{2}}. \quad (3.2)$$

E and $\mathbf{p}$ are related by

$$E=\pm(c^2p^2+\mu^2c^4)^{\frac{1}{2}}. \quad (3.3)$$

Four mutually orthogonal vectors are obtained by using the two signs of the energy and any pair of orthogonal σ state vectors.

We use henceforth the notation

$$G_{E,\mathbf{p}}(i)\equiv 1-c\boldsymbol{\alpha}(i)\cdot\mathbf{p}/(E+\mu c^2). \quad (3.4)$$

The argument i designates the particle on whose dichotomic variables G operates. The normalization constant C is a function of the sign and magnitude of the energy only; since all our calculating will be done in the center-of-mass system with particles of equal mass and equal positive energies, it will never be necessary to use distinguishing subscripts with C.

For a system of two particles, linear combinations of products of state vectors of the form (3.1), one for each particle, are solutions of the Dirac equation for two free particles. These have the form

$$\begin{aligned}|E_1, E_2, \mathbf{p}_1, \mathbf{p}_2, (S_{\text{rel}}(1,2))\rangle &= C^2G_{E1,\mathbf{p}1}(1)G_{E2,\mathbf{p}2}(2)|\rho_-'(1)\rangle|\rho_-'(2)\rangle \\ &\times|S_{\text{rel}}(1,2)\rangle|\mathbf{p}_1(1)\rangle|\mathbf{p}_2(2)\rangle, \quad (3.5)\end{aligned}$$

in which $|S_{\text{rel}}(1,2)\rangle$ is *any* linear combination of two-particle σ-spin state vectors. As in the nonrelativistic case,

$$|\mathbf{p}(1)\rangle|\mathbf{p}(2)\rangle=|\mathbf{P}\rangle|\mathbf{p}\rangle, \quad (3.6)$$

where $\mathbf{P}$ and $\mathbf{p}$ are the momenta conjugate to $\mathbf{R}=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$ and $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ respectively; i.e. $\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2$,

[7] L. Rosenfeld, *Theory of Nuclear Forces* (Interscience Publishers, Inc., New York, 1949). The notation we use is given in Rosenfeld's Chapter IV, particularly Sec. 4.2. However, the representation in which we write the Dirac equation differs from Rosenfeld's; it is obtained from his by transforming with the unitary operator ρ_1, i.e., essentially by the interchange of large and small components of the state vectors.

$\mathbf{p}=\frac{1}{2}(\mathbf{p}_1-\mathbf{p}_2)$. Thus, putting also $|\rho_-'(1)\rangle|\rho_-'(2)\rangle=|\rho_-\rangle$:

$$|E_1, E_2, \mathbf{p}_1, \mathbf{p}_2, (S_{\rm rel}(1,2))\rangle$$
$$=|E_1, E_2, \mathbf{p}, \mathbf{P}, (S_{\rm rel}(1,2))\rangle$$
$$=C^2G_{E1,\,\mathbf{p}1}(1)G_{E2,\,\mathbf{p}2}(2)|\rho_-\rangle|S_{\rm rel}(1,2)\rangle|\mathbf{P}\rangle|\mathbf{p}\rangle. \quad (3.7)$$

The form of the relativistic state vector that replaces the nonrelativistic one $|\mathbf{P}\rangle|\mathbf{p}\rangle|S\rangle$ follows from the customary formulation of the assumption that the Pauli-Schrödinger mechanics is the nonrelativistic limiting form of the Dirac mechanics, *viz.*: The Dirac operator $\mathbf{p}(i)$ corresponds to the observable, relativistic momentum of a particle; and the Dirac operator $\boldsymbol{\sigma}(i)$ corresponds to the Pauli operator $\boldsymbol{\sigma}(i)$ in the limit $v/c=0$. The values of the relativistic momenta $\mathbf{P}$ and $\mathbf{p}$ are determined by the experimental situation. The correspondence of the $\boldsymbol{\sigma}$ operators requires that $|S_{\rm rel}\rangle$ be chosen equal to the nonrelativistic $|S\rangle$, to within a unitary transformation; we put $|S_{\rm rel}\rangle=|S\rangle$. The signs of the energies E_1 and E_2 will, of course, in our case be positive (*first* Born approximation).

IV. COVARIANT INTERACTION FORMS

The relativistic generalization of U_p [Eq. (2.4)] that we shall use will be the simplest one satisfying conditions (a) and (b): On the right-hand side of (2.4), the vectors $\mathbf{p}_i$ and $\mathbf{p}_f$ in the state vectors $|\mathbf{p}_f\rangle$ and $\langle\mathbf{p}_i|$, and $\mathbf{dp}_i$ and $\mathbf{dp}_f$, are replaced by their relativistic *three*-vector counterparts; the normalization of the nonrelativistic state vectors is retained, and the integration over all values of the vectors $\mathbf{p}_i$ and $\mathbf{p}_f$ is also retained. In $\mathfrak{U}_p(\mathbf{p}_{if})$ considered as a function of $\mathbf{p}_{if}$, on the other hand, $\mathbf{p}_{if}$ is replaced by $(\mathbf{p}_{if}, iE_{if}/c)$, with

$$E_{if}=\tfrac{1}{2}[(E_{1i}-E_{2i})-(E_{1f}-E_{2f})], \quad (4.1)$$

to yield the relativistic function $\mathfrak{U}_{p\ \rm rel}(\mathbf{p}_{if})$; whatever the form of $\mathfrak{U}_p(\mathbf{p}_{if})$, whether scalar or tensor in $\mathbf{p}_{if}$, this replacement is straightforward for positive energy states. (It will be noted that this procedure gives a four-vector for $\mathfrak{U}_{p\ \rm rel}$ when $\mathfrak{U}_p$ is a three-vector, and similarly for higher rank tensors. Symbolically, if $U_p=\{T(\mathbf{p})\}_{\rm op}$, where $T(\mathbf{p})$ is a scalar, vector, dyadic, etc., function of $\mathbf{p}$, this gives for the relativistic generalization of U_p (to be called $U_{p\ \rm rel}$):

$$U_{p\ \rm rel}=\{T((\mathbf{p}, iE/c))\}_{\rm op}).$$

The invariance of all matrix elements (between state vectors of the form $|\mathbf{p})$) of $U_{p\ \rm rel}$ follows from the invariance of $\mathfrak{U}_{p\ \rm rel}(\mathbf{p}_{if})$, provided normalization of all state vectors of the form $|\mathbf{p}\rangle$ to one particle per unit volume is retained under Lorentz transformations. It will be noted that the change in normalization of actual solutions of the free-particle Dirac equation under Lorentz transformations is incorporated in the factor C^2 of Eq. (3.5).

The lack of uniqueness of the above choice of $U_{p\ \rm rel}$ is quite apparent. In particular, $\mathfrak{U}_{p\ \rm rel}(\mathbf{p}_{if})$ as above defined could be multiplied by any scalar function of $(\mathbf{p}_i, iE_i/c)$ and $(\mathbf{p}_f, iE_f/c)$ that reduces to unity in the nonrelativistic limit, without violating conditions (a) and (b).

We now turn to the matter of the relativistic generalization of U_σ [Eq. (2.8)], to be denoted to $U_{\sigma\ \rm rel}$. The necessary invariance of the matrix elements of $U_{\sigma\ \rm rel}$ when taken between the spinor parts of the state vectors (3.5) (the factor C, giving the Lorentz density change, being taken as included in the spinor part) restricts us to the operators $S=\beta=\rho_3$, $V=(\boldsymbol{\alpha}, -i)$, $T=\beta(\boldsymbol{\sigma}, \boldsymbol{\alpha})$, $A=(\boldsymbol{\sigma}, -i\rho_1)$, $P=\rho_2$, S being a scalar, V a polar four-vector, T an antisymmetric tensor, A an axial four-vector, and P a pseudoscalar.

The forms of U_pU_σ that appear in the nonrelativistic interactions with which we are concerned are invariant combinations of the Pauli operators $\mathbf{1}$ and $\boldsymbol{\sigma}$ and momentum operators of the form $\{f(|\mathbf{p}|)\}_{\rm op}$ or $\{f(|\mathbf{p}|)p_jp_k\}_{\rm op}$ (j, k being tensor indices). The corresponding relativistic operators will be obtained by combining the Dirac operators S, V, T, A, P with momentum operators obtained from the nonrelativistic momentum operators in the manner described previously. We shall retain in the relativistic domain the restriction that the latter be tensors of not higher than second rank in the momentum.

The desired operators will be written down systematically[8] in a way that exhibits the completeness of the collection subject to the above restrictions. For simplicity, we put f_0 for a scalar operator function of $(\mathbf{p}_{if}, iE_{if}/c)$; f_1 for the corresponding four-vector; and f_2 for a tensor. Of the two Dirac operators in each product, one belongs to one of the particles, and one to the other; some have to be symmetrized between the particles and this is indicated by the operator "s" appearing in front of such expressions. The factor i (square root of minus one) that appears in some cases is for the purpose of making the operator hermitian.

The two-particle operators are then (a) SSf_0, (b) $isSVf_1$, (c) $sSTf_2$, (d) VVf_0, (e) $sVVf_2$, (f) $iVTf_1$, (g) TTf_0, (h)[9] TTf_2, (i) AAf_0, (j) AAf_2, (k) $isAPf_1$, (l) PPf_0, (m) $isAT(\text{dual})f_1$. In all cases the operators are to be combined by summation of indices in such a way as to give an invariant; where this summation is ambiguous, it is uniquely defined by requiring that the result does not duplicate the form of another operator included in the list.

We note that three of the above operators, while included for completeness, could have been omitted, at least for our present purposes: (c) vanishes identically,

[8] The cases involving a sum over the indices of T by itself are omitted; this sum vanishes, since T is an antisymmetrical tensor.

[9] A Møller type calculation [Z. Physik **70**, 686 (1931)] of the interaction matrix element between momentum eigenstates for two Dirac particles leads to just this operator (h), if instead of using the complete charge-current density four-vectors one uses the parts of these that refer to the magnetization-polarization densities alone. Thus this operator is physically the relativistic form of the dipole-dipole potential energy interaction.

since T is an antisymmetric tensor and f_2 is symmetric, and (b) and (e) can be shown, by use of the one-particle Dirac equation, to have matrix elements which vanish between states of positive energy.

The Pauli operators to which the above operators reduce may be found by setting the Dirac operators α, ρ_1, and ρ_2 equal to zero and $\rho_3=1$. We then obtain, since E_{if} vanishes in the nonrelativistic limit (particles of equal mass): (a) $\rightarrow f_0$, (c) $=0$, (d) $\rightarrow -f_0$, (g) $\rightarrow \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 f_0$, (h) $\rightarrow\{[|\mathbf{p}_{if}|^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-\boldsymbol{\sigma}_1\cdot\mathbf{p}_{if}\boldsymbol{\sigma}_2\cdot\mathbf{p}_{if}]f(|\mathbf{p}_{if}|^2)\}_{\text{op}}$, (i) $\rightarrow\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 f_0$, (j) $\rightarrow\{\boldsymbol{\sigma}_1\cdot\mathbf{p}_{if}\boldsymbol{\sigma}_2\cdot\mathbf{p}_{if}f(|\mathbf{p}_{if}|^2)\}_{\text{op}}$ while all others approach zero. These equivalences hold, of course, only in the positive energy subspace.

V. DERIVATION OF THE RELATIVISTIC CORRECTIONS TO THE CROSS SECTION

We take as our relativistic interaction

$$V_{\text{rel}}=\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 U_{\text{rel}}, \tag{5.1}$$

where U_{rel} operates on momentum, ρ- and σ-spin coordinates only. In obtaining the relativistic matrix element, it is convenient to make an explicit separation of the three types of inner product corresponding to the three types of coordinates of U_{rel}. We define

$$\mathfrak{U}_{\text{rel}}=\langle\mathbf{p}_f|\,U_{\text{rel}}\,|\mathbf{p}_i\rangle, \tag{5.2}$$

and further

$$U_{\text{rel}}'=\langle\rho_-|\,G_f(1)G_f(2)\mathfrak{U}_{\text{rel}}G_i(1)G_i(2)\,|\rho_-\rangle, \tag{5.3}$$

in which the G_i and G_f are operators of the form (3.4) appropriate for the description of the initial and final states. $\mathfrak{U}_{\text{rel}}'$ is, formally, a Pauli operator, and will be called the *Pauli* equivalent operator of $\mathfrak{U}_{\text{rel}}$, since in terms of it the relativistic first Born approximation cross section (for positive energy states) is formally the same as the nonrelativistic one, except for constant factors. We shall use the prime and the term "Pauli equivalent operator" for any operator of this type.

To order v^2/c^2, the neutron–proton cross section for unpolarized initial and final particles is found to be[9a]

$$\sigma(\theta)=\sigma_{NR}(\theta)+\frac{\mu^2}{32\pi^2\hbar^4}Tr(Y\mathfrak{U})(YH^{(2)}). \tag{5.4}$$

The unfamiliar terms in this expression are defined as follows:

$$\sigma_{NR}(\theta)=\frac{\mu^2}{64\pi^2\hbar^4}Tr(Y\mathfrak{U})^2 \tag{5.5}$$

is the nonrelativistic cross section,

$$\mathfrak{U}=\mathfrak{U}(\mathbf{p}_{if})=\langle\mathbf{p}_f|\,U\,|\mathbf{p}_i\rangle \tag{5.6}$$

being the momentum matrix element of the nonrelativistic interaction. $H^{(2)}$ is that part of $\mathfrak{U}_{\text{rel}}'$ which is hermitian in σ-spin and of second order in v/c; we shall henceforth, where necessary, use superscripts in parentheses to indicate the order in v/c of any operator. The trace Tr is to be taken over a complete set of two-particle spin states. The dictates of the Pauli principle, and the effect of the operator $\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$, are contained in the operator

$$Y=-\tfrac{1}{2}(1+P_pP_\sigma), \tag{5.7}$$

where P_σ is the σ-spin exchange operator, and P_p interchanges $\mathbf{p}_{1i}$ and $\mathbf{p}_{1f}$ wherever they occur in $\mathfrak{U}$.

The way in which terms of higher than zero order in v/c are separated off, can be seen as follows. If the expression (3.4) is used explicitly for all the G's in (5.3) and the multiplication then carried out, one obtains a sum of terms of zero to fourth order in $cp/(E+\mu c)$, or, essentially, in v/c. The zero-order term must be just $\mathfrak{U}$. The first-order term vanishes for all operators that we shall use (see Sec. VI for criteria of selection of these operators). The second-order term produces the v^2/c^2 contribution, i.e., the second term of (5.4), which is seen to have the form of an interference term between the nonrelativistic scattering amplitude and that due to the "hermitian" part of the second-order Pauli equivalent operator of the interaction.

VI. APPLICATION TO TWO NONRELATIVISTIC INTERACTIONS; RESULTS

The nonrelativistic interactions that we shall deal with in this paper can be written in the general form

$$V(\mathbf{r})=\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 V_0[(1-\tfrac{1}{2}g)f_0(r)+\tfrac{1}{2}g\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 f_0(r)+\gamma S_{12}(\mathbf{r})f_2(r)], \tag{6.1}$$

where V_0, g and γ are constants; r means $|\mathbf{r}|$; and

$$S_{12}(\mathbf{r})=3[(\boldsymbol{\sigma}\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})/r^2]-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2. \tag{6.2}$$

$f_0(r)$ and $f_2(r)$ embody the spatial dependence of the central and tensor parts of the interaction respectively; these are so far abitrary.

In order to investigate the relativistic corrections, we need $\mathfrak{U}(\mathbf{p}_{if})$ (defined in Sec. V). For convenience in the relativistic work we break up the tensor force into its two components, defining

$$s(\mathbf{p}_{if})=(\boldsymbol{\sigma}_1\cdot\mathbf{p}_{if}\boldsymbol{\sigma}_2\cdot\mathbf{p}_{if})/p_{if}^2. \tag{6.3}$$

Then

$$\mathfrak{U}(\mathbf{p}_{if})=\frac{4\pi}{3}V_0\{\tfrac{3}{2}g\varphi_0(p_{if})+[(1-\tfrac{3}{2}g)\varphi_0(p_{if})+\gamma\varphi_2(p_{if})]\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-3\gamma s(\mathbf{p}_{if})\varphi_2(p_{if})\}, \tag{6.4}$$

[9a] *Note added in proof:* It will be noted that in the present formulation, the relativistic correction [second term of Eq. (5.4)] is dynamically independent of the nonrelativistic cross section since it contains the factor $H^{(2)}$ in the trace. The normalization factor C^8 [see Eq. (3.2)] coming from the Dirac state vectors cancels, to order v^2/c^2, the relativistic corrections contained in the density-of-states factor common to all terms in the cross section. The net result is that $\sigma_{NR}(\theta)$ of (5.4) would, if $\sigma(\theta)$ were calculated exactly, be multiplied by a factor differing from unity only in terms of order v^4/c^4, these terms amounting to only 0.03 percent at 90 Mev. This wholly negligible amount is the entire "kinematic" correction in our interpretation, which we think is recommended by the natural way in which it emerges from the above analysis (a completely non-arbitrary definition of the "kinematic" correction is evidently impossible). It has, of course, been customary to omit the C^8 factor in defining the correction; this explains the difference between our result and that of Case and Pais [K. M. Case and A. Pais, Phys. Rev. **80**, 203 (1950)].

where

$$\varphi_n(p_{if})=\int_0^\infty j_n(p_{if}r/\hbar)f_n(r)r^2dr;$$

j_0 and j_2 are spherical bessel functions,

$$j_n(x)=(\pi/2x)^{\frac{1}{2}}J_{n+\frac{1}{2}}(x).$$

We regard $\mathfrak{U}(\mathbf{p}_{if})$ as a linear combination of the Pauli operators $\mathbf{1}$, $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$, $s(\mathbf{p}_{if})$, and 0, with coefficients depending on $\mathbf{p}_{if}$ and on the constants g and γ. Then the relativistic generalization of $\mathfrak{U}(\mathbf{p}_{if})$ will be obtained by (a) replacing $\mathbf{p}_{if}$ by the magnitude of its corresponding four-vector (this four-vector has zero fourth component in the CM system, and its space part is equal to the nonrelativistic $\mathbf{p}_{if}$ and (b) making the operator replacements, for each Pauli operator,

Pauli operator $\rightarrow$ linear combination of relativistic equivalent Pauli operators;

where the "relativistic equivalent Pauli operators" are deduced from invariant operators (a) to (m), Sec. IV. Formally, the coefficients of all the relativistic operators on the right-hand side that reduce to the Pauli operator on the left-hand side must add up to unity, and the coefficients of relativistic operators that reduce to zero may have arbitrary magnitude (so far as the zero-energy limit is concerned; see Sec. I).

It did not seem worth while at present to attempt an exhaustive investigation, such as would consist of incorporating all the operators (a) to (m) into the generalized interaction in all allowable combinations, for example. The criterion of choice was to omit zero-reducing operators, with the exception of PPf_0, which is a particularly simple and fundamental case, and to replace each Pauli operator by a linear combination of only those two operators of the list which reduce to it. The operators that are omitted, other than those that vanish identically or have vanishing matrix elements between positive energy states, are *differences* between operators that reduce to the same Pauli equivalent operator, and (f), (k), and (m), all linear in the momentum. The last three operators have Pauli equivalent operators with nonvanishing terms of first order in v/c. In general, they will thus yield relativistic corrections to the cross section of order v/c, while the operators that we include yield corrections of order v^2/c^2. This v/c cross-section contribution turns out to be zero for (m), but is not zero for (f) and (k). So long as one hopes to adjust relativistic terms to fit theory to experiment at 90 Mev, without introducing appreciable relativistic effects in the moderate and low energy regions, operators that contribute to first order in v/c are substantially less important than those which contribute only in higher orders, since their relativistic terms decrease more slowly with decreasing energy. If the desired type of adjustment cannot be achieved with v^2/c^2 corrections, the same should hold still more strongly with v/c corrections.

Instead of writing down the Dirac spin-momentum operators that will be used, we shall give just the momentum matrix elements, derived from them, which are to be used as spin operators in $\mathfrak{U}_{\text{rel}}$. We use the following notation. The subscript u refers to a Dirac operator reducing to the unit Pauli operator; σ to the Pauli $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$; s to the Pauli $s(\mathbf{p}_{if})$; and n to the Pauli null operator. Further subscripts are used to distinguish between Dirac operators both of which reduce to the same Pauli operator. The operators to be used are then

$$\begin{aligned}
&\zeta_{u1}=\beta(1)\beta(2),\\
&\zeta_{u2}=1-\boldsymbol{\alpha}(1)\cdot\boldsymbol{\alpha}(2),\\
&\zeta_{\sigma1}=\beta(1)\beta(2)[\boldsymbol{\sigma}(1)\cdot\boldsymbol{\sigma}(2)+\boldsymbol{\alpha}(1)\cdot\boldsymbol{\alpha}(2)],\\
&\zeta_{\sigma2}=\boldsymbol{\sigma}(1)\cdot\boldsymbol{\sigma}(2)-\rho_1(1)\rho_1(2),\\
&\zeta_{s1}=\beta(1)\beta(2)[s(\mathbf{p}_{if})-\rho_1(1)\rho_1(2)s(\mathbf{p}_{if})+\boldsymbol{\alpha}(1)\cdot\boldsymbol{\alpha}(2)],\\
&\zeta_{s2}=s(\mathbf{p}_{if}),\\
&\zeta_n=\rho_2(1)\rho_2(2).
\end{aligned}\tag{6.5}$$

These are the forms obtained in the CM system for two particles of equal mass, i.e., with the fourth component of the momentum-energy vector $(\mathbf{p}_{if}, iE_{if}/c)$ equal to zero, from the following operators of Sec. IV, respectively: (a), (d), (g), (i), (g) minus (h), (*j*), and (l). (In the case of ζ_{s1} and ζ_{s2}, the original operators have been divided by $|\mathbf{p}_{if}|^2$.)

With our restrictions, the equivalent Pauli $\mathfrak{U}_{\text{rel}}'(\mathbf{p}_{if})$ corresponding to the $\mathfrak{U}(\mathbf{p}_{if})$ of (6.4) is

$$\begin{aligned}
\mathfrak{U}_{\text{rel}}'(\mathbf{p}_{if})=\frac{4\pi}{3}V_0\{&\tfrac{3}{2}g[a_{u1}\zeta_{u1}'+a_{u2}\zeta_{u2}']\varphi_0(p_{if})\\
&+[1-\tfrac{3}{2}g][a_{\sigma10}\zeta_{\sigma1}'+a_{\sigma20}\zeta_{\sigma2}']\varphi_0(p_{if})\\
&+\gamma[a_{\sigma12}\zeta_{\sigma1}'+a_{\sigma22}\zeta_{\sigma2}']\varphi_2(p_{if})\\
&-3\gamma[a_{s1}\zeta_{s1}'+a_{s2}\zeta_{s2}']\varphi_2(p_{if})\\
&+a_{n0}\zeta_n'\varphi_0(p_{if})+a_{n2}\zeta_n'\varphi_2(p_{if})\},
\end{aligned}\tag{6.6}$$

with the condition that the sum of the a's in any bracket is equal to unity. The ζ' in (6.6) are primed to indicate that equivalent Pauli operators are meant. Expressions for the equivalent Pauli operators of the ζ' are given in the Appendix. Note that the coefficients of $\zeta_{\sigma1}'$ and $\zeta_{\sigma2}'$ that are coupled with $\varphi_0(p_{if})$ are treated independently of those that are coupled with $\varphi_2(p_{if})$.

The cross section in the Born approximation [Eq. (5.4)] can be written as

$$\sigma(\theta)=\sigma_{NR}(\theta)+\sum_i a_i\sigma_i^{(2)}(\theta),\tag{6.7}$$

where $\sigma_i^{(2)}(\theta)$ is the second-order-in-v/c contribution due to the operator ζ_i with all factors that appear with it in Eq. (6.6) *except* the factor $a_i=a_{u1}, a_{u2}, a_{\sigma10}, \cdots$; e.g.,

$$\begin{aligned}
\sigma_{u1}^{(2)}(\theta)=(\mu^2/64\pi^2\hbar^4)\cdot\tfrac{3}{2}g\cdot 2Tr[(Y\mathfrak{U})\\
\cdot(4\pi/3)V_0\cdot YH^{(2)}(\zeta_{u1}')\varphi_0(p_{if})].
\end{aligned}\tag{6.8}$$

Simplified expressions for evaluating the traces are given in the Appendix. The expression for $\sigma_{NR}(\theta)$ is also given for reference.

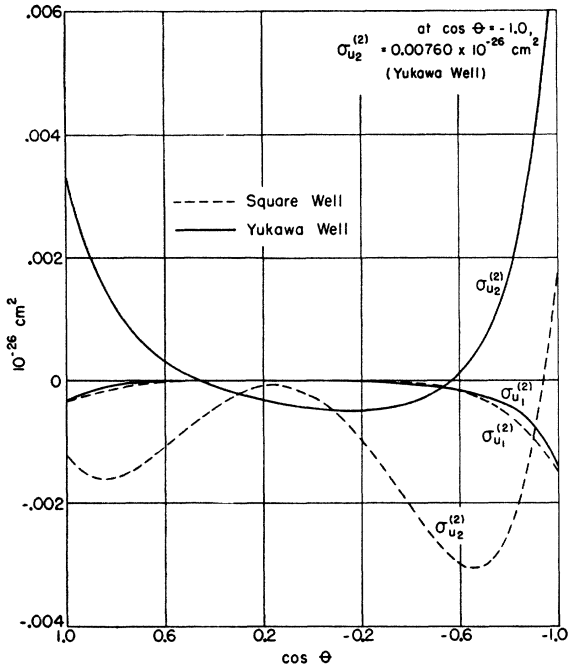

FIG. 1. Relativistic corrections $\sigma_{u1}^{(2)}$, $\sigma_{u2}^{(2)}$.

As stated in Sec. I, relativistic corrections have been calculated for two nonrelativistic interactions of the form (6.1): (1) The Rarita-Schwinger square-well interaction,[10] and (2) a Yukawa interaction with constants computed by Feshbach[11] for "best" fit of the low-energy data.

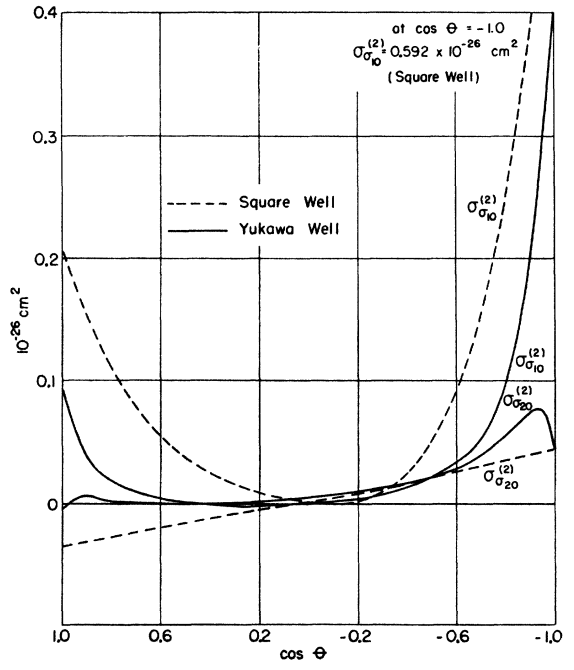

FIG. 2. Relativistic corrections $\sigma_{\sigma 10}^{(2)}$, $\sigma_{\sigma 20}^{(2)}$.

[10] W. Rarita and J. Schwinger, Phys. Rev. **59**, 436 (1941).
[11] Private communication (to be published).

The numerical results for the second-order contributions $\sigma_i^{(2)}(\theta)$ that appear in Eq. (6.7) are plotted in Figs. 1 to 5 for the two interactions. It will be noted that the $\sigma_i^{(2)}(\theta)$ are much more sensitive to the nonrelativistic interaction than is the nonrelativistic cross section. For purposes of comparison, the nonrelativistic cross sections given by the two interactions are plotted in Fig. 6; the experimental points are also shown. The nonrelativistic cross section for interaction (1) was obtained by extrapolation from the 100-Mev calculations of Ashkin and Wu,[12] using the 90- and 100-Mev Born approximation results to give the proportion. In this way a considerable improvement over the straight Born approximation should be obtainable. For interaction (2), the nonrelativistic cross section was obtained by combining a Born approximation calculation for the triplet scattering with an exact calculation for singlet scattering.[13] In the latter case a substantial error probably still remains, although, as will be argued later, it should not affect our conclusions. The nonrelativistic total cross sections obtained are

$$\sigma_{NR}=14.1\times10^{-26}\text{ cm}^2\text{ (Square well)},$$
$$\sigma_{NR}=9.59\times10^{-26}\text{ cm}^2\text{ (Yukawa well)};$$

these are to be compared with the experimental value[1] of 7.6×10^{-26} cm^2.

VII. ANALYSIS OF RESULTS

In this section we analyze the properties of the relativistic corrections obtained, by comparing them quantitatively with the discrepancies in total cross section

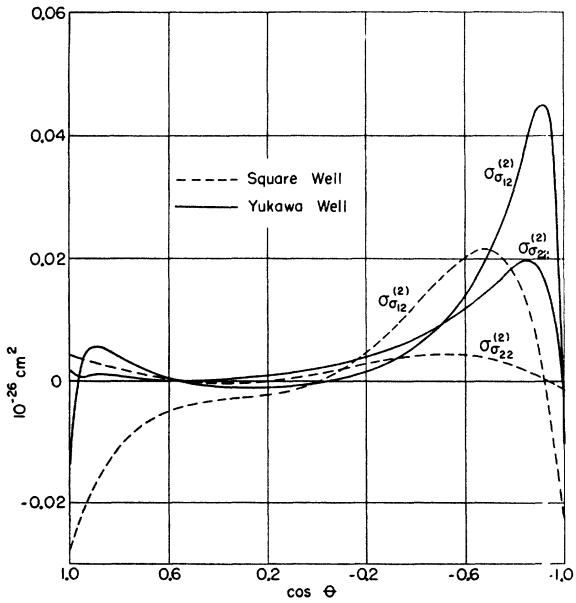

FIG. 3. Relativistic corrections $\sigma_{\sigma 12}^{(2)}$, $\sigma_{\sigma 22}^{(2)}$.

[12] Ashkin and Wu, Phys. Rev. **73**, 973 (1948).
[13] The author is indebted to Professor Feshbach for the results of the singlet scattering calculation. Its use eliminates the substantial error that would result from the very large repulsive singlet odd potential if the Born approximation were used.

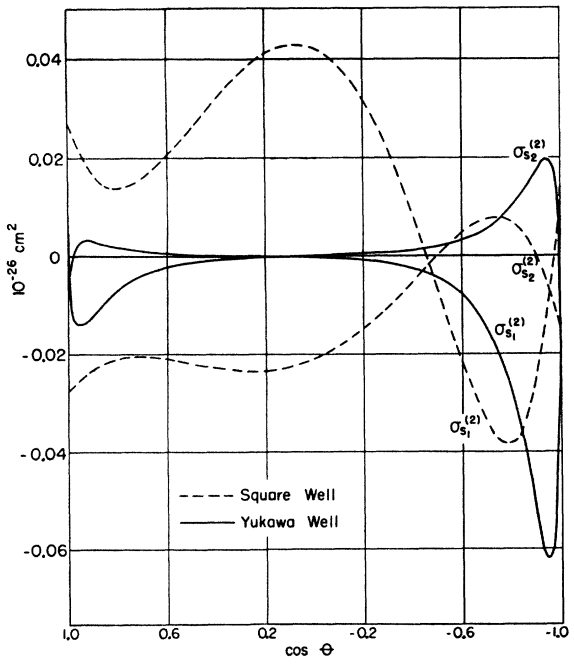

FIG. 4. Relativistic corrections $\sigma_{s1}^{(2)}$, $\sigma_{s2}^{(2)}$.

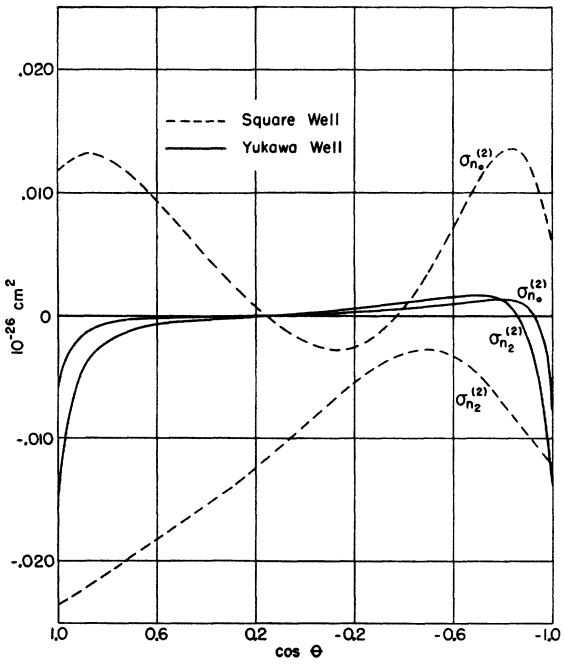

FIG. 5. Relativistic corrections $\sigma_{n0}^{(2)}$, $\sigma_{n2}^{(2)}$.

and angular distribution between nonrelativistic theory and experiment.

The shape characteristics of n–p scattering curves are usually expressed through the ratio of the 180° value to that at the minimum of the curve, and to that at 0°. In this case the point $\cos\theta=0.8$ will be used instead of $\theta=0°$, because the data do not extend to 0°. Furthermore, the Yukawa curves fluctuate very greatly in the immediate neighborhood of 180°, so that the point $\cos\theta=-0.9$, where they behave more quietly, will be used instead; this will result in a better measure of their effect in the back-scattering portion of the curve (any very rapid fluctuations of this kind if they occurred in nature, would of course be smoothed out in the experimental results, due to finite resolution).

The required shape improvements can now be summed up:

(a) Reduction of the ratio $\sigma(-0.9)/\sigma(0.2)$.
Experimental value: 2.6.
Square well, nonrel.: 7.9.
Yukawa well, nonrel.: 5.7 (Born approx. triplet, exact singlet).

(b) Reduction of the ratio $\sigma(-0.9)/\sigma(+0.8)$.
Experimental value: 1.4.
Square well, nonrel.: 3.0.
Yukawa well, nonrel.: 3.6 (Born approx. triplet, exact singlet).

A convenient criterion for the shape-improving characteristics of the relativistic corrections in these two respects is the following: Let

$$R_i(\cos\theta)=\sigma_i^{(2)}(\cos\theta)/\sigma_{NR}(\cos\theta).$$

Then we have the following conditions on the sign of any a_i in Eq. (6.7) for shape improvement with respect to the desiderata (a) and (b), respectively:

(a) $R_i(0.2)>(<)R_i(-0.9)$ requires $a>(<)0$. (7.1)
(b) $R_i(+0.8)>(<)R_i(-0.9)$ requires $a>(<)0$.

The correlation of the signs required for shape improvement with that required for total cross-section

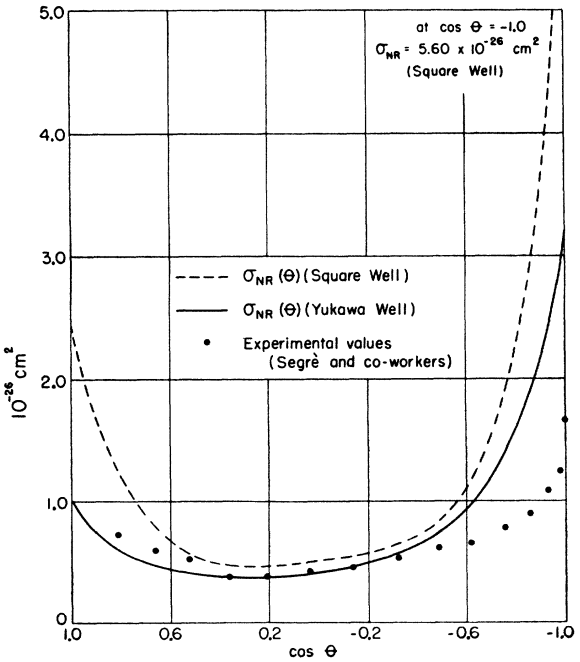

FIG. 6. Nonrelativistic cross sections.

TABLE I. Corrections to the total cross section. Values of $R_i(-0.9)$, $R_i(0.2)$, $R_i(+0.8)$ for the $\sigma_i^{(2)}$ of Eq. (7.2); $R_i(\text{total})=\sigma_i^{(2)}(\text{total})/\sigma_{NR}(\text{total})$.

		Yukawa well					Square-well correlations[a]			
	Corrections to total cross section (10^{-26} cm^2)	R_i(total) %	$R_i(-0.9)$ %	$R_i(0.2)$ %	$R_i(+0.8)$ %	Correlations[a] a	b	c	d	a	b	c	d
$\sigma_{u1}^{(2)}$	−0.00183	−0.019	−0.035	0.0017	−0.014	+	+	+	+	+	+	+	+
$\sigma_{u-}^{(2)}$	0.00934	0.097	0.21	−0.089	0.22	−	−	−	+?	+	−	−	−
$\sigma_{\sigma 10}^{(2)}$	0.427	4.5	8.0	−0.44	3.2	−	−	−	−	−	−	−	−
$\sigma_{\sigma 0-}^{(2)}$	−0.240	−2.5	−4.6	0.84	−2.9	+	+	+	+	+	?	+	−
$\sigma_{\sigma 12}^{(2)}$	0.090	0.96	2.0	−0.27	0.66	−	−	−	−	−	−	−	−
$\sigma_{\sigma 2-}^{(2)}$	−0.0259	−0.27	−1.1	0.53	−0.52	+	+	+	+	−	+	+	+
$\sigma_{s1}^{(2)}$	−0.1045	−1.1	−2.4	−0.085	−1.2	+	+	+	+	−	+	+	+
$\sigma_{s-}^{(2)}$	0.138	1.4	3.2	0.11	1.5	−	−	−	−	+	−	−	−
$\sigma_{n0}^{(2)}$	−0.000327	−0.0034	0.036	−0.0051	−0.12	+	−	−	−	−	?	−	+
$\sigma_{n2}^{(2)}$	−0.00817	−0.085	−0.067	−0.021	−0.35	+	?	+	−	+	−	−	−
$N-R$ theory minus experiment, ratio to theory (in percent)		21	53	0	−21								

[a] Column a: Sign of coefficient for improvement of total cross section. Column b: Net sign of coefficient for shape improvement. Column c: Sign for shape improvement with respect to ratio $\sigma(-0.9)/\sigma(+0.2)$. Column d: Same as c, but for ratio $\sigma(-0.9)/\sigma(+0.8)$.

improvement (minus for positive integrated corrections, plus for negative ones) is an index of the extent to which a given correction can be effective in improving the agreement of the theoretical and experimental cross sections.

For the present purpose it is convenient to eliminate the necessity of referring to the condition that the two a's of any pair add up to unity, by replacing a_{u1} by $1-a_{u2}$, $a_{\sigma 10}$ by $1-a_{\sigma 20}$, etc., in Eq. (6.7). We then obtain for the summation in (6.7):

$$\sum a_i\sigma_i^{(2)}=\sigma_{u1}^{(2)}+a_{u2}\sigma_{u-}^{(2)}+\sigma_{\sigma 10}^{(2)}+a_{\sigma 20}\sigma_{\sigma 0-}^{(2)}+\sigma_{\sigma 12}^{(2)}+a_{\sigma 22}\sigma_{\sigma 2-}^{(2)}+\sigma_{s1}^{(2)}+a_{s2}\sigma_{s-}^{(2)}+a_{n0}\sigma_{n0}^{(2)}+a_{n2}\sigma_{n2}^{(2)}; \quad (7.2)$$

where the second-order σ's with minus signs in the subscripts are *differences* between the members of those pairs in (6.7) that come from relativistic operators having a common nonrelativistic limiting form. All coefficients that appear in this expression are completely at our disposal at least as far as the limit $v=0$ is concerned, since they multiply terms that vanish in this limit.

Table I presents (for the Yukawa well) the information necessary for arriving at all correlations, along with the signs of the coefficients necessary for total cross-section improvement and [from (7.1)] the two kinds of shape improvement. For comparison, the signs for the square well are also given. In the bottom row of Table I are listed the percent discrepancies between nonrelativistic theory and experiment, referred to the theory. Each of these numbers is directly comparable with the quantities in the column above it.

TABLE II. Results obtained with trial adjustments of cross section, using relativistic corrections.

Discrepancy for*	Total cross section	$\cos\theta=-0.9$	$\cos\theta=+0.2$	$\cos\theta=+0.8$
(1)	+25	+61	−1	−18
(2)	−5	+6	+10	−53
(3)	0	+3	−3	−45

* (1) Discrepancies with respect to $N-R$ theory [Interaction (2)] after addition of "non-arbitrary" corrections. All in percent. (2) Same after addition of "non-arbitrary" corrections plus $12\sigma_{\sigma 0-}^{(2)}$. (3) Same after addition of "non-arbitrary" corrections minus $18\sigma_{s-}^{(2)}$.

The correlations of $\sigma_{u1}^{(2)}$, $\sigma_{\sigma 10}^{(2)}$, $\sigma_{\sigma 12}^{(2)}$, and $\sigma_{s1}^{(2)}$ are not particularly important, since these are mostly small and have coefficient unity in (7.2). The correlations of principal importance are those of $\sigma_{u-}^{(2)}$, $\sigma_{\sigma 0-}^{(2)}$, $\sigma_{\sigma 2-}^{(2)}$, $\sigma_{s-}^{(2)}$, $\sigma_{n0}^{(2)}$ and $\sigma_{n2}^{(2)}$; a perfect correlation for any of these consists in having the *same* sign, either plus or minus, across the last four columns of the table.

It will be seen that for the square-well interaction most of the correlations are rather poor, though not entirely unfavorable. Much better correlations are obtained with the Yukawa interaction; these are, in fact, qualitatively almost perfect except for $\sigma_{n0}^{(2)}$ and $\sigma_{n2}^{(2)}$.

We can utilize Table I to obtain a rough idea of the magnitudes of the coefficients needed to adjust the nonrelativistic cross sections to experiment. These estimates will contain errors from several sources, to be described later, but arguments will be given to show that the errors do not interfere with an order of magnitude estimate. Only a few of the most favorable corrections will be used; since the coefficients required with these are already large enough to introduce substantial relativistic corrections at low energies, it is not necessary to go any further.

The procedure used in estimating the magnitudes of the coefficients will become clear on considering Table II, in which the results are presented. The first row of the table shows the percent discrepancies *after* the addition of $\sigma_{u1}^{(2)}+\sigma_{\sigma 10}^{(2)}+\sigma_{\sigma 12}^{(2)}+\sigma_{s1}^{(2)}$ (termed "non-arbitrary" corrections) to the nonrelativistic cross section, these being necessarily included in the cross section, formally speaking, if we use (7.2). These discrepancies are obtained by adding the percent R_i's for these corrections to the percent discrepancies in the last row of Table I; the resulting numbers are percent discrepancies referred to the uncorrected nonrelativistic theoretical cross section. If we now add $12\sigma_{\sigma 0-}^{(2)}$ to this result, we reduce the discrepancies to the amounts shown in row 2, Table II. If, instead, we subtract 18

$\sigma_{s-}{}^{(2)}$, the remaining discrepancies are those in row 3. The two corrections $\sigma_{\sigma 0-}{}^{(2)}$ and $\sigma_{s-}{}^{(2)}$ are the largest relativistic corrections, as well as the best individual ones for over-all curve-fitting purposes.

The difficulty with the Yukawa interaction corrections is that their values at $\cos\theta=-0.9$ and $+0.8$, although such as to correct the *ratio* $\sigma(-0.9)/\sigma(+0.8)$ in harmony with the other ratios, are such as to lower the $+0.8$ value, while it should be at least unchanged, if not actually raised a little. This is not a large effect absolutely, however, since the cross section is itself small here.

The effect of relativistic contributions of the magnitudes obtained above, at low energies, can be illustrated by considering the deuteron. For the Yukawa well, the mean value of v^2/c^2 for either particle in the deuteron is about 0.016. The coefficients twelve and eighteen then imply relativistic effects on deuteron properties of about 20 percent or 30 percent. These are quite substantial, and could of course turn out to be even greater in actual calculations. Moreover, the estimates given are, if anything, lower limits on the basis of the available information.

In emphasizing the largeness of the coefficients, it is well not to lose sight of one aspect of our results, already mentioned in the introductory section, which is highly favorable to the construction of a workable phenomenological interaction incorporating relativistic corrections; namely, the strikingly consistent correlations shown by Table I for the Yukawa corrections. These will contribute greatly to simplifying any eventual interaction of this type.

The errors due to the method of calculating the relativistic corrections remain to be discussed in relation to their possible effect on the above conclusions. In the first place, there is the use of the Born approximation. Guided by the form of the calculation, we might expect this error, relatively, to be of the same order as the relative error of the nonrelativistic Born approximation, a high estimate of which is perhaps 100 percent. This amount would not change our qualitative conclusions. There is also the neglect of terms of order higher than v^2/c^2; but since these are of order v^4/c^4, the relative error involved is nominally only 5 percent at 90 Mev.

A gross error in the relativistic corrections might occur if some components of the imaginary part of the Born nonrelativistic amplitude had the wrong sign [v. N. F. Mott and H. W. S. Massey, *Theory of Atomic Collisions*, 2d Ed. (Oxford University Press, London, 1949), p. 24]. This can easily occur, without making the Born *intensity* wrong, if any phase shifts are close to 90°. The only available information in this is the values of the phase shifts at 80 Mev for a square well and central forces, but with symmetric exchange.[14] These are all well under 90°.

The author is indebted to Professor Herman Feshbach for his original suggestion of this problem, for his understanding guidance during the entire course of the work, and for many concrete suggestions. He also wishes to thank Professor Victor F. Weisskopf for his interest in the work.

The aid of Mrs. Barbara Siegle Levine, and the Misses Patricia Boland, Hannah Paul, and Elizabeth Campbell, who did the computational work, is gratefully acknowledged.

[14] Bethe and Camac, Phys. Rev. **73**, 191 (1948).

APPENDIX

(1) When the Dirac spin-momentum operators, from which the ζ of (6.5) are derived, are substituted for $U_{\rm rel}$ in (5.2), and (3.4) is used for the G's in (5.3), the resulting operators are of zero to fourth order in $cp_{if}/(E+\mu c^2)$. We shall here give the second-order part in $cp_{if}/(E+\mu c^2)$ of the momentum matrix element [see (5.3)] of the operator obtained when the Dirac spin-momentum operator related to each of the ζ is substituted for $U_{\rm rel}$ in (5.2); this will be a Pauli spin operator, and will be designated by $\zeta'^{(2)}$, along with the subscript of the particular ζ from which it is obtained.

Much simplicity is gained by working in the CM system, in which

$$\mathbf{p}_{2i}=-\mathbf{p}_{1i},\quad \mathbf{p}_{2f}=-\mathbf{p}_{1f};$$

hence also [see Eq. (2.2)]

$$\mathbf{p}_i=\mathbf{p}_{1i},\quad \mathbf{p}_f=\mathbf{p}_{1f}.$$

The momentum exchange operator P_p turns $\mathbf{p}_i$ into $-\mathbf{p}_i$; we define

$$\mathfrak{p}_{if}\equiv P_p\mathbf{p}_{if}=P_p(\mathbf{p}_i-\mathbf{p}_f)=-\mathbf{p}_i-\mathbf{p}_f.$$

With energy conservation, $|\mathbf{p}_i|=|\mathbf{p}_f|$, and $\mathfrak{p}_{if}\cdot\mathbf{p}_{if}=0$. The following expressions are then obtained in the CM system, for transitions in which energy is conserved ($D\equiv c^2/(E+\mu c^2)^2$):

$$\zeta_{u1}{}'^{(2)}=D[\tfrac{1}{2}(p_{if}{}^2-\mathfrak{p}_{if}{}^2)+i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{p}_i\times\mathbf{p}_f],$$

$$\zeta_{u2}{}'^{(2)}=D[-\tfrac{1}{2}p_{if}{}^2+\tfrac{3}{2}\mathfrak{p}_{if}{}^2-p_{if}{}^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+p_{if}{}^2s(\mathbf{p}_{if})-3i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{p}_i\times\mathbf{p}_f],$$

$$\zeta_{\sigma 1}{}'^{(2)}=D[-p_{if}{}^2+(\tfrac{3}{2}\mathfrak{p}_{if}{}^2-\tfrac{1}{2}p_{if}{}^2)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+p_{if}{}^2s(\mathbf{p}_{if})-2\mathfrak{p}_{if}{}^2s(\mathfrak{p}_{if})-3i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{p}_i\times\mathbf{p}_f],$$

$$\zeta_{\sigma 2}{}'^{(2)}=D[\tfrac{1}{2}(p_{if}{}^2-\mathfrak{p}_{if}{}^2)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-p_{if}{}^2s(\mathbf{p}_{if})+2\mathfrak{p}_{if}{}^2s(\mathfrak{p}_{if})+i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{p}_i\times\mathbf{p}_f],$$

$$\zeta_{s1}{}'^{(2)}=D[\mathfrak{p}_{if}{}^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+\tfrac{1}{2}(p_{if}{}^2+\mathfrak{p}_{if}{}^2)s(\mathbf{p}_{if})-\mathfrak{p}_{if}{}^2s(\mathfrak{p}_{if})-\mathfrak{p}_{if}{}^2s(\mathbf{p}_{if}\times\mathfrak{p}_{if})],$$

$$\zeta_{s2}{}'^{(2)}=-2Dp_i{}^2s(\mathbf{p}_{if}),$$

$$\zeta_{n}{}'^{(2)}=Dp_i{}^2s(\mathbf{p}_{if}).$$

(2) The second-order contributions to the cross section associated with any of the operators (6.5) are given by expressions constructed analogously to Eq. (6.8). We give here a compact expression for the result of the trace calculation, which is sufficiently general to be used with any of the $\zeta'^{(2)}$ listed in the first part of this Appendix.

Let

$$\mathfrak{IC}=\frac{4\pi}{3}V_0D\{f_1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2f_2+s_+f_3+s_-f_4+s_\times f_5\}$$

where $f_1\cdots f_5$ are arbitrary functions of p_{if} and $\mathfrak{p}_{if}$; and

$$s_+=\tfrac{1}{2}(1+P_p)s(\mathbf{p}_{if});\quad s_-=\tfrac{1}{2}(1-P_p)s(\mathbf{p}_{if});\quad s_\times=s(\mathbf{p}_{if}\times\mathfrak{p}_{if}).$$

Then if $\mathfrak{U}$ is given by (6.4), the following expression holds in the CM system ($\mathbf{p}_{if}$, $\mathfrak{p}_{if}$, and $\mathbf{p}_{if}\times\mathfrak{p}_{if}$ being mutually perpendicular):

$$Tr(Y\mathfrak{IC})(Y\mathfrak{U})=(4\pi/3)^2V_0{}^2D(A+4P_pA+2B+2P_pB),$$

where

$$A=2\{[3gf_1+(2-3g)(3f_2+f_3+f_5)]\varphi_0(p_{if})+\gamma[-f_3-3f_4+2f_5]\varphi_2(p_{if})\},$$

$$B=2\{[3(1-g)f_1+(-1+3g)(3f_2+f_3+f_5)]\varphi_0(\mathfrak{p}_{if})+\gamma[-f_3+3f_4+2f_5]\varphi_2(\mathfrak{p}_{if})\}.$$

(3) The nonrelativistic cross section in the Born approximation for the interaction (6.1) is

$$\sigma_{NR}(\theta)=\frac{\mu^2V_0{}^2}{3\hbar^4}\{[1-3g+3g^2][\varphi_0{}^2(p_{if})+4\varphi_0{}^2(\mathfrak{p}_{if})]+2[-1+6g-6g^2]\varphi_0(p_{if})\varphi_0(\mathfrak{p}_{if})+2\gamma^2[\varphi_2{}^2(p_{if})-2\varphi_2(p_{if})\varphi_2(\mathfrak{p}_{if})+4\varphi_2{}^2(\mathfrak{p}_{if})]\}.$$