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On the Space Exchange Magnetic Moments of Light Nuclei*

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The calculations were based on Sachs' phenomenological theory and on the independent particle model. Spin exchange moments were not included. Uncertainties in the parameters required to characterize the shell dimensions and the neutron-proton interaction, which was assumed to be of an entirely exchange nature, cause the calculated values to be uncertain by a factor of about 3; they are probably too large. For L-S coupling, the values found in the $1p$ shell for ^2P states for Li^7, Be^9, B^{11}, C^{13}, and N^{15}, and in the $1d$ shell for ^2D states for Ne^{21}, Na^{23}, Cl^{37}, and K^{39} were -0.04, -0.01, 0.1, 0.05, 0.1, -0.4, -0.1, 0.4, and 0.4 nuclear Bohr magnetons, respectively. For j-j coupling, the values found for $B^{11}(p_{3/2})^{-1}$, $C^{13}(p_{1/2})$, and $N^{15}(p_{1/2})$ were 0.2, -0.09, and 0.1, respectively. The signs and ratios of these values are such as to lead to the conclusion that the simple models used here cannot be brought into agreement with the data by the addition of space exchange contributions alone. The magnitudes are nevertheless reasonably large in some cases, and indicate that they must be included in any consistent theory of nuclear magnetic moments.

I. INTRODUCTION

No theory thus far has accounted adequately for the experimental magnetic moments of even the light nuclei. The measure of success enjoyed by the various models is due largely to the spin contribution, which is the same for most of them. While it may well be that these models are inadequate, the possibility remains that agreement with the data might be achieved by the inclusion of the space and spin exchange magnetic moment contributions.

The complete exchange moment operator can only be derived by meson theory and as such is rather questionable. However, Sachs showed that the space exchange part, henceforth denoted by M_x, can be arrived at phenomenologically;[1] the potential is still arbitrary. While no definitive conclusion can be reached by a calculation which omits the spin exchange contribution,[2] the values of M_x obtained for a given model should be helpful in judging the validity of that model.

Sachs calculated M_x for the case of H^3 and obtained an extremely small value.[1] This was to be expected, since, as will be shown, M_x vanishes for an S state so that its non-zero value in H^3 is due to the small admixture of D state in H^3. We shall consider nuclei whose ground states are predominantly other than S states. We shall treat our ground states as pure states.

II. PROPERTIES OF THE SPACE EXCHANGE MAGNETIC MOMENT OPERATOR

The expression derived by Sachs can be written in the form

$$M_x = b\sum_\nu\sum_\pi \int (\psi, O_{\pi\nu z}\psi)d\tau, \qquad (1)$$

where

$$O_{\pi\nu z} = (i/a_0^2 J_0)(\mathbf{r}_\pi \times \mathbf{r}_\nu)_z J_{\pi\nu} P_{\pi\nu},$$

and

$$b = J_0 a_0^2 M/\hbar^2.$$

Here M_x is in nuclear Bohr magnetons, the index x indicating its exchange nature, the indices ν and π refer to neutrons and to protons, respectively, $P_{\pi\nu}$ is the space exchange operator, $J_{\pi\nu}P_{\pi\nu}$ is the space exchange interaction, M is the proton mass, $\mathbf{r}$ is the distance of the nucleon from the center of gravity, and the subscript z represents the z component; a_0 and J_0, the range and strength of the space exchange interaction, have been introduced to make b and the integral dimensionless. The scalar product represents the summation over spins, and the space integration is over all particles.

There are a number of immediate consequences of the above form.

(a) To within the approximation of the equality of neutron-neutron and proton-proton forces, the M_x's of conjugate pairs of nuclei, that is, mirror nuclei, are equal in magnitude and opposite in sign.[1] This follows from the antisymmetry of the operator $O_{\pi\nu z}$ in neutrons and protons.

(b) As a special case of (a), we have that the M_x's of self-conjugate nuclei vanish.[1]

(c) The contribution to M_x of a neutron and a proton in a completely space symmetric or completely space antisymmetric state vanishes. This follows from the antisymmetry of the operator $O_{\pi\nu z}$ in neutrons and protons.

(d) The contribution to M_x of a neutron and a proton in space states which do not overlap vanishes. This follows from the presence of the operator $P_{\pi\nu}$.

(e) The M_x's of nuclei with zero orbital angular momentum vanish. In the independent particle model, since closed shells have no angular momenta, the contribution to M_x of a closed shell of neutrons interacting with a closed shell of protons vanishes. These follow from the fact that $O_{\pi\nu z}$ has the transformation properties of an $L=1$ function under space rotation.

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[1] R. G. Sachs, Phys. Rev. 74, 433 (1948).
[2] R. K. Osborne and L. L. Foldy have recently derived the spin exchange moments phenomenologically; the radial functions are however arbitrary. Phys. Rev. 79, 795 (1950).

The fact that a nucleon pair which contributes to M_x must consist of a neutron and a proton is physically evident since space exchange magnetic moments are due to exchange currents, and therefore charge must be exchanged.

III. MATRIX ELEMENTS ON THE INDEPENDENT PARTICLE MODELS[3]

The only models which we shall consider are the independent particle models, which are the ones which have enjoyed the most over all success. There are then three types of non-vanishing contributions to M_x:

(A) One of the nucleons is in a closed shell and the other is in an incomplete shell.

(B) The nucleons are in the same incomplete shell.

(C) The nucleons are in different incomplete shells. We shall have no occasion to consider contributions of this type.

We first study the case of *L-S coupling*. We consider type (A), and define $M_x(nl\gamma n'l')$ as the contribution to M_x due to the interaction between an incomplete shell of neutrons and protons with radial, orbital, and other quantum numbers n, l, and γ, respectively, and radial functions R_{nl}, and a complete shell of $2(2l'+1)$ neutrons and $2(2l'+1)$ protons with quantum numbers n' and l' and radial functions $R_{n'l'}$. We then find, as shown in Appendix I, that

$$M_x(nl\gamma n'l')=b\langle L_{\pi z}-L_{\nu z}\rangle_{\gamma l}2(2l'+1)I(nln'l'), \quad (2)$$

where $\langle L_{\nu z}\rangle_{\gamma l}$ and $\langle L_{\pi z}\rangle_{\gamma l}$ are the z components of the orbital angular momenta of the neutrons and protons, respectively, in the incomplete shell, and

$$I_{nln'l'}=(i/4\pi m a_0^2 J_0)\int R_{nln'l'}(\pi,\nu)(\mathbf{r}_\nu\times\mathbf{r}_\pi)_z J_{\pi\nu} \times Y_{lm}^*(\nu)Y_{lm}(\pi)P_{l'}(\cos\theta_{\pi\nu})d\tau_\pi d\tau_\nu.$$

m is any projection of l, Y_{lm} is a spherical harmonic, $P_{l'}$ is a Legendre function, and

$$R_{nln'l'}(\pi,\nu)=R_{nl}(r_\pi)R_{nl}(r_\nu)R_{n'l'}(r_\pi)R_{n'l'}(r_\nu).$$

Contributions of type (B) must usually be calculated explicitly by means of Eq. (1). However, if either the neutrons or the protons in the incomplete shell n, l form a closed subshell, Eq. (2) will be valid with $n'=n$, $l'=l$. Furthermore the number of explicit calculations can be reduced by means of hole theory. Thus the magnetic quantum numbers which are absent in the wave function of a nucleus are opposite in sign to those which are present in the nucleus which is its hole with respect to both neutrons and protons. For the contribution within an incomplete shell, $M_x(nl\gamma)$, if the nucleus and its hole have the same total angular momentum, J, we find[4]

$$M_x(nl\gamma)=-M_x(nl\gamma)(\text{hole}) +b\langle L_{\pi z}-L_{\nu z}\rangle_{\gamma l}2(2l+1)I_{nlnl}. \quad (3)$$

We turn now to *j-j coupling*. Though the elimination of the spin coordinates is no longer completely trivial,

TABLE I. Magnetic moments of the light nuclei with *L-S* coupling.

	M(exp)	State	$\langle L_{\pi z}\rangle$	M(non)	M_x	ΔM
Li^7	3.26	$^2P_{3/2}$	1/3	$\mu_p+1/3 = 3.12$	$-b(1/3)2I_{1110} \approx -0.04$	+0.14
Be^9	−1.18	$^2P_{3/2}$	1/3	$\mu_n+1/3 = -1.58$	$-b(1/3)2I_{1110}+b(3/5)I_{1111} \approx -0.01$	+0.40
B^{11}	2.69	$^2P_{3/2}$	2/3	$\mu_p+2/3 = 3.46$	$b(1/3)2I_{1110}+b(13/5)I_{1111} \approx 0.1$	−0.77
C^{13}	0.70	$^2P_{1/2}$	4/9	$-(1/3)\mu_n+4/9 = 1.08$	$b(2/9)2I_{1110}+b(4/3)I_{1111} \approx 0.05$	−0.38
N^{15}	(−)0.28	$^2P_{1/2}$	2/3	$-(1/3)\mu_p+2/3 = -0.26$	$b(2/3)(2I_{1110}+6I_{1111}) \approx 0.1$	−0.02
Ne^{21}		$^2D_{5/2}$	0	$\mu_n = -1.91$	$-b(2)(A) \approx -0.4$	
Ne^{21}		$^2D_{3/2}$	0	$-(3/5)\mu_n = 1.15$	$-b(9/5)(A) \approx -0.4$	
Na^{23}	2.22	$^2D_{3/2}$	3/5	$-(3/5)\mu_p+3/5 = -1.07$	$-b(3/5)(A) \approx -0.1$	+3.29
Cl^{37}	0.68	$^2D_{3/2}$	9/5	$-(3/5)\mu_p+9/5 = 0.13$	$b(9/5)(A+10I_{1212}) \approx +0.4$	+0.55
K^{39}	0.39	$^2D_{3/2}$	9/5	$-(3/5)\mu_p+9/5 = 0.13$	$b(9/5)(A+10I_{1212}) \approx +0.4$	+0.26

M(exp) is the experimental value of the magnetic moment in nuclear Bohr magnetons.
M(non) is the theoretical non-exchange value for *L-S* coupling.
$A=2I_{1210}+6I_{1211}+2I_{1220}$
$\Delta M=M(\text{exp})-M(\text{non})$.

TABLE II. Magnetic moments with *j-j* coupling.

	M(exp)	State	$\langle L_{\pi z}\rangle$	M(non)	M_x	ΔM
B^{11}	2.69	$(p_{3/2})^{-1}$	1	$\mu_p+1 = 3.79$	$b(2I_{1110}+4I_{1111}) \approx +0.2$	−1.10
C^{13}	0.70	$(p_{1/2})$	0	$-(1/3)\mu_n = 0.64$	$b(-2/3)(2I_{1110}+4I_{1111}) \approx -0.09$	+0.06
N^{15}	(−)0.28	$(p_{1/2})$	2/3	$-(1/3)\mu_p+2/3 = -0.26$	$b(2/3)(2I_{1110}+(4+2)I_{1111}) \approx +0.1$	−0.02

M(exp) is the experimental value of the magnetic moment in nuclear Bohr magnetons.
M(non) is the theoretical non-exchange value for *j-j* coupling.
$\Delta M=M(\text{exp})-M(\text{non})$.

[3] An excellent review of the merits of the various models and a useful list of references to papers dealing with shell models is contained in a recent article by E. Feenberg, Phys. Rev. **77**, 771 (1950). See also M. G. Mayer, Phys. Rev. **78**, 16, 22 (1950).

[4] This is a functional rather than a numerical relationship, the parameters appearing in the I_{nlnl} being different for the nucleus and for its hole.

arguments similar to those of Appendix I lead to the result that the contribution to M_x due to the interaction of an incomplete shell with quantum numbers n, l, j, and γ, and of a closed shell with quantum numbers n', l', and j', is given by

$$M_x(nlj\gamma n'l'j')=b\langle L_{\pi z}-L_{\nu z}\rangle_{\gamma lj}(2j'+1)I_{nln'l'}. \quad (2')$$

As is quite reasonable, this differs from $M_x(nl\gamma n'l')$ in that $2(2l'+1)$ has been replaced by $(2j'+1)$, the number of particles in a closed shell of particles with quantum number j'. Here too the expression is valid within an incomplete shell, if either the neutrons or the protons form a closed subshell, with $n'=n$, $l'=l$, $j'=j$.

IV. APPLICATION TO LIGHT NUCLEI WITH L-S COUPLING

By argument (*b*) above, the only stable nuclei in the $1p$ shell with non-vanishing M_x's are Li^7, Be^9, B^{11}, C^{13}, and N^{15}. The theoretical values of $\langle L_{\pi z}\rangle$ and of $\langle L_{\nu z}\rangle$ (and of the non-exchange magnetic moments) are known for these nuclei.[5] The contributions due to the interaction of the particles in the $1p$ shell with the particles in the $1s$ shell follow immediately from Eq. (2). By (*c*), $M_x(11\gamma)(Li^7)=0$, since the proton and the two neutrons in the $1p$ shell are assumed to be in a space symmetric state; this is true for $j=3/2$ and for $j=1/2$. Only in the case of Be^9 must an explicit calculation be performed; using the wave function in Appendix II, one finds $M_x(11\gamma)(Be^9)=(3/5)bI_{1111}$. The others now follow from Eqs. (2) and (3).

The M_x's of O^{17} and F^{19} vanish to a first approximation, by (*e*), since these are predominantly in S-states. In the $1d$ shell we choose only some simple cases at the beginning and end of the shell. For Ne^{21}, Cl^{37}, and K^{39}, we have one neutron, three proton holes, and one proton hole, respectively, in the $1d$ shell, so that the specification of J, L, and S, uniquely determines the non-exchange magnetic moment and the M_x, the latter following from Eq. (2). We assume that the $(1d)(1d)^2$ configuration in Na^{23} is completely symmetric, in which case the state is $^2D_{3/2}$, $M_x(12\gamma)=0$, and $L_{\pi z}=(1/3)L_z$. Table I gives our results.

The numerical evaluation of the M_x's is discussed in Appendix III; since all of the relevant $I_{nln'l'}$ are positive and since b is positive, the signs of all of the M_x's but that of Be^9, where a difference of terms appears, are determined. The uncertainties in the parameters required to characterize the exchange interaction and the radial functions make the values uncertain by a factor of about two for the $1p$ shell nuclei and of about three for the $1d$ shell nuclei.

V. APPLICATION TO LIGHT NUCLEI WITH j-j COUPLING

We restrict our calculations to the simple cases where the assumption that the $p_{3/2}$ and $p_{1/2}$ shells fill in that order uniquely determines the wave functions, namely, B^{11} ($p_{3/2}$ proton hole), C^{13} ($p_{1/2}$ neutron), and N^{15} ($p_{1/2}$ proton). N^{15} has the same wave function and therefore the same M_x as in the case of L-S coupling. By Eq (2′), we have the results of Table II. In general M_x can be expected to be greater in absolute magnitude for j-j than for L-S coupling since in the latter case the wave functions are so constructed as to be as space-symmetric as possible. To put it in another way, where M_x within a shell does not vanish under either assumption, the two values will be of the same order of magnitude, but there will be cases, such as Li^7, where M_x within a shell will vanish for L-S but not for j-j coupling.

VI. CONCLUSIONS

While the magnitudes of the M_x's are uncertain by a factor of as much as 3 in some cases (the values given are probably too large), the signs of the M_x's are uniquely determined for all cases except possibly Be^9. These signs are in the wrong direction for the three j-j coupling cases considered (Table II) and for all of the L-S coupling cases other than Cl^{37} and K^{39} (Table I). The last two nuclei have the same non-exchange magnetic moments and their M_x's differ only very slightly due to the small difference in their dimensions and hence in their integrals. They should therefore have the same magnetic moments, which is again in contradiction with the data. One must therefore conclude that while the space exchange magnetic moments can be significantly large, the simple models used here can not be brought into agreement with the data by the addition of space exchange contributions alone.

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APPENDIX I

Evaluation of the $M_x(nl\gamma n'l')$

We define $M_x(nl\gamma n'l')$ as the contribution to M_x due to the interaction between an incomplete shell of s neutrons and t protons with quantum numbers n, l, and γ and a closed shell of $2(2l'+1)$ neutrons and $2(2l'+1)$ protons with quantum number n' and l'. (Many of the terms used in this appendix have been defined in Section III.) We need not antisymmetrize with respect to particles in different shells, and we thus have as our normalized wave function

$$\psi=\chi_1\chi_2\sum_{q,r}C_{qr}\Theta_q\Omega_r.$$

$\chi_1(1,\cdots 4l'+2)$ and $\chi_2(1,\cdots 4l'+2)$ are the same normalized wave functions, antisymmetric in the space and spin coordinates of the neutrons and protons, respectively, and constructed from the one particle functions $\varphi_{n'l'm_l'm_s'}$, where m_l' and m_s' are the projections of the orbital and intrinsic spin momenta. All values of m_l' and of m_s' appear in the closed shell functions χ_1 and χ_2. $\Theta_q(4l'+2+1,\cdots 4l'+2+s)$ and $\Omega_r(4l'+2+1,\cdots 4l'+2+t)$ are each an orthonormal set of functions, antisymmetric in the space and spin coordinates, constructed from the one particle functions $\varphi_{nlm_l^qm_s^q}$ and $\varphi_{nlm_l^rm_s^r}$ respectively. The superscripts q and r

[5] L. Rosenfeld, *Nuclear Forces* (Interscience Publishers, Inc., New York), Vol. II, p. 409. These values can in fact be calculated without the use of normalized wave functions. See Appendix II.

specify which set of values of m_l and m_s appear in that particular function. The C_{qr} are the expansion coefficients.

$M_x(nl\gamma n'l')$ can be broken up into the contribution due to the interaction of the neutrons in the (n, l) shell and the protons in the closed shell, $M_{x\nu}(nl\gamma n'l')$, and the contribution due to the interaction of the protons in the (n, l) shell and the neutrons in the closed shell, $M_{x\pi}(nl\gamma n'l')$. Since $O_{\pi\nu z}$ affects only one neutron and one proton at a time, all cross terms vanish and we find

$$M_{x\nu}(nl\gamma n'l')=b\sum_{\pi=1}^{4l'+2}\sum_{q,r}|C_{qr}|^2\sum_{m_l{}^q}\sum_{m_s{}^q}\times\int(\varphi_{nl}(\nu)_{m_l{}^q m_s{}^q}\chi_2(1,\cdots 4l'+2),\, O_{\pi\nu z}\varphi_{nl}(\nu)_{m_l{}^q m_s{}^q}\chi_2(1,\cdots 4l'+2))d\tau.$$

Since the closed shell function χ_2 has no angular momenta, the angular momenta of $\varphi_{nlm_l{}^q m_s{}^q}\chi_2$ are those associated with $\varphi_{nlm_l{}^q m_s{}^q}$. We make use of the matrix relation[6]

$$(l, m_l{}^q, m_s{}^q|O_{\pi\nu z}|l, m_l{}^q, m_s{}^q)=m_l{}^q\times(1/m)(l, m|O_{\pi\nu z}|l, m),$$

where m is any projection of l. The spin coordinates give the factor unity, since $O_{\pi\nu z}$ is independent of spin. Proceeding along identical lines, we find for $\langle L_{\nu z}\rangle$, which is due to the neutrons in the (n, l) shell,

$$\langle L_{\nu z}\rangle_{\gamma l}=\sum_{q,r}|C_{qr}|^2\sum_{m_l{}^q}m_l{}^q.$$

Combining these results,

$$M_{x\nu}(nl\gamma n'l')=b\langle L_{\nu z}\rangle_{\gamma l}\sum_{\pi=1}^{4l'+2}(1/m)\int(\varphi_{nlm}(\nu)\chi_2,\, O_{\pi\nu z}\varphi_{nlm}(\nu)\chi_2)d\tau.$$

We now write $\varphi_{nlm}(\mathbf{r})=R_{nl}(r)Y_{lm}(\theta,\varphi)$, express $\chi_2(1,\cdots 4l'+2)$ in terms of its one particle components, use the Legendre addition theorem to perform the sum over m', the projection of l', and find, with a factor of 2 appearing due to summation over spins,

$$M_{x\nu}(nl\gamma n'l')=b\langle -L_{\nu z}\rangle_{\gamma l}2(2l'+1)I_{nln'l'}.$$

$I_{nln'l'}$ is independent of the m which appears in its definition. We write

$$I_{nln'l'}=(1/m)K_{nlmn'l'},$$

and shall choose values of m for convenience in integration.

To obtain $M_{x\pi}(nl\gamma n'l')$ we must replace $\langle -L_{\nu z}\rangle_{\gamma l}$ by $\langle L_{\pi z}\rangle_{\gamma l}$. The change in sign is due to the antisymmetry in neutrons and protons of $O_{\pi\nu z}$. Adding the expressions $M_{x\nu}(nl\gamma n'l')$ and $M_{x\pi}(nl\gamma n'l')$, we obtain Eq. (2) for $M_x(nl\gamma n'l')$.

APPENDIX II

Angular Dependence of the Ground-State Wave Functions of Odd Mass Nuclei in the $1p$ Shell for L–S Coupling

While many properties of these nuclei depend only on their quantum numbers and symmetry properties, other properties depend in greater detail on the wave functions, which do not seem to have been tabulated for the assumption of the equality of nuclear forces. We let nucleons 5, 9, 13 and 6, 10, 14 and 7, 11, 15 and 8, 12, 16 represent, when present, neutrons with spin up or down, neutrons with spin down or up, protons with spin up or down, and protons with spin down or up, respectively, in the $1p$ shell. The functions must, of course, be antisymmetric in space in the particles with the same spin and isotopic spin. For any vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, we define the completely symmetric vector

$$[\mathbf{a},\mathbf{b},\mathbf{c}]=(\mathbf{a}\cdot\mathbf{b})\mathbf{c}+(\mathbf{b}\cdot\mathbf{c})\mathbf{a}+(\mathbf{c}\cdot\mathbf{a})\mathbf{b}.$$

We let the unit vector $\mathbf{i}$ represent the i'th particle and $q=(4\pi)^{-1/2}$.

[6] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, London, 1935), p. 63.

We define the vectors

$$\boldsymbol{\omega}(\mathrm{Li}^7)=(9/5)^{1/2}q^3[\mathbf{5},\mathbf{6},\mathbf{7}]$$
$$\boldsymbol{\omega}(\mathrm{Be}^9)=(81/20)^{1/2}q^5(\mathbf{5}\times\mathbf{9})\times[\mathbf{6},\mathbf{7},\mathbf{8}]$$
$$\boldsymbol{\omega}(\mathrm{B}^{11})=(729/80)^{1/2}q^7[\mathbf{5}\times\mathbf{9},\mathbf{6}\times\mathbf{10},\mathbf{7}\times\mathbf{11}]\times\mathbf{8}$$
$$\boldsymbol{\omega}(\mathrm{C}^{13})=(3^7/80)^{1/2}q^9(\mathbf{5}\cdot\mathbf{9}\times\mathbf{13})[\mathbf{6}\times\mathbf{10},\mathbf{7}\times\mathbf{11},\mathbf{8}\times\mathbf{12}]$$
$$\boldsymbol{\omega}(\mathrm{N}^{15})=(3^8/16)^{1/2}q^{11}(\mathbf{5}\cdot\mathbf{9}\times\mathbf{13})(\mathbf{6}\cdot\mathbf{10}\times\mathbf{14})(\mathbf{7}\cdot\mathbf{11}\times\mathbf{15})(\mathbf{8}\times\mathbf{12}).$$

The functions $\omega^{\pm 1}=(\omega_x\pm i\omega_y)/2^{1/2}$ and $\omega^0=\omega_z$ which have $L=1$, $m_L=\pm 1$ and 0, respectively, must be multiplied by the spin functions, antisymmetrized, and combined to obtain the function with the desired J and J_z. These functions have the greatest possible space symmetry consistent with the exclusion principle.

These functions, which are readily constructed, are convenient to handle and exhibit the relation between a nucleus and its hole with respect to both neutrons and protons. Furthermore, the non-exchange moments follow rather easily from them. Thus, in the case of the $^2P_{3/2}$ ground state of Be^9, for example, we have $J=J_z=3/2$, $L=L_z=1$, and $S=S_z=S_{\pi z}=1/2$. The $L=1$ function $(\mathbf{5}\times\mathbf{9})\times[\mathbf{6},\mathbf{7},\mathbf{8}]$ is constructed from two functions, $(\mathbf{5}\times\mathbf{9})$ and $[\mathbf{6},\mathbf{7},\mathbf{8}]$, each with $L=1$, and each having $\langle L_z\rangle=1/2$. Since the function $[\mathbf{6},\mathbf{7},\mathbf{8}]$ is completely symmetric, the protons 7 and 8 have $L_{\pi z}=2/3\times 1/2=1/3$, and thus $M(\mathrm{non})(\mathrm{Be}^9, {}^2P_{3/2})=\mu_p+1/3$. The normalization factors are not used.

APPENDIX III

Evaluation of the Integrals

For the explicit evaluation of the $I_{nln'l'}$ we set

$$J_{12}=-J_0\exp(-r_{12}{}^2/a_0{}^2)$$

and take oscillator functions for the R_{nl}. In particular,

$$R_{1l}=\left(\frac{2^{3l+7/2}l!}{\pi^{1/2}(2l+1)!}\right)^{\frac{1}{2}}(\alpha_{nl}/a_0)^{3/2}\rho^l\exp(-\rho^2),$$

$$R_{20}=(2^{9/2}/3\pi^{1/2})^{1/2}(\alpha_{20}/a_0)^{3/2}(2\rho^2-3/2)\exp(-\rho^2),$$

where we have let $\rho=\alpha_{nl}r/a_0$. a_0/α_{nl} is a measure of the dimension of the nl shell. It is then always possible to throw $I_{nln'l'}$ into the form

$$I_{nln'l'}=\int g(\rho_1{}^2,\rho_2{}^2,\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2)\exp(-\beta\rho_1{}^2-\beta'\rho_2{}^2+p\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2)d\tau_1 d\tau_2$$
$$=g\left(-\frac{\partial}{\partial\beta},-\frac{\partial}{\partial\beta'},\frac{\partial}{\partial p}\right)8\pi^3(4\beta\beta'-p^2)^{-3/2},$$

where $\beta=\beta'=1+\alpha_{nl}{}^2+\alpha_{n'l'}{}^2$ and $p=2$. The desired form can be obtained by taking $I_{11n'l'}=(1/1)K_{111n'l'}$ and

$$I_{12n'l'}=1/5[(1/1)K_{121n'l'}+4(1/2)K_{122n'l'}].$$

One finds

$$I_{1110}=(1/4)\alpha_{11}{}^5\alpha_{10}{}^3\gamma_{1110}{}^5$$
$$I_{1111}=(5/24)\alpha_{11}{}^3(\alpha_{11}{}^2+1)^{-7/2}$$
$$I_{1210}=(1/8)\alpha_{12}{}^7\alpha_{10}{}^3\gamma_{1210}{}^7$$
$$I_{1211}=(1/48)\alpha_{12}{}^7\alpha_{11}{}^5\gamma_{1211}{}^9(7+4\gamma_{1211}{}^{-2})$$
$$I_{1220}=(1/192)\alpha_{12}{}^7\alpha_{20}{}^3\gamma_{1220}{}^9\{(40\alpha_{20}{}^2+36\alpha_{20}{}^2\alpha_{12}{}^2)+(7\alpha_{20}{}^4+4\alpha_{20}{}^2+4\alpha_{20}{}^2\alpha_{12}{}^2-6\alpha_{12}{}^2-3\alpha_{12}{}^4)^2\}$$
$$I_{1212}=(7/32)\alpha_{12}{}^3(\alpha_{12}{}^2+1)^{-11/2}(9+4\alpha_{12}{}^2+4\alpha_{12}{}^4)$$

where

$$\gamma_{nln'l'}=2(\alpha_{nl}{}^4+2\alpha_{nl}{}^2+2\alpha_{nl}{}^2\alpha_{n'l'}{}^2+2\alpha_{n'l'}{}^2+\alpha_{n'l'}{}^4)^{-1/2}.$$

The nuclear range is taken as $a_0=1.5\times 10^{-13}$ cm, and the magnitude of the exchange interaction is taken as $J_0=55$ Mev. This J_0 is some average over the singlet and triplet state interactions, where it has been assumed that the entire interaction is of an exchange nature, so that J_0 may be too large by a factor of as much as two. The α_{nl} are fixed by setting $\bar{r}=1.4\times 10^{-13}B^{1/3}$, where $\bar{r}$ is the average value of r and B is the number of nucleons up through and including the shell under consideration. The $I_{nln'l'}$ are uncertain to within a factor of about two.