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The Phenomenological Theory of Liquid Helium II*

SADAO NAKAJIMA, KAZUHISA TOMITA, AND TUNEMARU USUI**
Physics Department, Tokyo University, Tokyo, Japan
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The purpose of the present paper is to refine the "two-fluid model" as a phenomenological theory of liquid He II. General thermohydrodynamical equations are formulated in Part I, following the treatment of viscous fluids by Eckart and Tolman. The previous phenomenological theories, which have been proposed by Landau, Tisza, F. London and Zilsel, and Gorter, are included in the present general formalism, and their intrinsic relations are discussed from a unified point of view.

The thermodynamic equations of motion are applied in Part II to the stationary flow in He II under the reversible process approximation. Adopting Gorter's type of mutual friction, we can fully describe the fountain pressure (thermomechanical effect) and the anomalous heat transfer (heat-superconductivity). The theoretical predictions concerning the maximum fountain pressure or maximum heat current against temperature, and the critical current appointing the validity limit of H. London's thermostatical relation $dp/dT=\rho s$, are fairly consistent with observed tendencies.

PART I. THE GENERAL THEORY

1. Introduction

As Tisza[1] has insisted, it is very desirable to formulate the phenomenological equations of motion of liquid He II, which correspond to the well-known theory of superconductivity due to F. London and H. London, in order to give a consistent description of various experimental facts and establish a starting point for the considerations from a molecular point of view. Among such phenomenological theories the "two-fluid model" seems to be most promising. This idea was first treated qualitatively by F. London and Tisza, who were guided by the theory of Bose-Einstein condensation, and was then formulated successfully by Landau[2] and Tisza[1] in the case of infinitesimal motions, resulting in the theory of "second sound."

According to their theory, liquid He II is considered as a sort of fluid mixture consisting of the normal and the superfluids. Let the density and the velocity of the normal fluid be ρ_n and $\mathbf{V}_n$, and those of the superfluid ρ_s and $\mathbf{V}_s$. Then the total density

$$\rho=\rho_n+\rho_s$$

and the velocity of the center of gravity

$$\mathbf{V}_1=(\rho_n/\rho)\mathbf{V}_n+(\rho_s/\rho)\mathbf{V}_s$$

have the ordinary hydrodynamical meanings. On the other hand, the velocity of the normal fluid relative to the center of gravity

$$\mathbf{V}_2=\mathbf{V}_n-\mathbf{V}_1$$

is essential to the peculiarity of He II.

Recently F. London and Zilsel[3] have shown that the anomalous heat current in He II can be explained by adding to the Landau-Tisza equation a term which represents the ordinary viscosity of the normal fluid (a term of Navier-Stokes type). On the other hand, Gorter[4] has pointed out that the mutual friction between the normal and the superfluids plays an important role.

The purpose of the present paper is to formulate the idea of "two-fluid model" as generally as possible, and to put in order the various conception which have been discovered by many people, clarifying the intrinsic relations among them. Putting aside premature speculations about the molecular mechanism, we try to refine the two-fluid model as a phenomenological theory. In particular, we adopted Tisza's point of view that in He II, parameters such as the viscosity coefficient, coefficient of thermal conduction, etc., have similar values to those of He I, whereas the differential equations which characterize the motion of He II must be modified. To obtain such fundamental equations, we follow the treatment of a viscous fluid due to Eckart[5] and Tolman.[6] Thus we consider a region τ which is fixed in space and apply the conservation laws of mass and momentum and the first and second laws of thermodynamics to He II in this region. In the following sections, σ is the surface of the region τ, $d\boldsymbol{\sigma}$ the surface element (inward) normal vector to σ, $\mathbf{V}_n\cdot d\boldsymbol{\sigma}$ the scalar product, and $\Pi d\boldsymbol{\sigma}$ a vector which is obtained by operating with a tensor Π on the vector $d\boldsymbol{\sigma}$.

2. Foundations of the Theory

Experiments of Kapitza[7] indicate that no entropy is transferred by the flow of He II through a sufficiently narrow capillary. As F. London[8] has shown, one can define the fraction of the normal part for a given state of He II on the basis of this experimental evidence and without referring to any specific molecular model.

The result is

$$\rho_n/\rho=s/s_\lambda, \tag{2.1}$$

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** Not at the Physics Department, Nagoya University, Nagoya, Japan.

[1] L. Tisza, Phys. Rev. **72**, 838 (1947).

[2] L. Landau, J. Phys. U.S.S.R. **5**, 71 (1941).

[3] F. London and P. R. Zilsel, Phys. Rev. **74**, 1148 (1948).

[4] C. J. Gorter, Phys. Rev. **74**, 1544 (1948).

[5] C. Eckart, Phys. Rev. **58**, 267, 269 (1940).

[6] R. C. Tolman and P. C. Fine, Rev. Mod. Phys. **20**, 67 (1948).

[7] P. Kapitza, J. Phys. U.S.S.R. **5**, 59 (1941).

[8] F. London, Rev. Mod. Phys. **17**, 310 (1945).

where s denotes the entropy per unit mass and s_λ its value at the λ-point $T_\lambda(=2.19°\mathrm{K})$. Experimentally, the entropy proves to be a function of temperature only, under sufficiently low pressures, and is given by[1,8]

$$s=s_\lambda(T/T_\lambda)^r. \tag{2.2}$$

Experiments at Leiden give the values $s_\lambda=0.403$ cal./g-deg. and $r=5.6$, while those of Kapitza give the values $s_\lambda=0.405$ cal./g-deg. and $r=5.5$.

Now, using (2.1), we find the fundamental equation of *thermostatics* for the variation of the internal energy of a fixed volume τ,

$$\delta U=T\delta S+[u-Ts+(p/\rho)]\delta M,$$

is transcribed as

$$\delta U=u\delta M+(Ts_\lambda\delta M_n-Ts\delta M)+(p/\rho)\delta M, \tag{2.3}$$

where M, S, M_n, p, and u are, respectively, the total mass, the total entropy, the mass of the normal fluid, hydrostatic pressure, and the internal energy per unit mass. In order to step into *thermodynamics*, it is necessary to specify the detailed mechanism which governs the change of M and M_n. Especially, it is to be noted that M_n may vary not only by heat conduction (transformation of the superfluid into the normal, see later sections), but also through the flow of the normal fluid which is governed by the different velocity field from that of the total mass flow.

Thus, corresponding to each term of Eq. (2.3), we propose for the energy flow through the surface σ of volume τ to be classified as follows:

(*a*) *Ordinary Convection*

$$\int_\sigma u\rho\mathbf{V}_1\cdot d\boldsymbol{\sigma}, \tag{2.4}$$

which corresponds to the first term of (2.3).

(*b*) *Internal Convection*

$$\int_\sigma Ts_\lambda\rho_n\mathbf{V}_n\cdot d\boldsymbol{\sigma}-\int_\sigma Ts\rho\mathbf{V}_1\cdot d\boldsymbol{\sigma}=\int_\sigma \rho sT\mathbf{V}_2\cdot d\boldsymbol{\sigma}, \tag{2.5}$$

which corresponds to the third term of (2.3) and the part of the second term due to the flow of the normal fluid. The energy flow appeared in (2.5),

$$\mathbf{q}=\rho sT\mathbf{V}_2 \tag{2.6}$$

is peculiar to He II and responsible for its anomalous thermal properties. To emphasize its important role, we propose to use a special terminology, *internal convection*, which was used already by F. London and Zilsel.[3] In ordinary fluids, where the flow of entropy and that of mass are described by the same velocity field, the internal convection does not appear.†

† This is incompatible with above-mentioned experiments of Kapitza in the case of He II.

(*c*) *Heat Conduction*

$$\int_\sigma \mathbf{q}_0\cdot d\boldsymbol{\sigma} \tag{2.7}$$

which corresponds to the rest of the second term of (2.3). For $\mathbf{q}_0$, we assume Fourier's expression

$$\mathbf{q}_0=-\kappa\,\mathrm{grad}T, \tag{2.8}$$

where κ is, from our fundamental point of view, expected to be of the same order of magnitude as the heat conductivity of He I ($\sim10^{-5}$ cal./cm deg. sec.).

(*d*) *Work Done by the Hydrostatic Pressure*

$$\int_\sigma p\mathbf{V}_1\cdot d\boldsymbol{\sigma}, \tag{2.9a}$$

which corresponds to the last term of (2.3). In the case of motion of finite velocity, we must also take into account work done by the viscous stress,

$$\int_\sigma \mathbf{V}_n\cdot(\Pi' d\boldsymbol{\sigma}), \tag{2.9b}$$

where the viscous stress Π' is assumed to be caused only by the normal fluid and given by

$$\left.\begin{aligned}\Pi_{xx}'&=\tfrac{2}{3}\eta_n\,\mathrm{div}\mathbf{V}_n-2\eta_n(\partial V_{nx}/\partial x),\\ \Pi_{xy}'&=-\eta_n[(\partial V_{nx}/\partial y)+(\partial V_{ny}/\partial x)],\quad\cdots.\end{aligned}\right\} \tag{2.10}$$

As Tisza has proposed, and we also support from our general point of view, the temperature dependence of moment of inertia measured by Andronikashvilli[9] and the viscosity measurements by Keesom and Mac Wood[10] indicate the evidence that the viscosity of the normal fluid may be obtained by a suitable extrapolation from that of He I and that the superfluid has no viscosity of this kind.

Finally, we must take account of

(*e*) *Flux of Kinetic Energy*

$$\int_\sigma(\tfrac{1}{2}\rho_n\mathbf{V}_n^2)\mathbf{V}_n\cdot d\boldsymbol{\sigma}+\int_\sigma(\tfrac{1}{2}\rho_s\mathbf{V}_s^2)\mathbf{V}_s\cdot d\boldsymbol{\sigma}. \tag{2.11}$$

Summing up the five kinds of energy flow, (2.4), (2.5), (2.7), (2.9), and (2.11), and equating to the rate of change of the total energy contained in the volume τ, one obtains the first law. Similarly, one can formulate the conservation laws of mass and momentum and the second law. These fundamental laws are written as follows:

mass conservation law,

$$\frac{\partial}{\partial t}\int_\tau(\rho_n+\rho_s)d\tau=\int_\sigma(\rho_n\mathbf{V}_n+\rho_s\mathbf{V}_s)\cdot d\boldsymbol{\sigma}; \tag{2.12}$$

[9] E. Andronikashvilli, J. Phys. U.S.S.R. **10**, 201 (1946).
[10] W. H. Keesom and G. E. Mac Wood, Physica **5**, 737 (1938).

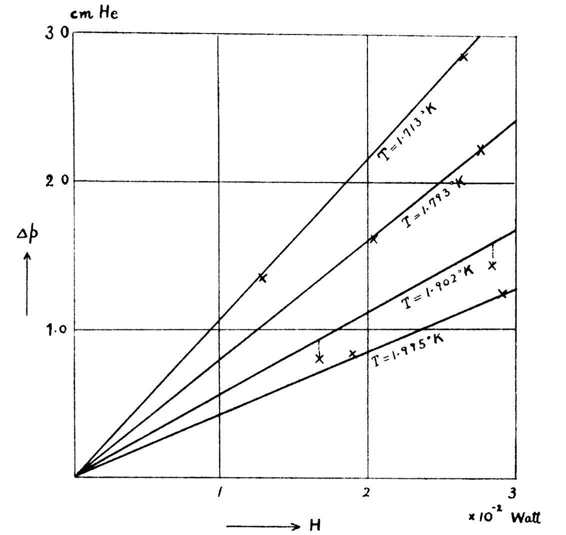

FIG. 1. Fountain effect against heat current, for constant temperature of helium bath. x: observed (KD); ——: calculated $d=19\mu$.

momentum conservation law

$$\frac{\partial}{\partial t}\int_\tau(\rho_n\mathbf{V}_n+\rho_s\mathbf{V}_s)d\tau=\int_\sigma(\rho_n\mathbf{V}_n)\mathbf{V}_n\cdot d\boldsymbol{\sigma}+\int_\sigma(\rho_s\mathbf{V}_s)\mathbf{V}_s\cdot d\boldsymbol{\sigma}+\int_\sigma\Pi d\boldsymbol{\sigma},\quad(2.13)$$

where

$$\Pi_{xx}=p+\Pi_{xx}',\quad \Pi_{xy}=\Pi_{xy}',\quad\cdots;\quad(2.14)$$

the first law

$$\frac{\partial}{\partial t}\int_\tau(\tfrac{1}{2}\rho_n\mathbf{V}_n^2+\tfrac{1}{2}\rho_s\mathbf{V}_s^2+\rho u)d\tau$$
$$=\int_\sigma(\tfrac{1}{2}\rho_n\mathbf{V}_n^2)\mathbf{V}_n\cdot d\boldsymbol{\sigma}+\int_\sigma(\tfrac{1}{2}\rho_s\mathbf{V}_s^2)\mathbf{V}_s\cdot d\boldsymbol{\sigma}$$
$$+\int_\sigma(\rho u\mathbf{V}_1+\rho sT\mathbf{V}_2+\mathbf{q}_0+p\mathbf{V}_1)\cdot d\boldsymbol{\sigma}$$
$$+\int_\sigma\mathbf{V}_n\cdot(\Pi' d\boldsymbol{\sigma});\quad(2.15)$$

the second law

$$\frac{\partial}{\partial t}\int_\tau\rho s d\tau=\int_\sigma\rho s\mathbf{V}_n\cdot d\boldsymbol{\sigma}+\int_\sigma\frac{\mathbf{q}_0\cdot d\boldsymbol{\sigma}}{T}+\int_\tau\left(\frac{dS}{dt}\right)_{\text{irr}}d\tau,\quad(2.16)$$

where the first term on the right expresses the fact that the entropy is transferred only by the normal fluid; the second is the rate at which the entropy varies through the conduction of heat; the third is the rate of irreversible production of entropy per unit volume. Thus, it is evident that the above formulation of the first and the second laws is consistent with the fundamental equation of thermostatics, (2.3).

3. Thermodynamic Equations of Motion

Now, we transform the laws of integral form in Section 2 into differential equations by means of Gauss' theorem. From Eq. (2.12) we obtain the equation of

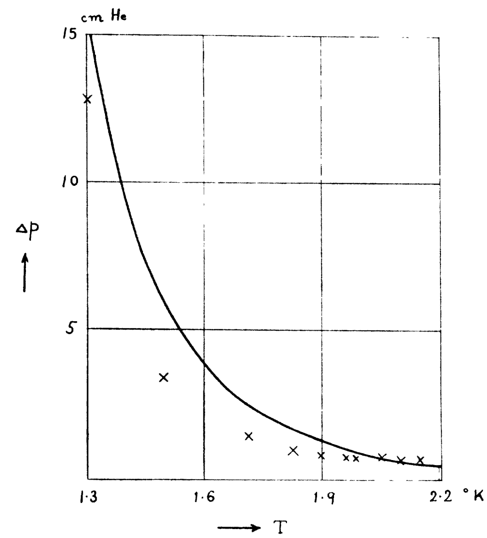

FIG. 2. Fountain effect against temperature. $d=10.5\mu$, $q=0.18$ watt cm^{-2}. x: observed (M); ——: calculated.

continuity

$$(\partial/\partial t)(\rho_n+\rho_s)+\text{div}(\rho_n\mathbf{V}_n+\rho_s\mathbf{V}_s)=0\quad(3.1)$$

or

$$(\partial\rho/\partial t)+\text{div}(\rho\mathbf{V}_1)=0\quad(3.2)$$

which may be decomposed into

$$\begin{aligned}(\partial\rho_n/\partial t)+\text{div}(\rho_n\mathbf{V}_n)&=\Gamma,\\(\partial\rho_s/\partial t)+\text{div}(\rho_s\mathbf{V}_s)&=-\Gamma.\end{aligned}\quad(3.3)$$

Γ is the rate of production of the normal fluid per unit volume which is always accompanied by annihilation of an equal amount of the superfluid, so that it may also be considered as the rate of transformation of the superfluid into the normal fluid. From (2.13), taking account of Eqs. (3.3) (2.10), and (2.14), one obtains

$$\rho_n\left(\frac{\partial}{\partial t}+\mathbf{V}_n\cdot\text{grad}\right)\mathbf{V}_n+\rho_s\left(\frac{\partial}{\partial t}+\mathbf{V}_s\cdot\text{grad}\right)\mathbf{V}_s$$
$$+\Gamma(\mathbf{V}_n-\mathbf{V}_s)+\text{grad}p+\eta_n\{\text{curl curl}\mathbf{V}_n$$
$$-(4/3)\text{ grad div}\mathbf{V}_n\}=0.\quad(3.4)$$

Here it is to be noted that the term $\Gamma(\mathbf{V}_n-\mathbf{V}_s)$ represents that part of the momentum which must be supplied where the above-mentioned transformation occurs.

Similarly, transforming Eq. (2.15) into a differential form, and using Eqs. (3.3), (3.4), and the thermostatic relation

$$du=(p/\rho^2)d\rho+Tds,$$

and carrying out a somewhat lengthy calculation, one finds that

$$(\mathbf{V}_s-\mathbf{V}_n)\cdot\left[\rho_s\left(\frac{\partial}{\partial t}+\mathbf{V}_s\cdot\mathrm{grad}\right)\mathbf{V}_s+\frac{\rho_s}{\rho}\,\mathrm{grad}p\right.$$
$$\left.-\rho_s s\,\mathrm{grad}T+\frac{\Gamma}{2}(\mathbf{V}_n-\mathbf{V}_s)\right]$$
$$+T\left[\frac{\partial(\rho s)}{\partial t}+\mathrm{div}(\rho s\mathbf{V}_n)\right]+\mathrm{div}\mathbf{q}_0-\epsilon=0, \quad (3.5)$$

where

$$\epsilon=\eta_n[-\tfrac{2}{3}(\mathrm{div}\mathbf{V}_n)^2+(\mathrm{curl}\mathbf{V}_n)^2-2(\mathbf{V}_n\cdot\mathrm{grad}\,\mathrm{div}\mathbf{V}_n)$$
$$-2\,\mathrm{div}(\mathbf{V}_n\times\mathrm{curl}\mathbf{V}_n)+\Delta\mathbf{V}_n^2] \quad (3.6)$$

is Rayleigh's dissipation function, and is essentially positive.

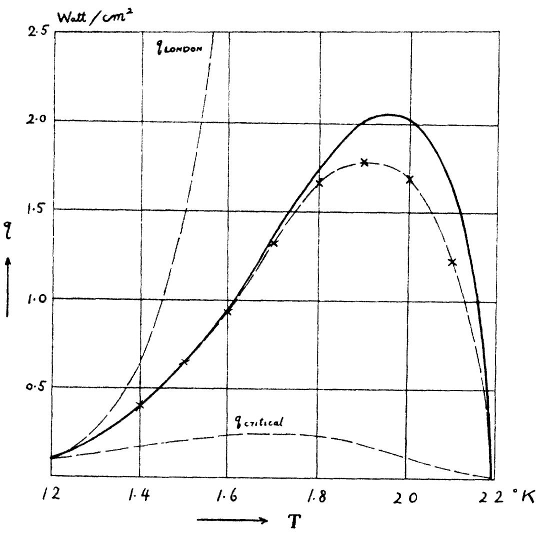

FIG. 3. Heat current density against temperature. $d=19\mu$, $\theta=0.006$ deg. cm^{-1}. x: observed (KD); ——: calculated.

Thus, if the equation of motion of the superfluid takes the form

$$\rho_s\left(\frac{\partial}{\partial t}+\mathbf{V}_s\cdot\mathrm{grad}\right)\mathbf{V}_s=-\frac{\rho_s}{\rho}\,\mathrm{grad}p+\rho_s s\,\mathrm{grad}T$$
$$-\frac{\Gamma}{2}(\mathbf{V}_n-\mathbf{V}_s)+\mathbf{f} \quad (3.7)$$

then Eq. (3.5) reduces to

$$\frac{\partial(\rho s)}{\partial t}+\mathrm{div}(\rho s\mathbf{V}_n)=\frac{1}{T}[-\mathrm{div}\mathbf{q}_0+\epsilon+\mathbf{f}\cdot(\mathbf{V}_n-\mathbf{V}_s)]. \quad (3.5')$$

Meanwhile, the second law (2.16) reduces to the differential form:

$$\frac{\partial}{\partial t}(\rho s)+\mathrm{div}(\rho s\mathbf{V}_n)+\frac{\mathrm{div}\mathbf{q}_0}{T}-\frac{\mathbf{q}_0\cdot\mathrm{grad}T}{T^2}=\left(\frac{dS}{dt}\right)_{\mathrm{irr}}. \quad (3.8)$$

This and (3.5′) lead to

$$\left(\frac{dS}{dt}\right)_{\mathrm{irr}}=-\frac{\mathbf{q}_0\cdot\mathrm{grad}T}{T^2}+\frac{\epsilon}{T}+\frac{\mathbf{f}\cdot(\mathbf{V}_n-\mathbf{V}_s)}{T}. \quad (3.9)$$

The separation parameter $\mathbf{f}$ is, for the time being, undetermined and must be found by considering the physical mechanism of the irreversible production of entropy. At least, however, since the condition $\mathbf{f}\cdot(\mathbf{V}_n-\mathbf{V}_s)>0$ must be always valid, $\mathbf{f}$ may be expected to be $(\mathbf{V}_n-\mathbf{V}_s)$ multiplied by a positive scalar function, which means physically a mutual friction per unit volume existing between the normal and the supercomponent of He II. While we were endeavoring to find out a definite form of this term, Gorter[4] had pointed out that the irreversible production of entropy due to such a mutual friction should play an important role.

If one puts

$$\mathbf{f}=A\rho_n\rho_s(\mathbf{V}_n-\mathbf{V}_s)^2(\mathbf{V}_n-\mathbf{V}_s), \quad (3.10)$$

one can include the Gorter's mutual friction in the present general formalism. Equation (3.10) might be an approximation to the more complicated expression.

To collect the equations obtained, we have

$$(\partial\rho_n/\partial t)+\mathrm{div}(\rho_n\mathbf{V}_n)=\Gamma,$$
$$(\partial\rho_s/\partial t)+\mathrm{div}(\rho_s\mathbf{V}_s)=-\Gamma, \quad (\mathrm{I})$$

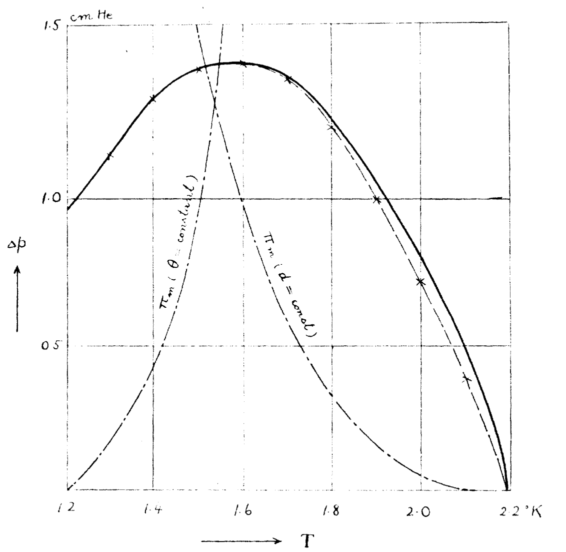

FIG. 4. Fountain pressure against temperature. $d=19\mu$, $\theta=0.006$ deg. cm^{-1}. x: observed (KD); ——: calculated.

$$\rho_n\left(\frac{\partial}{\partial t}+\mathbf{V}_n\cdot\text{grad}\right)\mathbf{V}_n+\rho_s\left(\frac{\partial}{\partial t}+\mathbf{V}_s\cdot\text{grad}\right)\mathbf{V}_s$$
$$+\Gamma(\mathbf{V}_n-\mathbf{V}_s)=-\text{grad}p-\eta_n\{\text{curl curl}\mathbf{V}_n$$
$$-(4/3)\text{ grad div}\mathbf{V}_n\},\quad \text{(II)}$$

$$\rho_s\left(\frac{\partial}{\partial t}+\mathbf{V}_s\cdot\text{grad}\right)\mathbf{V}_s+\frac{\rho_s}{\rho}\text{ grad}p-\rho_s s\text{ grad}T$$
$$+\frac{\Gamma}{2}(\mathbf{V}_n-\mathbf{V}_s)-\mathbf{f}=0,\quad \text{(III)}$$

$$\left.\begin{aligned}&\frac{\partial(\rho s)}{\partial t}+\text{div}(\rho s\mathbf{V}_n)+\text{div}\left(\frac{\mathbf{q}_0}{T}\right)=\left(\frac{dS}{dt}\right)_{\text{irr}},\\&\left(\frac{dS}{dt}\right)_{\text{irr}}=\mathbf{q}_0\cdot\text{grad}\left(\frac{1}{T}\right)+\frac{\epsilon}{T}+\frac{1}{T}\mathbf{f}\cdot(\mathbf{V}_n-\mathbf{V}_s).\end{aligned}\right\}\text{(IV)}$$

When it is necessary, one can obtain the equation of motion for the normal part from (II) and (III):

$$\rho_n\left(\frac{\partial}{\partial t}+\mathbf{V}_n\cdot\text{grad}\right)\mathbf{V}_n+\frac{\rho_n}{\rho}\text{ grad}p+\rho_s s\text{ grad}T$$
$$+(\Gamma/2)(\mathbf{V}_n-\mathbf{V}_s)+\mathbf{f}+\eta_n\{\text{curl curl}\mathbf{V}_n$$
$$-(4/3)\text{ grad div}\mathbf{V}_n\}=0.\quad \text{(III}')$$

4. Discussion

Putting $\Gamma=0$ and $\mathbf{f}=0$, and neglecting at the same time the non-linear terms, $(\mathbf{V}_n\cdot\text{grad})\mathbf{V}_n$ and $(\mathbf{V}_s\cdot\text{grad})\mathbf{V}_s$, and the viscosity term, too, we can obtain the well-known Landau-Tisza equations from Eqs. (I)–(IV). When we take account of the viscosity term, we obtain the fundamental equations of F. London and Zilsel, which may be expected to be useful for the description of the viscosity measurement by Keesom and Mac Wood.[10] Thus, if one assumes that the superfluid moves with the same velocity as that of the normal fluid under uniform pressure and temperature, the calculation of Mac Wood[11] is still valid in the present formalism. On the other hand, if the superfluid is perfectly at rest, we have to put $\mathbf{V}_n$ and η_n/ρ_n in place of $\mathbf{V}$ and η/ρ, respectively, in his calculation. The result is just the same as Tisza has analyzed, i.e., we must multiply the values of Keesom and Mac Wood by the factor ρ/ρ_n. These corrected values of the viscosity coefficient do not agree with the values extrapolated smoothly from those of He I in contradiction to the suggestion of Tisza. Since there exists mutual friction between the normal and the superfluids, it might be unreasonable to assume the superfluid to be perfectly at rest. However, the effect of the mutual friction has been estimated to be negligibly small in this case. The pressure gradient and the temperature gradient neglected in the previous discussion would presumably play an important role. At any rate, the analysis of the viscosity measurement with the oscillating disk seems to be so complicated that one cannot obtain a definite result from it. It is preferable to make use of the fountain pressure measurement. We shall give the detailed description of this method in Part II.

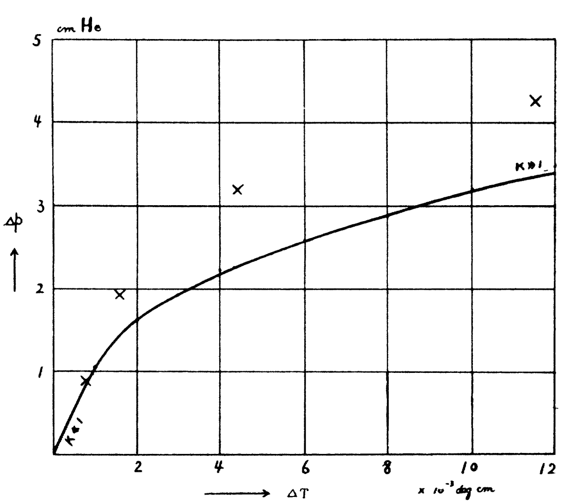

FIG. 5. Heat current density against temperature difference. $d=19\mu$, $T=1.391°$K. x: observed (KD); ——: calculated.

Now, one may assume s_λ to be constant, at least under sufficiently low pressures. Then, from (I) and (IV), one obtains

$$\Gamma=(1/s_\lambda)[(dS/dt)_{\text{irr}}-\text{div}(\mathbf{q}_0/T)],$$

which shows that the transformation of the superfluid into the normal fluid should be caused through the irreversible production of entropy or the conduction of heat. For instance, the transformation, which occurs in the constant volume of He II immersed in a heat bath by quasistatic heating, corresponds to the second term of the expression. Thus, Γ may also be significant near the heat source in the experiment of heat transfer.

Under ordinary conditions, however, the internal convection (2.6) overwhelms the heat conduction $\mathbf{q}_0$, and $\rho s\mathbf{V}_n$ turns out to be much greater than $(dS/dt)_{\text{irr}}$. Therefore, with the exception of singularities such as the heat sources, the reversible process approximation

$$[\partial(\rho s/\partial t]+\text{div}(\rho s\mathbf{V}_n)=0$$

may be valid. All authors have succeeded in describing the essential features of phenomena by means of this approximation.

It might seem rather trivial to have introduced the apparently insignificant term $\mathbf{q}_0$, which, however, appears to us particularly important, for if this term is omitted it follows that the normal fluid flows up the temperature gradient in the case of heat transfer through a slit. The detailed accounts will be given in Part II.

[11] G. E. Mac Wood, Physica 5, 375 (1938).

Finally, we also take up the conventional boundary conditions that

$$\mathbf{V}_n=0 \quad \text{and} \quad \mathbf{V}_{s\perp}=0$$

at the solid wall. F. London and Zilsel have suggested the possibility that the normal fluid also slips along the wall. We shall show, in Part II, that such a modification of the boundary conditions is quite unnecessary.

PART II. HEAT TRANSFER

5. Introduction

As mentioned in Part I, the internal convection $\rho s\mathbf{V}_n$ is much greater than the term $(dS/dt)_{\text{irr}}$ and $\mathbf{q}_0/T$ under ordinary conditions, and therefore the reversible process approximation to Eqs. (IV) of Section 3,

$$(\partial/\partial t)(\rho s)+\text{div}(\rho s\mathbf{V}_n)=0 \text{ (conservation of entropy)},$$

may be considered to be valid. As Γ is explicitly given by the expression

$$\Gamma=(1/s_\lambda)[(dS/dt)_{\text{irr}}-\text{div}(\mathbf{q}_0/T)], \qquad (5.1)$$

we must disregard Γ at the same time in this approximation. All the previous treatments,[1–3] which have used this approximation and succeeded, though only partially, in clarifying some part of the peculiar behavior of He II, prove the validity of this approximation. Therefore, we also assume it and put $\Gamma=0$ in our fundamental equations. Then we have only the $\mathbf{f}$ term left among the newly introduced terms.

According to our general theory, the explicit form of the term is $(\mathbf{V}_n-\mathbf{V}_s)$ multiplied by a positive scalar function α. For more detailed knowledge concerning α we must proceed, by analyzing suitable experimental phenomena.

There are now numerous reports on the many peculiar properties of liquid He II; but many of them are in default of any precise description of the conditions under which the experiments were performed, and few are sufficiently quantitative to be compared with the theory. Among these experiments, those of the Leiden experimenters, Keesom, Saris, and Meyer (K-S-M);[12] Keesom and Duykaerts (K-D);[13] Mellink (M);[14] and Meyer and Mellink (M-M)[15] are so systematic and exhaustive that we can choose their data as source material for investigating the mutual friction.

These investigators measured the stationary heat current and the fountain pressure which were caused by electrically heating liquid He II in one of the two regions separated by a capillary or a slit which is adjustable between two zonal surfaces of a sphere. There the radius of the capillary or the width of the slit is so much smaller than the other dimensions that it may be legitimate to describe it as a two-dimensional case. In the following analysis we take the x axis along the longer side and the y axis along the width. The plane $y=0$ divides the width d equally.

[12] Keesom, Saris, and Meyer, Physica **7**, 817 (1940).
[13] W. H. Keesom and G. Duykaerts, Physica **13**, 153 (1947).
[14] J. H. Mellink, Physica **13**, 180 (1947).
[15] L. Meyer and J. H. Mellink, Physica **13**, 197 (1947).

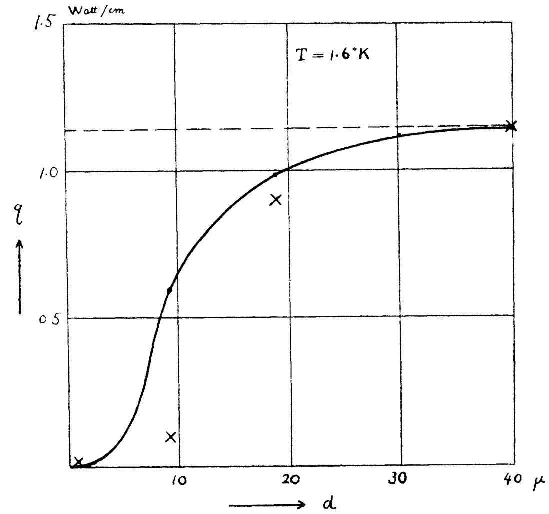

FIG. 6. Heat current against slit width. $\theta=0.006$ deg. cm^{-1}, $T=1.6$°K. x: observed (KD); ——: calculated.

To collect the important results of the Leiden experimenters:

(A) The heat current density $\mathbf{q}$ is exactly proportional to the fountain pressure Δp (Fig. 1).

(B) The fountain pressure decreases with increasing temperature T under constant heat current, but does not vanish at the λ-point T_λ (Fig. 2).

(C) The heat current under constant temperature difference ΔT increases with the temperature, and then, through a maximum, decreases agin, vanishing at the λ-point (Fig. 3). A similar variation is observed for the fountain pressure (Fig. 4).

(D) Under constant temperature, the heat current is proportional to ΔT in the case of a narrow slit or of a small temperature difference. But in the case of wide slit or larger temperature difference it is proportional to $(\Delta T)^{\frac{1}{3}}$ (Fig. 5).

(E) In the case of a capillary under constant temperature difference, the heat current is proportional to the cross section at lower temperatures, but is independent of the cross section at higher temperatures (Fig. 6).

Adding only the viscosity term of the normal fluid to the theory of Landau and Tisza and F. London and Zilsel[3] (abbreviated as L-Z in the following) have tried to explain the Leiden experiments. There appears no Γ in their treatment, and moreover, concerning the velocity fields of two fluids they make the assumption

$$\mathbf{V}_n=(u_n(y),0,0), \quad \mathbf{V}_s=(u_s(y),0,0). \qquad (5.2)$$

The latter assumption includes the existence of two sorts of uniform velocity fields along the length of slit, resulting in a pure *convection* if the thermal energy is carried only by the normal part. In other words, they assumed that irreversible process, e.g., the transformation of super- into normal fluid, are negligibly small compared to the reversible convection, which may be considered as an expression of the above-mentioned approximation.

Based on the above picture, called "internal convection," they succeeded in attributing the anomalous heat transfer in He II to the Poiseuille flow of normal fluid. Moreover, in their treatment the *thermostatic* relation of H. London[16]

$$dp/dT=\rho s \tag{5.3}$$

still remains valid, and consequently it is expected that

$$q\propto \Delta p\propto \Delta T. \tag{5.4}$$

Equation (5.4) is fairly consistent with the experimental data, when the temperature is low or the slit width is small. However, as already reported by K-D, there appears a region in which $q\propto(\Delta T)^{\frac{1}{3}}$ when the temperature is high [item (D)]. On the other hand, the relation $q\backsim\Delta p$ holds always [item (A)], and therefore the above fact shows that H. London's relation becomes invalid in the high temepratur region.

L-Z determined the critical heat current q_c empirically, fixing the region in which a discrepancy appears between experimental data and H. London's formula, and expected that the hydrodynamical description might fail above this current density. But, if so, the "two-fluid model," as discussed by Tisza,[2] would lose its secure basis, even without criticism from the molecular point of view. Of course, avoiding such a hasty conclusion, L-Z proposed several refinements within the scope of the "two-fluid model," but in each case they expected the solution of the problem to come ultimately in refinements and perfection of the fundamental equations of hydrodynamics.

Returning to our thermodynamic equations, we see that they fulfill L-Z's requirement of derivation. However, even under the reversible process approximation, our equations do not coincide with L-Z's in containing the mutual friction term $\mathbf{f}$, and if this plays a role *mechanically* analogous to ordinary viscosity, H. London's formula becomes invalid as is clearly seen from the equation of superfluid motion (III) in Part I, Section 3. Therefore, by giving a suitable form to the mutual friction, we tried to describe the region left unexploted by L-Z and to explain the related phenomena.

On the other hand, Gorter[17] has suggested that in order to explain the empirical formula $q\propto(\Delta T)^{\frac{1}{3}}$ in wide slits, a term of mutual friction of the form

$$A\rho_n\rho_s(V_n-V_s)^3+\cdots$$

per unit volume should be assumed. Of course, we do not know how Gorter has treated the fundamental equations, but, as is readily seen, his form of mutual friction satisfies the requirement of our general theory on the $\mathbf{f}$ term. Therefore, we included his mutual friction as our $\mathbf{f}$,

$$\mathbf{f}=A\rho_n\rho_s(\mathbf{V}_n-\mathbf{V}_s)^2(\mathbf{V}_n-\mathbf{V}_s) \tag{5.5}$$

and examined the resulting conclusion, which will be presented later. Here A is the parameter which characterizes the mutual friction, having dimensions

$$[A]=[M^{-1}LT]=[\eta_n^{-1}].$$

6. Two Velocity Fields in He II

Following the line stated in the preceding section, we also assume $\Gamma=0$ and a stationary solution of laminar flow, (5.2). Then, from (III′) and (III) the velocity fields are governed by the equations

$$(\rho_n/\rho)\,\mathrm{grad}p+\rho_s s\,\mathrm{grad}T+A\rho_n\rho_s(\mathbf{V}_n-\mathbf{V}_s)^2(\mathbf{V}_n-\mathbf{V}_s)=\eta_n\Delta\mathbf{V}_n \tag{III′}$$

and

$$(\rho_s/\rho)\,\mathrm{grad}p-\rho_s s\,\mathrm{grad}T-A\rho_n\rho_s(\mathbf{V}_n-\mathbf{V}_s)^2(\mathbf{V}_n-\mathbf{V}_s)=0. \tag{III}$$

In the reversible process approximation, Eqs. (I) and (IV), expressing only the mass conservation of both fluids, do not give any explicit condition which restricts the phenomena, Note that the $\mathbf{f}$ term should have disappeared, if we neglect the quadratic terms in the velocities as did L-Z. Our approximation is of no such mathematical nature, but a physical one based on the evidence observed in He II.

As is easily seen, these equations should not be confined to the case of heat transfer only, being valid for any stationary laminar flow.

In the first place, eliminating relative motion from these equations, one obtains

$$\mathrm{grad}p=\eta_n\Delta\mathbf{V}_n. \tag{6.1}$$

Thus the velocity field of the normal fluid is directly connected with the pressure gradient, as Gorter[17] has pointed out.

The y-component of (6.1) gives

$$\partial p/\partial y=0,\text{ i.e., }p=p(x),$$

that is, the pressure is uniform in the plane of a cross section.

Then, the x-component of the same equation gives

$$\frac{dp}{dx}=\eta_n\frac{\partial u^2{}_n}{\partial y^2},$$

which results in the well-known Poiseuille flow, under the ordinary boundary condition that the normal fluid sticks to the wall,

$$u_n=(\pi/2\eta_n)(\tfrac{1}{4}d^2-y^2) \tag{6.2}$$

where

$$\pi\equiv-dp/dx.$$

In the second place, Eq. (III) determines the velocity of the superfluid relative to the normal fluid. Its y-component gives

$$\partial T/\partial y=0,\text{ i.e., }T=T(x).$$

[16] H. London, Nature **142**, 612 (1938).
[17] G. J. Gorter, Phys. Rev. **74**, 838 (1947).

Thus, our solution implies that the temperature variation over the cross section does not play an essential role in this phenomena, contrary to the suggestion of L-Z.

The x-component of (III) gives

$$A\rho\rho_s(u_n-u_s)^3=\pi-\rho s\theta,$$

where

$$\theta\equiv -dT/dx.$$

Then, one obtains as the velocity field of superfluid

$$u_s=u_n+\left(\frac{\pi-\rho s\theta}{A\rho\rho_n}\right)^{\frac{1}{3}}. \tag{6.3}$$

While the first term on the right-hand side represents Poiseuille flow, which vanishes on the wall, the second is constant over the cross section and therefore represents a *slip flow*. It is to be noted that the slip flow becomes appreciable only when the fountain effect deviates from H. London's thermostatic relation. Thus, we may expect that the slip flow should play an essential role for the heat current greater than q_c estimated by L-Z. Ordinary heat conduction q_0, which was introduced in our general formalism, is really necessary to secure the existence of slip flow, because omitting q_0, Eq. (IV) leads to a conclusion that $u_s=0$ on the wall under the strict application of second law.

As the temperature T is constant over the cross section, so also ρ_n and ρ_s. Taking account of this fact, one can calculate the mean velocities as

$$\langle V_n\rangle=(d^2/12\eta_n)\pi, \tag{6.4}$$

and

$$\langle V_s\rangle=\langle V_n\rangle+[(\pi-\rho s\theta)/A\rho\rho_n]^{\frac{1}{3}}. \tag{6.5}$$

Finally, the mass flow density is given by the expression

$$\begin{aligned}\langle\rho V_1\rangle&=\langle\rho_nV_n\rangle+\langle\rho_sV_s\rangle\\&=(\rho d^2/12\eta_n)\pi+\rho_s[(\pi-\rho s\theta)/A\rho\rho_n]^{\frac{1}{3}}.\end{aligned} \tag{6.6}$$

7. Heat Transfer in Liquid He II

As discussed in Part I, the convection part of the heat flow is generally given by

$$\langle\rho uV_1\rangle+\langle\rho sTV_2\rangle, \tag{7.1}$$

where $V_2=V_n-V_1$.

If, however, the net mass flow vanishes, i.e.,

$$\langle\rho\mathbf{V}_1\rangle=0, \tag{7.2}$$

as in the case of Leiden measurements, (7.1) reduces to

$$\mathbf{q}=\rho sT\langle\mathbf{V}_n\rangle, \tag{7.3}$$

which was called by L-Z "internal convection."

Substituting for $\langle\mathbf{V}_n\rangle$ by (6.4) of the preceding section, one obtains

$$q=(\rho sTd^2/12\eta_n)\pi. \tag{7.4}$$

As Gorter has pointed out, (7.4) expresses the fact that $q\propto\Delta p$ always, which may be compared with the experimental results (A) and (B).

TABLE I. Relationship of fountain effect, heat current, and temperature in He II.

T°K	$H\times10^4$(watt)	Δp_{calc}(cm He)	Δp_{obs}(cm He)
1.995	189.7	0.806	0.82
1.995	291.1	1.24	1.25
1.902	167.4	0.947	0.80
1.902	283.5	1.60	1.41
1.793	204.1	1.64	1.62
1.793	275.9	2.22	2.22
1.713	264.5	2.80	2.88
1.713	130.0	1.38	1.36

Then, from (7.2) and (6.6) one can obtain the relation of π and θ under the condition of no net mass flow:

$$(\rho d^2/12\eta_n)\pi+\rho_s[(\pi-\rho s\theta)/A\rho\rho_n]^{\frac{1}{3}}=0, \tag{7.5}$$

or with the abbreviations

$$\xi\equiv\rho_n/\rho=s/s_\lambda=(T/T_\lambda)^r \text{ (reduced temperature)}, \tag{7.6}$$

$$\gamma\equiv(\rho/8)(A/\eta_n{}^3)^{\frac{1}{2}} \text{ (material constant)}. \tag{7.7}$$

Equation (7.5) is rewritten as

$$\pi^3=\frac{27}{\gamma^2}\frac{(1-\xi)^3(\rho s_\lambda\theta\xi-\pi)}{d^6\xi}. \tag{7.8}$$

This equation involves the results of our *thermodynamical* equations of motion under no net mass flow condition, which should substitute H. London's *thermostatic* relation $\pi=\rho s_\lambda\theta\xi$. To obtain a direct comparison, we may solve (7.8) explicitly for π as follows:

$$\begin{aligned}\pi=\rho s_\lambda\theta\xi\cdot(3/2K)\{&[K+(1+K^2)^{\frac{1}{2}}]^{\frac{1}{3}}\\&+[K-(1+K^2)^{\frac{1}{2}}]^{\frac{1}{3}}\},\end{aligned} \tag{7.9}$$

where we have put for the sake of convenience

$$K\equiv(\rho s_\lambda\gamma/2)\cdot\theta d^3[\xi/(1-\xi)]^{\frac{3}{2}} \text{ (dimensionless)}. \tag{7.10}$$

Finally, corresponding to the extreme values of parameters, the expression (7.9) reduces to

$$\pi=\rho s_\lambda\theta\xi, \quad (K\ll1) \tag{7.11}$$

when $K\ll1$, i.e., the temperature gradient θ, the slit width d, or the temperature T are sufficiently small. The result is equivalent to H. London's relation.

For the other extremum $K\gg1$, (7.9) reduces to

$$\pi=3(\rho s_\lambda/\gamma)^{\frac{1}{3}}\cdot\theta^{\frac{1}{3}}d^{-2}(1-\xi). \quad (K\gg1). \tag{7.12}$$

Or, converted into heat current using Eq. (7.4),

$$q=[(\rho s_\lambda)^2/12\eta_nT_\lambda{}^{2r}]\theta d^2T^{2r+1}, \quad (K\ll1) \tag{7.13}$$

and

$$q=[(\rho s_\lambda)^{4/3}/4\eta_n\gamma^{2/3}T_\lambda{}^{2r}]\theta^{\frac{1}{3}}T^{r+1}(T_\lambda{}^r-T^r). \quad (K\gg1). \tag{7.14}$$

These formulas should be compared with the experimental results (C), (D), and (E).

TABLE II. Relationship of γ and T for various slit widths.

$d=19\mu$		$d=10.5\mu$		$d=5\mu$	
T(°K)	γ(10^6 c.g.s.)	T(°K)	γ(10^6 c.g.s.)	T(°K)	γ(10^6 c.g.s.)
1.391	1.34	1.514	2.84	1.328	3.4
1.713	1.82	1.514	2.68	1.716	5.44
1.793	2.24	1.845	3.43	1.725	6.12
1.902	2.69	1.845	4.31	1.902	4.48
1.995	2.35	1.975	3.29	2.036	4.0
2.077	1.39	2.054	2.93	2.119	1.58
2.148	1.30	2.054	2.63	2.120	1.84

Various tendencies in the behavior of thermal current and fountain pressure corresponding to the extreme values of adjustable parameters, given in the above formulas, have been observed by many independent authors experimentally.

K-S-M observed, with capillaries wider than 0.35 mm and in the temperature range $1.6>T>1$°K, that thermal current is not proportional to temperature difference but obeys an empirical formula

$$q=0.623T^5\theta^{\frac{1}{3}} \text{ watt cm}^{-2}.$$

This should correspond to the case $K\gg1$, i.e., Eq. (7.14) (Fig. 5).

K-D performed experiments using thinner capillaries and narrower slits than those of K-S-M. With 9.3μ slit with and $\theta=0.02$ deg. cm^{-1}, they observed the same behavior as K-S-M with temperatures higher than 1.9°K, while at lower temperatures there appeared a region in which q was proportional to ΔT. This means that they have gone through the region $K\sim1$ (Fig. 5).

Then M-M refined the experiments with slits narrower than 1μ and found that q is fairly proportional to ΔT in the entire temperature region. This corresponds to the case $K\ll1$ (Fig. 5). In this case the validity of H. London's formula is secured by Eq. (7.11) for all temperatures, and it is this extreme case only that the theory of L-Z describes. The theory predicts that heat current and conductivity should vary as $T^{2r+\frac{1}{2}}=T^{11\cdot5}$ and $T^{2r-\frac{1}{2}}=T^{10\cdot5}$, respectively, with results which are fairly consistent with the empirical data of K-D and M.

To the detailed comparison with the empirical data, we will contribute Section 8.

8. Comparison with Experiments

In comparing the fountain pressure and heat current, given by Eqs. (7.9) and (7.10), with experimental data, we adopted the following values of equilibrium constants.

Total mass density

$$\rho=0.146 \text{ g cm}^{-3}, \tag{8.1}$$

which was found by Keesom and Keesom[18] to be fairly constant below the λ-point.

The entropy density s varies as a function of temperature according to the empirical formula

$$s=s_\lambda(T/T_\lambda)^r, \tag{8.2}$$

where we adopted Kapitza's values[19]

$$r=5.5 \quad\text{and}\quad s_\lambda=0.405 \text{ cal. deg.}^{-1}\text{ g}^{-1}, \tag{8.3}$$

taking account of the results obtained by M-M who used the values of Keesom and Keesom[18] in the case of extremely narrow slits.

With regards to the *normal viscosity* coefficient there is already the oscillating disk measurement of Keesom and MacWood,[20] which may be reliable at least in the region of He I. But, as we have mentioned in Part I, there remain various difficulties in analyzing the data below the λ-point, possibly because of the mechanocaloric effect. On the contrary, we saw that the analysis of heat transfer in (7.4) is very simple.

We have extrapolated the value of η_n, corresponding to Tisza's opinion,[1] smoothly from the He I region, and obtained as an empirical formula

$$\eta_n=1.45\times10^{-5}T^{\frac{1}{2}} \text{ g cm}^{-1}\text{ sec.}^{-1}. \tag{8.4}$$

From this value of η_n and the observed value of q, we have calculated the fountain pressure Δp according to Eq. (7.4), which is compared in Table I and Fig. 1 with the observed data, showing satisfactory agreement [see item A, Section 5]. It is to be remembered that (7.4) is free from any effect of the mutual friction, and therefore we think that this is the most suitable phenomena for the determination of η_n.

In the case $q=\text{constant}=0.18$ watt cm^{-2}, the temperature variation of fountain pressure is compared

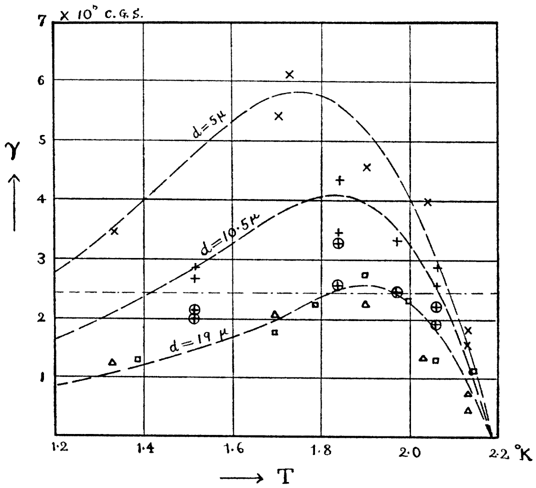

FIG. 7. The values of γ against temperature, with constant slit width.

□: Observed data for $d=19\mu$
+: Observed data for $d=10.5\mu\rightarrow\oplus$: Modified as $d=11.5\mu$
×: Observed data for $d=5.0\mu\rightarrow\triangle$: Modified as $d=6.5\mu$
The modified data show that the γ's have the same temperature dependence for the three different slit widths.

[18] W. H. Keesom and A. P. Keesom, Physica 1, 128 (1934).

[19] P. Kapitza, J. Phys. U.S.S.R. 5, 59 (1941).

[20] W. H. Keesom and MacWood, Physica 5, 737 (1938).

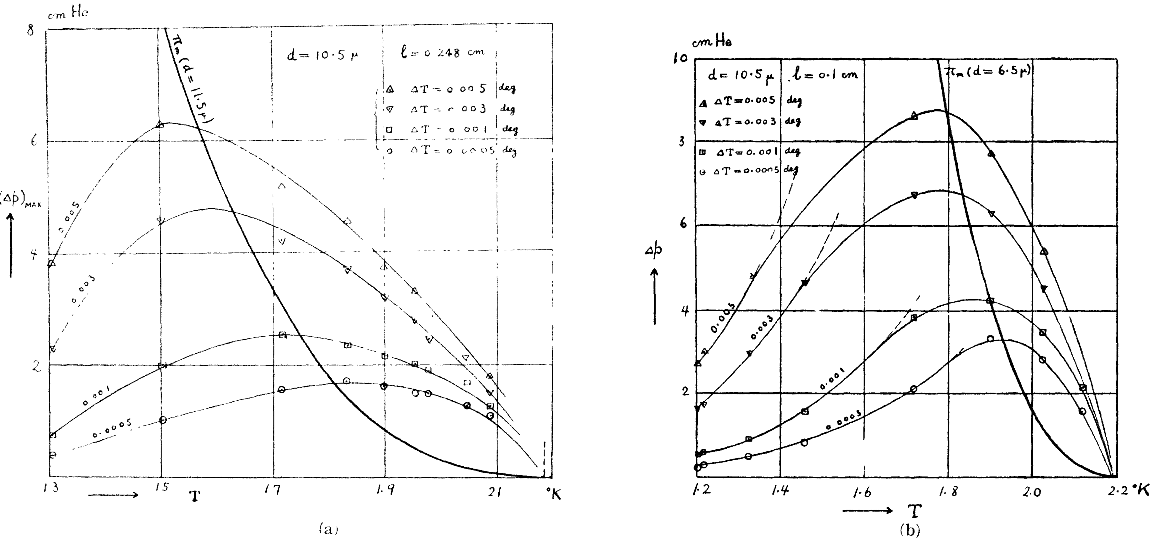

FIG. 8. Maximum fountain pressure against temperature. The numbers attached to the curves give the temperature differences in degrees. Solid curve represents the calculated maximum trace. (a) $d=10.5\mu$, (b) $d+5.0\mu$ (erroneous in the drawing).

with Mellink's data in Fig. 2 [item (B)]. Note that fountain pressure converges to a finite value

$$(\Delta p)_\lambda = (0.248)\pi_\lambda = (0.248)\frac{12\eta_n q}{\rho s_\lambda T_\lambda d^2} = 0.54 \text{ cm He}, \quad (8.5)$$

when the temperature approaches to λ-point, just as observed by M.

With this knowledge of η_n we shall be able to determine the parameter A, which characterizes the *mutual friction*, by analyzing the heat current near λ-point.

It is to be noted, however, that A appears in the result only in the form

$$\gamma \equiv (\rho/8)(A/\eta_n)^{\frac{1}{2}} \quad (8.6)$$

as is clear in Eq. (7.8). The dimension of γ is $[M^{-1}L^{-1}T^2]$.

Observing this situation, we put the Leiden data on π, ξ, and d straightforwardly into Eq. (7.8) and determined the value of γ at various temperatures. The result is shown in Table II and in Fig. 7.

While the values of γ show similar tendencies in temperature dependence for various slit width, their absolute magnitudes cannot be said to coincide in each case. But, as the experimenters themselves admit, the width of slit is the most ambiguous among the observed data. Moreover, it appears in our determination of γ as a factor d^3, being magnified in its ambiguity. Therefore, we cannot conclude hastily from these results that γ depends on slit width; in other words, we cannot deny Gorter's idea that the parameter A should be a material constant.

Observing this situation, we have adopted for the present

$$\gamma = 2.33\times 10^6 \text{ g}^{-1}\text{ cm}^{-1}\text{ sec.}^2, \quad (8.7)$$

and calculated the fountain pressure π and thermal current q in the case of 19μ slit. Calculated results, shown in Figs. 3–6 fairly reproduce the experimental data (C)–(E).

In this case parameter A is estimated as

$$A = 50.4\times T^{\frac{3}{2}} \text{ g}^{-1}\text{ cm sec.}, \quad (8.8)$$

which is nearly the same as Gorter's estimation $A=50$, showing a similar tendency in temperature variation. In any case, the investigation of A is a problem for the future, requiring further experimental measurements.

In the case of constant temperature and temperature difference, K-S-M observed that the heat current is proportional to the cross section for a thin capillary, whereas for a wide capillary it becomes independent of the cross section. K-D also observed similar tendencies

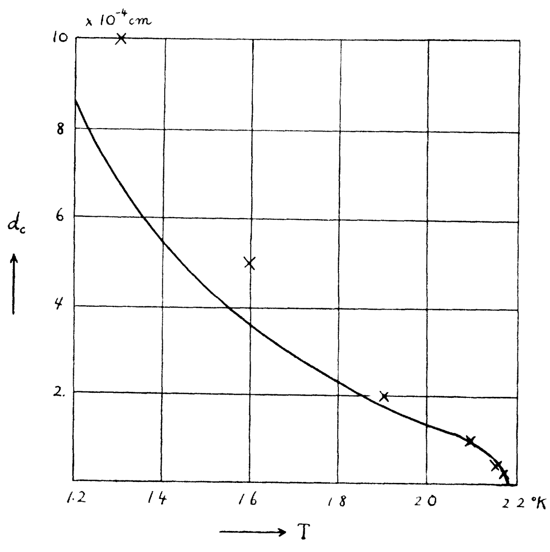

FIG. 9. Critical slit width against temperature. $\theta=7.51\times 10^{-2}$ deg. cm^{-1}. x: observed (MM); ——: calculated.

with varying slit width. On the other hand, M-M, observing the fountain pressure under similar conditions, found that for wide slits the fountain pressure increases with decreasing slit width, but for narrower ones it becomes insensitive to slit width variation. These slit width dependences are clearly expressed in Eqs. (7.11), (7.12), (7.14), and (7.15). To obtain the case of a capillary of radius R, we have only to replace d^2 by $\frac{3}{2}R^2$ in the above formulas (see Fig. 6).

We may inquire concerning the behavior of the characteristic maximum shown by the heat current and fountain pressure against temperature (see Figs. 3 and 4). With constant radius of capillary, K-S-M reported that maximum current temperature does not appreciably change with varying temperature difference, whereas Allen and Reekie[21] and K-D observed that the maximum current temperature increases with decreasing temperature difference; the latter also found that the fountain pressure also shows a similar tendency but more strongly, a fact which is clearly expressed in Fig. 8 (due to M).

For the sake of convenience we may first obtain the maximum fountain pressure from (7.8) and $d\pi/d\xi=0$. Then under constant slit width, and varying temperature difference, we obtain

$$\pi_m=\frac{3}{\gamma}\frac{(1-\xi)^2}{d^3\xi},\quad (d=\text{constant}),\tag{8.9}$$

and under constant temperature difference and varying slit width,

$$\pi_m=\rho s_\lambda\theta\frac{3\xi^2}{(1+2\xi)}.\quad (\theta=\text{constant}).\tag{8.10}$$

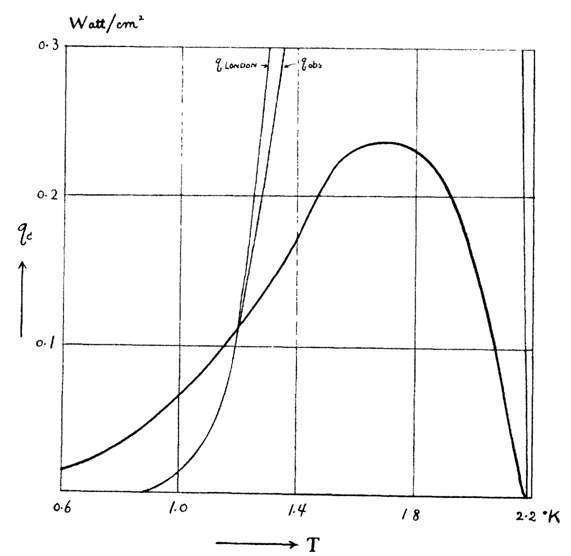

FIG. 9. Critical heat current against temperature. $d=19\mu$, $\theta=0.006$ deg. cm^{-1}.

[21] J. F. Allen and J. Reekie, Proc. Camb. Phil. Soc. 35, 114 (1939).

(See Fig. 4.) These loci are fairly consistent with M's data with 5 and 10.5μ slits as is shown in Fig. 8, and consequently supports the tendency in K-D's observation. As the ratio of q to π is monotonic with temperature, it is quite reasonable that $q_{\max}$ indicates a same tendency as π_m.

The apparent discrepancy of K-S-M's observation from others means only that they observed the nearly vertical region of (8.9), Fig. 4, corresponding to rather low temperature.

It is especially important that (8.10) does not involve such kinetic parameters as η_n or A, thus affording a way of testing the validity of the theory free from any ambiguity in parameter values. Moreover, these two formulas are analytically much simpler than that for π itself, so we may make a brief estimation of the value of π_m, using the empirical value of π_m.

We have recognized in the above a clear distinction between two different regions, i.e., the fountain pressure is proportional to ΔT in the one region, while in the other it is proportional to $(\Delta T)^{\frac{1}{3}}$. Accordingly, there must be a region in which the characteristics of these two regions alternate. We may call this the "*critical region*" and will investigate the behavior of this region for parameter variation.

Though, of course, there is no unique definition of critical region, we may be allowed to appint it by putting $K=1$. Then we obtain from (7.10) as the critical condition

$$\tfrac{1}{2}\gamma\rho s_\lambda\cdot\theta d^3(\xi/(1-\xi))^{\frac{3}{2}}=1.\tag{8.11}$$

In this case the fountain pressure is given from (7.9) as

$$\pi_c=c\cdot\rho s_\lambda\theta\xi,\tag{8.12}$$

where c is a numerical constant

$$c=\tfrac{3}{2}\{(1+\sqrt{2})^{\frac{1}{3}}+(1-\sqrt{2})^{\frac{1}{3}}\}=0.894.\tag{8.13}$$

Examination of this critical curve may give another possibility of testing the theory.

Keeping the temperature gradient constant M-M observed at each temperature the *critical slit width* d_c at which the fountain pressure becomes insensitive to slit width. Their results are shown in Fig. 9. Correspondingly, from Eq. (8.11) we obtain for d_c

$$d_c=\left(\frac{2}{\gamma\rho s_\lambda\theta}\right)^{\frac{1}{3}}\left(\frac{1-\xi}{\xi}\right)^{\frac{1}{2}},\tag{8.14}$$

which indicate a tendency quite similar to that of observed data (see Fig. 9). Unfortunately, the specification of temperature gradient is absent in *M*, we took tentatively $\theta=7.5\times10^{-2}$ deg. cm^{-1}.

If we consider a slit of definite width we obtain from (8.11) and (8.12)

$$\pi_c=\frac{2c}{\gamma}\frac{(1-\xi)^{\frac{3}{2}}}{d^3\xi^{\frac{1}{2}}},\quad (d=\text{constant}),\tag{8.15}$$

and referring to (6.4)

$$\langle V_n\rangle_c=\frac{c}{6\eta_n\gamma}\frac{(1-\xi)^{\frac{1}{2}}}{d\xi^{\frac{1}{2}}}\tag{8.16}$$

as the *critical flow velocity.*

Consequently, the critical heat flow is given by

$$q_c=\rho sT\langle V_n\rangle_c=\frac{c\rho sT}{6\eta_n\gamma d}\frac{(1-\xi)^{\frac{3}{2}}}{\xi^{\frac{1}{2}}},\tag{8.17}$$

which is shown in Fig. 10, being the substitute for the curve q_c which was empirically appointed as critical by L-Z. Its maximum current temperature is given by

$$\xi=13/46 \quad \text{or} \quad T=1.74°\text{K},$$

which is independent of slit width and free from any ambiguity in the definition of critical region.

Using no net mass flow condition we obtain further from (8.16)

$$d\langle V_s\rangle_c=-\frac{c}{6\gamma\eta_n}\xi^{\frac{1}{2}}(1-\xi)^{\frac{1}{2}},\tag{8.18}$$

which secures the remarkable fact that $d\langle V_s\rangle_c$ is independent of the slit width itself, being discovered empirically by L-Z. As they neglect the temperature variation of $d\langle V_s\rangle_c$, it is reasonable that their curve q_c does not coincide with ours (8.17), Fig. 10. In Fig. 11 it is shown that the critical region appointed by (8.18) is in good agreement with the empirical data given by L-Z, where $d\langle V_s\rangle_c$ are, as is expected, expressed as finite segments, not as points.

In this way we could clearly substantiate the empirical facts concerning critical region discovered by L-Z; while we cannot readily agree with their physical interpretation. First, it is already evident that the critical region does not indicate the validity limit of hydrodynamical description. Second, there is but obscure ground in the present stage for their attempt to correlate the critical flow velocity directly to the surface flow. Their relation, if any, may be the remaining problem to be clarified in future by the phenomenological theory.

9. Concluding Remarks

In the above-developed treatments we have seen that the phenomena of stationary heat flow in liquid He II is fully described by the general thermodynamical equations of motion, which we derived in Part I, adopting Gorter's type of mutual friction. This fact proves clearly that the so-called "two-fluid model" is quite satisfactory, at least in the phenomenological part of the theory. Therefore, it is our thought that this model, without being discarded, should be developed from the molecular point of view. For this purpose, the temperature variation of γ, which is clearly expressed in Table II but neglected in our first approximation, should be fully analyzed.

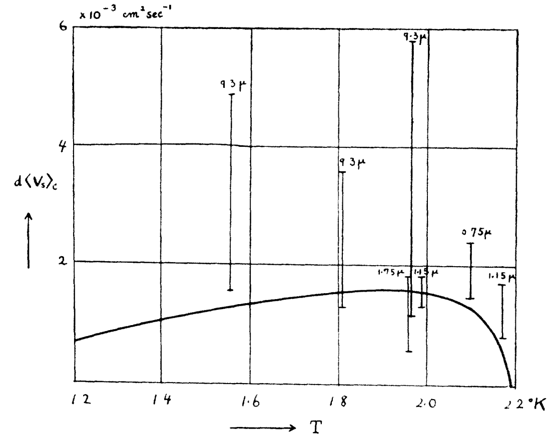

FIG. 11. $d\langle V_s\rangle_c$ against temperature.

In order to investigate the effect of mutual friction, we will take the problem of "capillary flow," having net mass flow, and of "surface flow." As regards the latter phenomenon one may expect the possibility that its description is beyond the present phenomenological treatment. It is to be noted, however, that surface flow can also cause the thermomechanical effect, as was shown first by Daunt and Mendelssohn, [22] and moreover, the ratio of the measured heat flow to that through the slit is just equal to the ratio of film thickness to slit width which was recognized by M-M.

As for the capillary flow, it should be pointed out that the boundary condition is different from that for the internal convection. In the case of capillary flow there is no heat source, but instead the pressure difference is maintained mechanically.

Unfortunately, no special caution was taken about the temperature difference, which is expected to be accompanied by capillary flow and surface flow.

Under stationary conditions our calculation gives the general expression (6.6) for the total mass flow in the reversible process approximation. To check this result, it is necessary to know the empirical relation between π and θ. It should be possible to observe the *mechanocaloric effect* as systematically and exhaustively as in Leiden measurements,[13–15] and is highly desirable from the theoretical point of view.

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[22] J. G. Daunt and K. Mendelssohn, Nature 143, 719 (1939).