averaged over the wire surface is given by [5,24]

$$\varphi^{**}=\sum_i w_i\varphi_i^{**}, \tag{1}$$

where w_i is the fraction of the total emission which comes from the ith type of crystal surface whose apparent work function is φ_i^{**}. The corresponding value of the emission constant A^{**} is given by

$$\log A^{**}=\log j_0/T^2+e\varphi^{**}/2.303kT, \tag{2}$$

where j_0 is the average current per unit area extrapolated to zero field. Figure 2 is a plot of the current measured as a function of the angle about the wire axis corresponding to a temperature of 1465°K. Using Eq. (1) in connection with Figs. 1 and 2, graphical integration yields a value of 4.45 volts for φ^{**} and a corresponding value of 41 amp/cm² deg² for A^{**}. These approximate results are in agreement with those of Table I.

These results further emphasize[2,5] the fact that the average thermionic constants from polycrystalline emitters depend upon the types and relative amounts of the various crystal surfaces exposed and are generally not characteristic of a uniform surface.

[24] C. Herring, Phys. Rev. 59, 889 (1941).

TABLE V. Results of spectroscopic analysis of specimens after heat treatment.

Element	K Tungsten	218 Tungsten
Ca	p^{++}	p^{+}
Fe	p^{+}	p^{+}
Mo	p	p^{+}
Ti	p^{+}	p^{+}
Cr	p^{--}	p^{--}
Ni	p	p
Al	p^{-}	p^{-}
Mg	p^{+}	p
Si	A?	p^{-}
Na and K — no difference		

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The Radiative Collisions of Positrons and Electrons*

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The general matrix elements for the collision of two electrons (*e-e* case) or a positron and an electron (*p-e* case) accompanied by the emission of a quantum of radiation are formulated. It is expected that the two cross sections will differ considerably both because of the existence of a dipole for the *p-e* case, and because of the essentially different nature of the exchange phenomenon in the *p-e* case arising there from the possibility of virtual pair creation or annihilation rather than the indistinguishability of the particles. Two cases are calculated in detail; the non-relativistic limit and the extreme relativistic limit in which the quantum energy is restricted to $\leqq E^{\frac{1}{2}}$ in the laboratory system. In the former case the cross section for the *e-e* case is found to vanish if the momentum of the quantum is neglected, while the *p-e* cross section is comparable to that for the "bremsstrahlung" process. In the latter case the cross sections of the two cases are of the same order of magnitude the *e-e* cross section being greater because of the exchange terms, the exchange terms of the *p-e* case being negligible in all of the cases considered.

I. INTRODUCTION

THE elastic scattering of electrons by electrons has been treated by Møller,[1] Breit,[2] and Bethe and Fermi,[3] using both correspondence principle methods and the quantum electrodynamics. The results of Møller have been adapted by Bhabha[4] to the case when one of the colliding particles is a positron.

The scattering of two charged particles by each other with the emission of a quantum of radiation has been treated by Møller[5] for particles of arbitrary masses. This author only carries out the complete calculation of the collision cross section for the limiting case that the mass of one of the particles becomes infinite, and thus obtains an expression identical to that obtained by Bethe and Heitler[6] for the "bremsstrahlung" process. The process is generalized in a paper by Havas[7] for the case that any number of quanta are emitted.

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[1] C. Møller, Ann. d. Physik 126, 259 (1930).
[2] G. Breit, Phys. Rev. 34, 553 (1929).
[3] H. Bethe and E. Fermi, Zeits. f. Physik 77, 296 (1932).
[4] H. J. Bhabha, Proc. Roy. Soc. A154, 195 (1936).
[5] C. Møller, Proc. Roy. Soc. A152, 481 (1935).
[6] H. Bethe and W. Heitler, Proc. Roy. Soc. A146, 83 (1934).
[7] P. Havas, Phys. Rev. 68, 214 (1945).

The present investigation treats the radiative collisions for the case that both of the particles are electrons (e-e case) and the case that one of the particles is an electron and the other a positron (p-e case). Classical considerations lead one to expect a higher probability for the latter process as a classically-defined dipole moment can exist which is absent for the case that both particles possess the same sign of charge. Also the exchange phenomenon attending the collision of identical particles is likewise present for the collision of particles and anti-particles as is pointed out by Bhabha.[4] The exchange phenomenon in the latter case, however, is of an essentially different nature than that arising in the collision of identical particles since it rises from the virtual annihilation and creation of a pair. It is these two points, the existence of a classically defined dipole moment and an essentially different type of exchange effect, that lead one to the expectation that the cross sections for the two processes should differ considerably.

We shall use so-called "natural units" throughout the calculation. Such units take as the unit of mass the electronic mass m, the unit of length the Compton wave-length, $\hbar/mc$, and the unit of velocity the velocity of light in vacuum, c. In such units the units of momentum and energy are mc and mc^2, respectively.

As a general solution of the problem is extremely laborious, we shall confine outselves to two limiting approximations; first, the non-relativistic limit, and second, the extreme relativistic limit, with the restriction that the energy of the emitted quantum in the center-of-gravity system is of the order of magnitude of the proper electron energy.

II. FORMULATION OF THE MATRIX ELEMENT FOR THE ELECTRON-ELECTRON COLLISION

The matrix element for the electron-electron collision with the emission of a quantum of energy k according to Møller[8] is:

$$\left(\frac{32\pi^3\alpha^3}{L^9k}\right)^{\frac{1}{2}}\sum_{''}\frac{(u_2'^*\boldsymbol{\alpha}\cdot\mathbf{f}u_2'')}{E_1+E_2-E_2''-E_1'}\left\{-\frac{(u_2''^*u_2)(u_1'^*u_1)-(u_2''^*\boldsymbol{\alpha}u_2)\cdot(u_1'^*\boldsymbol{\alpha}u_1)}{(E_1-E_1')^2-q_1^2}\right\}$$
$$+\sum_{'''}\left\{-\frac{(u_2'^*u_2''')(u_1'^*u_1)-(u_2'^*\boldsymbol{\alpha}u_2''')\cdot(u_1'\boldsymbol{\alpha}u_1)}{(E_1-E_1')^2-q_1^2}\right\}\frac{(u_2'''^*\boldsymbol{\alpha}\cdot\mathbf{f}u_2)}{E_2'+E_1'-E_1-E_2'''}$$
$$+(\text{term with 1 and 2 interchanged})-(\text{exchange terms}). \quad (1)$$

The subscripts 1 and 2 refer to quantities belonging to one or the other of the particles while the presence or absence of a prime refers to the final or initial state. The double or triple primes refer to the intermediate state. α is the fine-structure constant while L is the edge length of the normalization cube. The summations $\sum_{''}$ and $\sum_{'''}$ refer to sums over those intermediate states for which momentum is conserved. The term written explicitly refers to the emission of the quantum by the particle labeled 2. There will be a similar term for the particle labeled 1 which is identical except for an interchange of subscripts. From these "direct" terms there can be obtained the two exchange terms by changing the subscripts of all quantities referring to the final state (primed quantities). This procedure corresponds to the use of anti-symmetric wave functions to describe the electrons as required by the exclusion principle.

The sum over intermediate states is carried out with the aid of the so-called "selection operators."[9] These are operators which give vanishing results when applied to a Dirac amplitude which refers to a state of positive (negative) energy and zero when applied to a Dirac amplitude referring to a state of negative (positive) energy. We obtain for the sum over the intermediate state of the first term in (1) the following expression:

$$\sum_{''}\frac{(u_2'^*\boldsymbol{\alpha}\cdot\mathbf{f}u_2'')(u_2''^*u_2)(u_1'^*u_1)}{(E_1+E_2-E_2''-E_1')[(E_1-E_1')^2-q_1^2]}=\frac{(u_2'^*\boldsymbol{\alpha}\cdot\mathbf{f}}{(E_1-E_1')^2-q_1^2}\left\{\frac{|E_2''|-\boldsymbol{\alpha}\cdot\mathbf{p}_2''-\beta}{2|E_2''|}\frac{1}{E_1+E_2-|E_2''|-E_1'}\right.$$
$$\left.+\frac{|E_2''|+\boldsymbol{\alpha}\cdot\mathbf{p}_2''+\beta}{2|E_2''|}\frac{1}{E_1+E_2+|E_2''|-E_1'}\right\}u_2'')(u_2''^*u_2)(u_1'^*u_1). \quad (2)$$

By use of the momentum conservation equation $\mathbf{p}_2''=\mathbf{p}_2'+\mathbf{k}$ this expression may be simplified to the following:

$$\frac{(u_2'^*\{I_2\}u_2)(u_1'^*u_1)}{(E_1-E_1')^2-q_1^2};\quad \{I_2\}=\left\{\frac{k(\boldsymbol{\alpha}\cdot\mathbf{f})+ik(\boldsymbol{\sigma}\cdot\mathbf{f}')-2(\mathbf{f}\cdot\mathbf{p}_2')}{2(E_2'k-\mathbf{p}_2'\mathbf{k})}\right\};\quad \mathbf{f}\times\mathbf{f}'=\frac{\mathbf{k}}{|k|}=\mathbf{n}. \quad (3)$$

If one carries out this summation for all the terms in (1), one obtains the following expression for the matrix element for the collision of two electrons accompanied by the emission of a quantum of energy k. $\boldsymbol{\alpha}$, β, and $\boldsymbol{\sigma}$

[8] C. Møller, Proc. Roy. Soc. **A152**, 481 (1935).
[9] H. Casimir, Helv. Phys. Acta **6**, 287 (1933).

are the usual Dirac matrices.

$$H_{e-e}=\left(\frac{32\pi^3\alpha^3}{L^9k}\right)^{\frac{1}{2}}\left(-\frac{1}{(E_1-E_1')^2-q_1^2}\left[(u_2'^*\{I_2\}u_2)(u_1'^*u_1)-(u_2'^*\{I_2\}\boldsymbol{\alpha}u_2)\cdot(u_1'^*\boldsymbol{\alpha}u_1)\right.\right.$$
$$\left.\left.+(u_2'^*\{II_2\}u_2)(u_1'^*u_1)-(u_2'^*\boldsymbol{\alpha}\{II_2\}u_2)\cdot(u_1'^*\boldsymbol{\alpha}u_1)\right]\right)+(1\text{ and }2\text{ interchanged})-(\text{exchange terms}) \quad (4)$$

$$\{II_2\}=\left\{\frac{k(\boldsymbol{\alpha}\cdot\mathbf{f})-i(\boldsymbol{\sigma}\cdot\mathbf{f}')+2(\mathbf{f}\cdot\mathbf{p}_2)}{2(E_2k-\mathbf{p}_2\cdot\mathbf{k})}\right\}. \quad (5)$$

III. FORMULATION OF THE MATRIX ELEMENT FOR THE POSITRON-ELECTRON COLLISION

We shall describe the transitions of a positron in terms of a negative-energy energy electron which carries out the positron transition in the opposite sense. Thus the transition of the positron from a certain initial state to a certain final state is viewed as the transition of a negative-energy electron from the final state of the positron, evacuating a "hole" there in the infinite sea of negative-energies, and filling the "hole" corresponding to the positron initially present. We shall denote the Dirac amplitudes, energies and momenta referring to this negative energy electron by the subscript 1, while the corresponding quantities for a positive energy electron remain unsubscripted. We shall denote the energies and momenta of the positron by the subscript $+$. The following well-known relations connect the energies and momenta of the positron to those of the corresponding negative-energy electron.

$$|E_1|=E_+ \quad \mathbf{p}_1=-\mathbf{p}_+. \quad (6)$$

Further we shall denote by the absence or presence of a prime the initial and final states respectively of the *positron* which are just reverse from those of the negative-energy electron.

The direct matrix element for the case that the positron emits the quantum becomes:

$$H_{p-e}=\left(\frac{32\pi^3\alpha^3}{L^9k}\right)^{\frac{1}{2}}\left\{\sum_{''}\frac{(u_1^*\boldsymbol{\alpha}\cdot\mathbf{f}u_1'')}{E-E'-|E_1'|-E_1''}\left[\frac{(u_1''^*u_1')(u'^*u)-(u_1''^*\boldsymbol{\alpha}u_1')\cdot(u'^*\boldsymbol{\alpha}u)}{(E-E')^2-q^2}\right]\right.$$
$$\left.+\sum_{'''}\left[-\frac{(u_1^*u_1''')(u'^*u)-(u_1^*\boldsymbol{\alpha}u_1''')\cdot(u'^*\boldsymbol{\alpha}u)}{(E-E')^2-q^2}\right]\frac{(u_1'''^*\boldsymbol{\alpha}\cdot\mathbf{f}u_1')}{E'-E-|E_1|-E_1'''}\right\}. \quad (7)$$

The above expression is similar to (1) except for the interchange of denominators between terms, the interchange of primes on the Dirac amplitudes, and the sign of the energies of the intermediate states. The first two differences are due to the description of the process in terms of a particle making the reverse transition. The change in the sign of the intermediate state energy stems from the fact that a positive-energy intermediary state of the positron corresponds to a negative-energy state of the corresponding electron, while a negative-energy intermediary state of the positron corresponds to a positive-energy state for the electron.

The sum over the intermediate energy states is carried out with the help of the selection operators exactly as in the case cf the electron. After carrying out the summation one obtains for this term:

$$H_{p-e}=-\left(\frac{32\pi^3\alpha^3}{L^9k}\right)^{\frac{1}{2}}\frac{1}{(E-E')^2-q^2}$$
$$\times\{(u_1^*[I_1]u_1')(u'^*u)-(u_1^*\boldsymbol{\alpha}[I_1]u_1')\cdot(u'^*\boldsymbol{\alpha}u)$$
$$+(u_1^*[II_1]u_1')(u'^*u)-(u_1^*[II_1]\boldsymbol{\alpha}u_1')\cdot(u'^*\boldsymbol{\alpha}u)\} \quad (8)$$
$$[I_1]=\left[\frac{-k(\boldsymbol{\alpha}\cdot\mathbf{f})-ik(\boldsymbol{\sigma}\cdot\mathbf{f}')+2(\mathbf{f}\cdot\mathbf{p}_1')}{2(|E_1'|k+\mathbf{p}_1'\cdot\mathbf{k})}\right]$$
$$[II_1]=\left[\frac{-k(\boldsymbol{\alpha}\cdot\mathbf{f})+ik(\boldsymbol{\sigma}\cdot\mathbf{f}')-2(\mathbf{f}\cdot\mathbf{p}_1)}{2(|E_1|k+\mathbf{p}_1\cdot\mathbf{k})}\right].$$

If we insert the values for the energy and momentum of the positron as given by (6) we obtain the following relations between the operators for the emission by a positron (9) and the operators for emission by an electron (3) and (5).

$$[I_+]=\left[-\frac{k(\boldsymbol{\alpha}\cdot\mathbf{f})-ik(\boldsymbol{\sigma}\cdot\mathbf{f}')-2(\mathbf{f}\cdot\mathbf{p}_+')}{2(E_+'k-\mathbf{p}_+'\cdot\mathbf{k})}\right]=-\{I_+\}^*$$
$$[II_+]=\left[-\frac{k(\boldsymbol{\alpha}\cdot\mathbf{f})+ik(\boldsymbol{\sigma}\cdot\mathbf{f}')+2(\mathbf{f}\cdot\mathbf{p}_+)}{2(E_+k-\mathbf{p}_+\cdot\mathbf{k})}\right]=-\{II_+\}^*. \quad (10)$$

The matrix element for the emission of the quantum by the electron can be written down at once:

$$H_{p-e}=-\left(\frac{32\pi^3\alpha^3}{L^9k}\right)^{\frac{1}{2}}\frac{1}{(E_+-E_+')^2-q_+^2}$$
$$\times[(u'^*\{I\}u)(u_1^*u_1')-(u'^*\{I\}\boldsymbol{\alpha}u)\cdot(u_1^*\boldsymbol{\alpha}u_1')$$
$$+(u'^*\{II\}u)(u_1^*u_1')-(u'^*\boldsymbol{\alpha}\{II\}u)\cdot(u_1^*\boldsymbol{\alpha}u_1')]. \quad (11)$$

It is seen to be virtually identical to the first term in (5).

If one compares the terms (8) and (11) noting the relation (10) it is seen that the two terms are negative Hermitian conjugates of each other. Hence the square of their sum will be the same as the square of the sum of the first two terms of (5) except for the sign of the cross-product term. For the e-e case this cross-product term is positive while for the p-e case it is negative because of (10). This is just what one would expect from the algebraic signs of the charges, but we have obtained it without introducing the explicit sign of charge.

We now must consider the exchange terms for the p-e case. Unlike the case of two electrons where the exchange terms correspond to the non-identifiability of the two electrons, the exchange terms for the p-e case correspond to the annihilation or creation of a pair. These exchange terms are found to be:

$$\left[\frac{(u'^*\{I\}u_1')(u_1^*u)-(u'^*\{I\}\boldsymbol{\alpha}u_1')\cdot(u_1^*\boldsymbol{\alpha}u)}{(E+E_+)^2-(\mathbf{p}+\mathbf{p}_+)^2}\right]$$
$$+\left[\frac{(u_1^*\{II\}u)(u'^*u_1')-(u_1^*\boldsymbol{\alpha}\{II\}u)\cdot(u'^*\boldsymbol{\alpha}u_1')}{(E'+E_+')^2-(\mathbf{p}'+\mathbf{p}_+')^2}\right]$$
$$+\left[\frac{(u'^*[I_+]u_1')(u_1^*u)-(u'^*\boldsymbol{\alpha}[I_+]u_1')\cdot(u_1^*\boldsymbol{\alpha}u)}{(E+E_+)^2-(\mathbf{p}+\mathbf{p}_+)^2}\right]$$
$$+\left[\frac{(u_1^*[II_+]u)(u'^*u_1')-(u_1^*[II_+]\boldsymbol{\alpha}u)\cdot(u'^*\boldsymbol{\alpha}u_1')}{(E'+E_+')^2-(\mathbf{p}'+\mathbf{p}_+')^2}\right]. \tag{12}$$

The denominators of these terms are seen to differ from the e-e case in that they contain energies and momenta referring solely to the initial or final states and not to both, and that the corresponding quantities are added. This difference arises from the fact that the intermediate state may contain all four particles or none at all. This means that the energy denominators will be quite large and the probability for such virtual pair creation or annihilation very small. We shall see that the exchange terms are negligible compared to the direct terms in the (p-e) collision for the cases considered.

IV. THE NON-RELATIVISTIC LIMIT

The first of the two limiting cases for which we shall carry out explicit calculations is the non-relativistic approximation. We shall obtain the cross section for radiative collisions when the velocities of the particles are small compared to the velocity of light. In natural units such as we are using, this means that all energies are approximately equal to unity and all momenta are numbers small compared to unity.

In (1) and (7) we note that each term consists of two parts

$$(u_2''^*u_2)(u_1'^*u_1)-(u_2''^*\boldsymbol{\alpha}u_2)\cdot(u_1'^*\boldsymbol{\alpha}u_1). \tag{13}$$

The first term gives the Coulomb or longitudinal interaction and the second the interaction by means of a virtual quantum or transverse interaction. This latter term corresponds to the retardation. The expectation value of the operator $\boldsymbol{\alpha}$ is $\mathbf{v}/c$ so the transverse interaction is of the order v^2/c^2 compared to the longitudinal interaction and hence may be neglected in the non-relativistic approximation.

If we examine the invariant energy-momentum denominators (which we shall hereafter refer to as the Møller denominators) we note that the exchange terms in the p-e case become negligible for this approximation though not in the e-e case. As:

$$\begin{array}{lcc} & \text{Exchange} & \text{Direct} \\ e\text{-}e \text{ case} & \dfrac{1}{(E_1-E_2')^2-(\mathbf{p}_1-\mathbf{p}_2')^2}\to-\dfrac{1}{(\mathbf{p}_1-\mathbf{p}_2')^2} & \dfrac{1}{(E_2-E_2')^2-(\mathbf{p}_2-\mathbf{p}_2')^2}\to-\dfrac{1}{(\mathbf{p}_2-\mathbf{p}_2')^2} \\ p\text{-}e \text{ case} & \dfrac{1}{(E+E_+)^2-(\mathbf{p}+\mathbf{p}_+)^2}\to\dfrac{1}{4} & \dfrac{1}{(E-E')^2-(\mathbf{p}-\mathbf{p}')^2}\to\dfrac{1}{4}. \end{array} \tag{14}$$

Since all momenta are numbers small compared to unity, we are justified in neglecting the exchange terms in the (p-e) case as small compared to the direct terms.

This result leads to the conclusion that the positron and electron behave like independent particles in all collisions where the energies involved are small compared to the proper energy of the particles; i.e., the non-relativistic limit. This is exactly what one would expect as the connection between the particles and their anti-particles is given by the Dirac "hole" theory which is an essentially relativistic theory, and such a connection should be expected to disappear in the non-relativistic limit.

We shall carry out the calculation initially for the center-of-gravity system, i.e. that reference system such that $\mathbf{p}_2=-\mathbf{p}_1$. The conservation equation then reads:

$$\frac{p_2^2}{2}+\frac{p_1^2}{2}=\frac{p_2'^2}{2}+\frac{p_1'^2}{2}+k \quad k=p_2^2-\frac{p_2'^2}{2}-\frac{p_1'^2}{2}. \tag{15}$$

These equations show that $k\sim p^2$. Since $p\ll 1$ we are justified in neglecting the momentum of the light quantum as compared to that of the particles. Hence $\mathbf{p}_2'\approx\mathbf{p}_1'$. This approximation simplifies the calculations considerably. Though we neglect the momentum of emitted quantum, we do not neglect its energy, i.e. $|\mathbf{p}_1|\neq|\mathbf{p}_1'|$; $|\mathbf{p}_2|\neq|\mathbf{p}_2'|$.

Since $\mathbf{p}_1=-\mathbf{p}_2$, $\mathbf{p}_1'=-\mathbf{p}_2'$ the matrix element referring to particle 2 is related to that referring to particle 1 by the following relationship:

$$(u_1'^*\{I_1\}u_1)(u_2'^*u_2)=R[(u_2'^*\{I_2\}u_2)(u_1'^*u_1)]. \quad (16)$$

R is the operator which inverts the space coordinates of the electron alone leaving the coordinates of the light quantum propagation vector unaltered. The matrix element for the e-e collision can be then written as:

$$H_{e-e}=\left(\frac{32\pi^3\alpha^3}{L^9k}\right)^{\frac{1}{2}}\left[\frac{1}{(\mathbf{p}_1-\mathbf{p}_1')^2}-\frac{1}{(\mathbf{p}_1+\mathbf{p}_1')}\right](1+R)$$
$$\times[(u_1'^*\{I_1\}u_1)(u_2'^*u_2)+(u_1'^*\{II_1\}u_1)(u_2'^*u_2)]. \quad (17)$$

The corresponding matrix element for the p-e collision can be written, using the identities (10) as:

$$H_{p-e}=\left(\frac{32\pi^3\alpha^3}{L^9k}\right)^{\frac{1}{2}}\left[\frac{1}{(\mathbf{p}-\mathbf{p}')^2}\right](1-R)$$
$$\times[(u'^*\{I\}u)(u_1^*u_1')+(u'^*\{II\}u)(u_1^*u_1')].$$

As is well known the operator R has the eigenvalue $+1$ when applied to the components of an axial vector or a scalar, while it has the eigenvalue -1 when applied to the components of a polar vector. The operators $\{I\}$ and $\{II\}$ are sums of the operators $(\boldsymbol{\alpha}\cdot\mathbf{f})$, $(\mathbf{p}\cdot\mathbf{f})$, $(\mathbf{p}'\cdot\mathbf{f})$ and $(\boldsymbol{\sigma}\cdot\mathbf{f}')$. All except the last are components of polar vectors referring to the electrons and change sign under application of the Operator R. The components of $\mathbf{f}$ and $\mathbf{f}'$ are unchanged as the operator R does not invert the coordinates of the quantum propagation vector. The operator $(\sigma\cdot\mathbf{f}')$ will cancel between the terms $\{I\}$ and $\{II\}$ as it occurs with opposite signs in them. Hence we see that (17) is identically zero. The transition probability for the radiative collision for the electrons is zero in the non-relativistic limit if the momentum of the emitted quantum is neglected.

This result is entirely in accord with the classical radiation theory. For, in the limit of wave-lengths large compared to the dimensions of the radiating system, the radiated intensity is calculated by coherent superposition of the radiation emitted by each of the accelerated particles. In the e-e collision the acceleration of the particles is always equal and opposite when the momentum of the emitted quantum is neglected. Hence the waves will destructively interfere and the net intensity radiated is negligible. If, however, one of the particles is positively charged, the two waves will constructively interfere and the net intensity radiated will be considerable.

The matrix element for the p-e collision is:

$$H_{p-e}=\left(\frac{32\pi^3\alpha^3}{L^9k}\right)^{\frac{1}{2}}\left[\frac{2}{(\mathbf{p}-\mathbf{p}')^2}\right]$$
$$\times[(u'^*\{2(\boldsymbol{\alpha}\cdot\mathbf{f})+2(\mathbf{p}-\mathbf{p}'\cdot\mathbf{f})\}u)(u_1^*u_1')]. \quad (18)$$

The transition probability is given by the well-known expression[10] which in natural units is

$$w=2\pi|H|^2\rho_F. \quad (19)$$

Dividing by an electron flux of one electron per volume L^3 moving with the velocity $v=p$ we get for the differential cross section

$$d\phi=\frac{2\pi L^3}{p}|H|^2\rho_F. \quad (20)$$

ρ_F is the density of final states into which the scattered electron can go and into which the quantum can be emitted. For the non-relativistic approximation and neglecting k compared with p' we have for ρ_F:

$$\rho_F=\frac{L^3p'k^2d\Omega_{p'}d\Omega_k}{2(2\pi)^6}. \quad (21)$$

Inserting this expression and integrating over the solid angle of p' and k we obtain for the total cross section:

$$\Phi=\frac{128}{3}\phi_0\frac{1}{p^2}\log\frac{(p+p')}{(p-p')}\frac{dk}{k} \quad (22)$$

$$\Phi_0=\text{natural cross section}=(\hbar/mc)^2\alpha^3.$$

We introduce the variable $\epsilon=k/p^2$ which gives the fraction of the kinetic energy which is radiated. The cross section now becomes:

$$\Phi=\frac{128}{3}\Phi_0\frac{1}{k}\log\left[\frac{1+(1-\epsilon)^{\frac{1}{2}}}{1-(1-\epsilon)^{\frac{1}{2}}}\right]d\epsilon. \quad (23)$$

Transforming this cross section to the laboratory system (in which we denote quantities with a star) this expression becomes:

$$\Phi^*=\frac{256}{3}\Phi_0\frac{1}{k}\log\frac{1+(1-2\epsilon^*)^{\frac{1}{2}}}{1-(1-2\epsilon^*)^{\frac{1}{2}}}d\epsilon^*. \quad (24)$$

It is easily seen that ϵ^* can only range from 0 to $\frac{1}{2}$ as compared to 0 to 1 for ϵ. In the center-of-gravity system the positron and electron can effectively come to rest giving all of their kinetic energy up as radiation. Conservation of momentum forbids this in the laboratory system. The maximum energy which can be radiated is then half the initial total energy.

The expression (24) is the same except for the numerical factor of 32 as that given by Heitler[11] for the non-relativistic "brehmsstrahlung." Since both particles radiate the square of the matrix element will thus contribute a factor of 4. Further, another factor of 4 comes from the reduced mass of the positron and electron. Finally a factor of 2 is introduced in passing from the center-of-gravity to the laboratory system.

[10] W. Heitler, *Quantum Theory of Radiation* (Clarendon Press, Oxford, 1944), second edition, p. 90.
[11] W. Heitler, reference 10, p. 166.

The cross section vanishes in the limit $\epsilon=1$ or $\epsilon^*=\frac{1}{2}$. This is due to the failure of the Born approximation (plane wave functions) in the limit of low velocities. The criterion for the validity of the Born approximation is $e^2/hv=\alpha/(v/c)\ll 1$. The Born approximation is valid far into the non-relativistic region because of the smallness of α. However it fails for those collisions in which the colliding particles slow down to the velocities of the order of $(1/137)c$. In this region the exact theory of Sommerfeld[12] gives a non-vanishing cross section for $\epsilon=1$. This result is well known and we will not discuss it further.

In the limit $\epsilon\rightarrow 0$ we see that the cross section diverges logarithmically (in addition to the well-known infra-red catastrophe due to the factor $1/k$; it being convenient to isolate this latter effect by discussing instead of the cross section Φ, the radiated intensity $k\Phi$).

This logarithmic divergence of the radiated intensity $k\Phi$ in the limit of very soft quanta is a Rutherford-type of infinity which arises from the implicit assumption that the Coulomb field extends to infinity. Although the interaction potential falls off directly with the impact parameter, the density of final states increases with the square of the impact parameter leading to a very large probability for collisions in which a very soft quantum is emitted. In the theory of the "brehmsstrahlung"[6] the screening of the nuclear field by the atomic electrons restricts the effective range of the Coulomb potential to a finite value and thus eliminates this form of divergence. In our problem the idea of screening is replaced when considering the passage of positrons through matter by the restriction that the positrons interact only with the nearest electron, thus restricting the impact parameter to the order of magnitude of the mean distance between electrons.

This idea may be illustrated by introducing instead of the Coulomb potential, the modified potential $e^{-r/a}/r$. Such a potential behaves like the Coulomb potential for $r<a$ and drops rapidly to zero for $r>a$. a is then the effective range of the screened Coulomb field. We obtain for the cross section in the center-of-gravity system:

$$\Phi=\frac{128}{3}\Phi_0\left\{\frac{1}{2p^2}\log\left(\frac{[1+(1-\epsilon)^{\frac{1}{2}}]^2+\dfrac{1}{a^2p^2}}{[1-(1-\epsilon)^{\frac{1}{2}}]^2+\dfrac{1}{a^2p^2}}\right)-\frac{2}{a^2p^2\left[(1+(1-\epsilon)^{\frac{1}{2}})^2+\dfrac{1}{a^2p^2}\right]\left[(1-(1-\epsilon)^{\frac{1}{2}})^2+\dfrac{1}{a^2p^2}\right]}\right\}. \tag{25}$$

The parameter $1/a^2p^2=\lambda\!\!\!^{-2}/a^2$ where $\lambda\!\!\!^-$ is the deBroglie wave-length of the incident particle. So long as the effective range of the Coulomb field is large compared to the deBroglie wave-length of the particles we may discard the second term as well as the term $1/a^2p^2$ in the numerator of the logarithm. Equation (25) then becomes

$$\Phi=\frac{64}{3}\Phi_0\log\left[\frac{[1+(1-\epsilon)^{\frac{1}{2}}]^2}{[1-(1-\epsilon)^{\frac{1}{2}}]^2+(1/a^2p^2)}\right]\frac{d\epsilon}{k}. \tag{26}$$

This expression is the same as (23) except in the limit $\epsilon\rightarrow 0$ it converges to the value $(128/3)\log(2ap)$.

V. THE EXTREME RELATIVISTIC LIMIT

We turn now to the second of the two limiting cases which we shall treat, the extreme relativistic limit. Since in natural units $p\gg 1$ in this limit and $E=(1+p^2)^{\frac{1}{2}}$ we shall be able in most parts of the calculation to take $p\cong E$. As a further simplification we shall assume that the maximum energy of the emitted quantum $k\sim 1$ so that k can be considered small compared to all energy and momentum quantities. The simplification that this assumption introduces is obvious when we examine the numerator of expressions (3) and (5), the radiation operators. The first two terms are of order k while the third is of order p. By restricting the magnitude of k in the center-of-gravity system to values ≤ 1 these first two terms are negligible compared to the third term. This restriction is not as narrow as would first appear since in going over to the laboratory system by means of a Lorentz transformation a forward emitted quantum is hardened by the Doppler effect so that its energy $k\sim E^{\frac{1}{2}}$. As virtually the total emission takes place in a narrow cone about the axis of initial velocities this result will give the emitted spectrum up to energies of the order of the square root of the energy of the incident particle.

In this approximation the expressions (3) and (5) now have the matrix dependence of the unit matrix and can be factored out of the Dirac amplitudes. Since the energy and momentum of the emitted quantum is small we can make the further approximation $E_2'=E_1'$; $\mathbf{p}_2'=-\mathbf{p}_1'$. As we are working in the center-of-gravity reference system we have $E_2=E_1$; $\mathbf{p}_2=-\mathbf{p}_1$. Thus we may drop the subscripts on the energy and momentum quantities. The matrix element for the e-e collision is then

$$\begin{aligned}H_{e-e}=-\left(\frac{32\pi^3\alpha^3}{L^9k^3}\right)^{\frac{1}{2}}\Bigg[&(\mathbf{p}\cdot\mathbf{f})\left(\frac{1}{E-\mathbf{p}\cdot\mathbf{n}}-\frac{1}{E+\mathbf{p}\cdot\mathbf{n}}\right)\\&-(\mathbf{p}\cdot\mathbf{f})\left(\frac{1}{E'-\mathbf{p}'\cdot\mathbf{n}}-\frac{1}{E'+\mathbf{p}'\cdot\mathbf{n}}\right)\Bigg]\\&\times\left\{\frac{(u_2'^*u_2)(u_1'^*u_1)-(u_2'^*\boldsymbol{\alpha}u_2)\cdot(u_1'^*\boldsymbol{\alpha}u_1)}{(E-E')^2-(\mathbf{p}-\mathbf{p}')^2}\right.\\&\left.-\frac{(u_2'^*u_1)(u_1'^*u_2)-(u_2'^*\boldsymbol{\alpha}u_1)\cdot(u_1'^*\boldsymbol{\alpha}u_2)}{(E-E')^2-(\mathbf{p}+\mathbf{p}')^2}\right\}.\end{aligned} \tag{27}$$

[12] A. Sommerfeld, Ann. d. Physik 11, 251 (1931).

We note that the curly-bracketed quantity is, except for a constant factor just the elastic scattering cross section for electrons as formed by Møller.[13] Our approximation is then seen to correspond to the procedure employed in classical radiation theory; to calculate the orbits of the particles under the assumption that energy is conserved, and, using the accelerations so obtained to calculate the intensity of the emitted radiation. Such a procedure is justified so long as the energy radiated is only a small fraction of the total energy of the system.

The expression (27) must be squared and summed over the spin directions of the particles and the polarization directions of the quantum. We then obtain

$$\sum_{f}\sum_{\sigma_1,\sigma_1',\sigma_2,\sigma_2'}|H_{e-e}|^2=\frac{128\pi^3\alpha^3}{L^9k}\left[\frac{p^4\cos^2\delta\sin^2\delta}{[1+p^2\sin^2\delta]^2}+\frac{p'^4\cos^2\gamma\sin^2\gamma}{[1+p'^2\sin^2\gamma]^2}-\frac{p^2p'^2[\cos\theta-\cos\delta\cos\gamma]\cos\delta\cos\gamma}{[1+p^2\sin^2\delta][1+p'^2\sin^2\gamma]}\right]$$
$$\times\left[\frac{8+2(1+\cos\theta)^2}{[(E-E')^2-(p^2+p'^2-2pp'\cos\theta]^2}+\frac{8+2(1-\cos\theta)^2}{[(E-E')^2-(p^2+p'^2+2pp'\cos\theta]^2}\right.$$
$$\left.-\frac{16}{[(E-E')^2-(p^2+p'^2-2pp'\cos\theta)][(E-E')^2-(p^2+p'^2+2pp'\cos\theta)]}\right] \quad (28)$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{p}'$ (the scattering angle), and δ and γ the angle between $\mathbf{k}$ and $\mathbf{p}$ and $\mathbf{p}'$ respectively.

Before attempting to integrate the expression over all angles we shall try to get some idea of its general behavior. Since $k\ll E$ we can equate primed and unprimed quantities except in the Møller denominators. If this is done for the denominators it is seen that they vanish for $\theta=0$ or $\theta=\pi$. To avoid this infinity (which corresponds to a complete miss) we shall expand these denominators in powers of k/E and retain only the lowest power that is necessary to prevent them from vanishing in the limit $\theta\rightarrow 0$ or $\theta\rightarrow\pi$. Introducing the parameters

$$w=1-\cos\theta,\quad y=1-\cos\gamma,\quad z=1-\cos\delta, \quad (29)$$

we then obtain for the expression (28):

$$\sum_{f}\sum_{\sigma_1,\sigma_1',\sigma_2,\sigma_2'}|H_{e-e}|^2=\frac{128\pi^3\alpha^3}{L^9k^3}\left[\frac{y(2-y)(1-y)^2}{[1+p^2y(1-y)]^2}+\frac{z(2-z)(1-z)^2}{[1+p^2z(1-z)]^2}-\frac{[y+z-yz-w](1-y)(1-z)}{[1+p^2y(1-y)][1+p^2z(1-z)]}\right]$$
$$\times\left(\frac{8+2(2-w)^2}{\left[\frac{ky}{2E^3}+2w\right]^2}+\frac{8+2w^2}{\left[\frac{ky}{2E^3}+2(2-w)\right]^2}-\frac{16}{\left[\frac{ky}{2E^3}+2w\right]\left[\frac{ky}{2E^3}+2(2-w)\right]}\right). \quad (30)$$

As far as the w-dependence is concerned the function will have large maxima for $w=0$ and $w=2$. Physically this corresponds to the fact that the cross section is appreciable only for very small scattering angles. The y and z dependence gives zero for y and $z=0$ or 2. However, there are steep maxima for values of y or z which are very close to 0 or 2, namely, y or $z=(1/2p^2)$ or $2-(1/2p^2)$. The behavior of these terms corresponds physically to the fact that there must be a component of momentum transverse to the propagation vector of the quantum if there is to be emission which one expects from classical radiation theory. However the Doppler denominators $(E-\mathbf{p}\cdot\mathbf{n})$, $(E+\mathbf{p}\cdot\mathbf{n})$ reduce the probability for the emission of a quantum whose propagation vector makes an angle greater than $1/(2p^2)$ with the momentum vector virtually to zero. So while there must be a transverse component of momentum if a quantum is to be emitted at all it must be very small.

The combination of the Doppler and Møller denominators makes it evident that the quantum will be emitted almost directly forward or backward and the emergent particles will come off at a very small angle with the incident particles. Although the Møller denominators give a maximum for zero scattering angle, there can be no emission, for conservation of momentum then demands that the light quantum propagation vector be parallel to the momentum vectors. There is then no transverse component of momentum, and hence there can be no emission of the light quantum.

This "hollow-peak" behavior of the radiative cross section leads one to question the original approximation. For should the neglected terms in the curly-bracketed expression give a finite probability for radiation when the scattering angle is zero, then it is not correct to discard them. Even if their contribution is of a smaller order of magnitude then the term considered, the steep maxima of Møller denominators for the scattering angles 0 and π will make their contribution to the total cross section considerable. A brief calculation, however, shows these neglected terms to give no contribution to the cross section for scattering angles of 0 or π.

[13] C. Møller, Ann. d. Physik **126**, 259 (1930).

Integrating over the angle variables we obtain for the cross section of the *e-e* collision in the limit $E \gg k$:

$$\Phi/\Phi_0 = (512/k^2)Edk, \qquad (31)$$

where Φ_0 is the natural cross section as defined in (22).

We note that the emitted intensity $k\Phi$ diverges like $1/k$ in the limit that the energy of the emitted quantum goes to zero. This again is a Rutherford-type infinity corresponding to an infinite probability for a "complete miss." Such an infinity may be excluded as in the non-relativistic limit by restricting the value that the impact parameter may assume. The divergence in part arises from the retardation terms which correspond to the emission by one particle of a virtual quantum and the absorption of the virtual quantum by the other particle. The momentum of the virtual quantum k' is just equal to q, the momentum transfer between the interacting particles. If the quantum is to be a virtual quantum it must be indetectable from the vacuum fluctuations of the radiation field. Thus utilizing the uncertainty relation between energy and time measurements a relation can be established between the energy of the virtual quantum and the distance between the interacting particles. For the uncertainty in time must be r/c the "time of flight" of the virtual quantum. In ordinary units we have

$$(\Delta E)(\Delta t) = k'(r/c) \geq \hbar, \quad k' > c\hbar/r. \qquad (32)$$

If r be restricted to be less than some maximum value, then $k'=q$ must always be greater than some minimum value hence likewise the Møller denominator $(E-E')^2 - q^2$ must remain finite. Thus the restriction upon the impact parameter has the same effect upon the terms arising from the retarded interaction as upon the Coulomb terms that was shown in Part IV, although it is more difficult to display this behavior explicitly. Thus the divergence of the cross section in the limit of soft quanta may be eliminated by restricting the value of the impact parameter to the order of the mean distance between electrons.

We turn now to the *p-e* collision. We see at once that the exchange terms will be small compared to the direct terms except for the case that the emergent particles come off almost parallel to each other. This corresponds to the emission of a very energetic quantum; the opposite situation to the one considered.

The matrix element for the *p-e* collision, omitting the exchange terms becomes:

$$H_{p-e} = -\left(\frac{32\pi^3\alpha^3}{L^9 k}\right)^{\frac{1}{2}}\left[(\mathbf{p}\cdot\mathbf{f})\left(\frac{1}{E-\mathbf{p}\cdot\mathbf{n}}+\frac{1}{E+\mathbf{p}\cdot\mathbf{n}}\right) - (\mathbf{p}'\cdot\mathbf{f})\left(\frac{1}{E'-\mathbf{p}'\cdot\mathbf{n}}+\frac{1}{E'+\mathbf{p}'\cdot\mathbf{n}}\right)\right] \times\left[\frac{(u'^*u)(u_1^*u_1')-(u'^*\alpha u)\cdot(u_1^*\alpha u_1')}{(E-E')^2-(\mathbf{p}-\mathbf{p}')^2}\right]. \qquad (33)$$

It is seen to differ from the matrix element (27) for the *e-e* collision only in the omission of the exchange term and in the addition, instead of subtraction, of the Doppler factors. This latter difference is unimportant in the extreme relativistic limit as $[1/(E-\mathbf{p}\cdot\mathbf{n})] \gg [1/(E+\mathbf{p}\cdot\mathbf{n})]$ and the sum of these two terms will differ little from their difference. One sees thus that coherence effects are no longer important in the extreme relativistic limit. Such coherent superposition of the radiation emitted by each particle is permissible only if the wave-length of the emitted radiation is large compared with the physical dimensions of the radiating system. Such is not the case in the limit considered. The typical dimension of our radiating system is the impact parameter. The minimum value which the impact parameter can assume is $\hbar/mc$; smaller values are beyond the limit of measureability. This is of the order of magnitude of the wave-length of the quanta considered. The Møller denominators give an appreciable probability for the process only for very small scattering angles. Since we are considering the collision to be quasi-elastic ($k \ll E$) the impact parameter varies inversely with the scattering angle. So for a very small scattering we should expect a large impact parameter ($b > \hbar/mc = \lambda$). Also if the energy of the emitted quantum decreases (and the wave-length increases) the maxima of the Møller denominators become steeper, and the scattering angle for an appreciable probability of radiation gets smaller corresponding to a still larger impact parameter. So $b > \lambda$ for all but a negligible fraction of the emitted radiation. Hence the radiation from the two particles will be incoherent, and the charge of the emitting particle is inconsequential.

After integrating over-all angles we obtain finally for the positron cross section:

$$\Phi/\Phi_0 = (96/k^2)Edk. \qquad (34)$$

We now want to transfer the cross section over into the laboratory system of reference. In the center-of-gravity system half the radiation is emitted in a narrow cone about the direction of the initial velocities in the forward direction and half is emitted in a similar cone in the backward direction. The half-angle of the cone is of the order of $1/2p^2$. Upon transferring to the laboratory system the backward scattered radiation is softened by a factor $1/(2E^*)^{\frac{1}{2}}$ and is nearly isotropic. The forward scattered radiation is hardened by a factor $(2E^*)^{\frac{1}{2}}$ and is emitted into a still narrower cone of half-angle of the order of $1/E^*$. Only the forward scattered radiation will be significant. The cross section for the forward-scattered radiation in the laboratory system is for the *e-e* case:

$$\frac{\Phi^*}{\Phi_0} = \frac{256}{k^{*2}}\left[\frac{E^*}{2}\left(1+\frac{p^*}{E^*}\right)\right]dk^* \qquad (35)$$

and for the p-e case

$$\frac{\phi^*}{\phi_0}=\frac{48}{k^{*2}}\left[\frac{E^*}{2}\left(1+\frac{p^*}{E^*}\right)\right]dk^*. \tag{36}$$

These expressions are valid for $k\leq(E^*)^{\frac{1}{2}}$.

VI. CONCLUSIONS

The two cases calculated in detail do not, unfortunately, include the most important experimental situations. We may, however, draw certain conclusions regarding the general behavior of the cross section. The constructive interference of the radiation from oppositely charged particles and the corresponding annulment of radiation from particles of the same charge has been verified for the cases that the radiation from the two particles is coherent. Also the exchange effects were found to be negligible in the p-e case except in the extreme relativistic limit when a very energetic quantum is emitted.

The cross section for the e-e collisions was found to be negligible in the non-relativistic limit. For p-e collisions in this limit it is seen to agree with the expression for the "bremsstrahlung" from impacts of electrons on protons ($Z=1$), except for a numerical factor. We note further that the natural cross section for the "bremsstrahlung" is $\Phi_0=(h/mc)^2\alpha^3Z^2$, that is Z^2 times the natural cross section for the p-e collision. It is then apparent that for light atoms the radiative collisions of positrons with orbital electrons will be of the same importance for energy loss as radiative collisions with the nucleus. The former process depends on the number of electrons per atom hence is proportional to Z while the latter is proportional to Z^2. For heavy atoms radiative impacts with the nucleus will predominate. Since the energy loss of electrons from radiative collisions with orbital electrons is zero we would expect the stopping of positrons of moderate energies in absorbers of low atomic weight to be greater than that of electrons even when the annihilations of the positrons in matter are not taken into account.

It is recognized that we have treated the electron in the collision as free so when the positron energy is of the same order as the ionization energy of the atom, the treatment breaks down. In such low energy regions, however, the most important cause of energy loss comes from exciting and ionizing impacts rather than radiative collisions.[14]

We can only infer a limited amount of information about the collisions in the extreme relativistic limit. All we can assert is that the cross sections for the emission of comparatively low energy quanta $k\leq E^{\frac{1}{2}}$ is of the same order of magnitude for both p-e and e-e collisions, the latter being larger by virtue of the exchange phenomenon. The quanta are emitted in the forward direction in a very narrow cone about the direction of the incident particle. The cross section increases with the energy instead of the logarithm of the energy as in the case of the "bremsstrahlung" cross section.

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[14] W. Heitler, reference 10, p. 217.