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Relativistic Motion in a Magnetic Field

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A simple integral of the relativistic equations of motion for a charge in a magnetic field is used to reduce the Dirac equations to two simultaneous first-order differential equations. Eigenstates of this integral are the basis for a treatment of degeneracy and, in the case of a homogeneous field, for the construction of energy eigenfunctions from non-relativistic solutions. For a free particle the same integral leads directly to the only possible energy states with exactly defined spin projections.

I. INTRODUCTION

IN a recent paper[1] a treatment of the NR motion of a charge in a homogeneous magnetic field has been given which contains an exhaustive classification of the energy states in terms of integrals of the motion. While eigenfunctions were also given for the relativistic motion, there remained a twofold degeneracy in the energy, "accidental" in the sense that the degenerate states were not connected with an integral of motion. In the present paper this degeneracy is derived from and the degenerate states described by certain integrals which have not been previously considered.

In an inhomogeneous magnetic field (Section II) the relativistic equations of motion admit a simple integral, I, by which the problem of finding energy eigenfunctions can be reduced to the solution of two simultaneous first-order differential equations. For positive energy states with positive (negative) eigenvalues of I, the spin density is parallel (antiparallel) to the matter current. The energy is degenerate if a second integral exists which anticommutes with I or if the magnetic field satisfies certain symmetry conditions.

In a homogeneous magnetic field (Section III) explicit expressions are found for other integrals connected with the spin degeneracy. One of these involves only transverse motion and leads to a separation of the energy into transverse and longitudinal parts, each of which is a constant of motion. General solutions for the eigenfunctions of I are obtained in terms of NR wave functions. From these solutions spinor representations, as well as matter, spin and current densities, are derived for the energy eigenstates of the various integrals of motion.

For a free particle (Section IV) the energy eigenstates of I prove to be the only possible energy states in which a projection of the spin can be exactly defined. Solutions for the eigenfunctions of I are easily obtained for plane waves from which spinor representations are derived for the energy eigenstates of several integrals of motion. Free particle eigenfunctions can also be obtained from solutions in a homogeneous magnetic field by passing to the limit $\mathcal{H}\rightarrow 0$.

[1] M. H. Johnson and B. A. Lippmann, Phys. Rev. **76**, 828 (1949). The notation of this paper will be followed here and equation references will be indicated by the suffix A.

II. INHOMOGENEOUS MAGNETIC FIELD

The Dirac Hamiltonian for a charge in any magnetic field is, in conventional notation,

$$H=c\rho_1\boldsymbol{\sigma}\cdot\boldsymbol{\pi}+\rho_3mc^2, \tag{1}$$

where $\boldsymbol{\pi}$ is the kinetic momentum which satisfies the commutation relation of Eq. (3-A). Since $\boldsymbol{\sigma}$ commutes with ρ_1 and ρ_3,

$$I=c\boldsymbol{\pi}\cdot\boldsymbol{\sigma} \tag{2}$$

commutes with H and is therefore an integral of the motion. Simultaneous eigenfunctions for H and I necessarily exist and, as we shall see, represent states in which the spin is parallel or antiparallel to the current.

The Hamiltonian expressed in terms of I is

$$H=\rho_1I+\rho_3mc^2. \tag{3}$$

Thus the transformation from any representation in which I is diagonal to the principal axis of H involves only ρ_1 and ρ_3. In such a representation

$$H=\rho_1F+\rho_3mc^2, \tag{4}$$

where F is an eigenvalue of I. If the quantities ν_i are introduced

$$\nu_1=[\rho_1mc^2-\rho_3F][F^2+m^2c^4]^{-\frac{1}{2}}, \tag{5a}$$

$$\nu_2=\rho_2, \tag{5b}$$

$$\nu_3=[\rho_1F+\rho_3mc^2][F^2+m^2c^4]^{-\frac{1}{2}}, \tag{5c}$$

the Hamiltonian becomes

$$H=\nu_3[F^2+m^2c^4]^{\frac{1}{2}}. \tag{6}$$

Since $\nu_i^2=1$ and ν_i satisfy exactly the same commutation relations as ρ_i $(i=1, 2, 3)$, the eigenvalues of ν_3 are ± 1 and of H are

$$E=\pm[F^2+m^2c^4]^{\frac{1}{2}}. \tag{7}$$

Hence the energy eigenvalues occur in pairs $\pm E$ and, in magnitude are greater than or equal to mc^2.

The eigenfunctions of H are readily found from eigenfunctions of I by using the conventional matrix representation of $\boldsymbol{\sigma}$ and ρ_i with σ_z and ρ_3 diagonal. If

$$\psi_F=\begin{pmatrix}au_1\\au_2\\bu_1\\bu_2\end{pmatrix}, \tag{8}$$

then ψ_F is an eigenfunction of I provided that

$$\pi_- u_2 + (\pi_z - F/c)u_1 = 0, \tag{9a}$$

$$\pi_+ u_1 + (-\pi_z - F/c)u_2 = 0. \tag{9b}$$

The constants a and b can be so chosen that ψ_F is an eigenfunction of H,

$$H\psi_{E,F} = (\rho_1 I + \rho_3 mc^2)\psi_{E,F} = E\psi_{E,F}. \tag{10}$$

We thus find

$$\psi_{E,F} = \begin{Bmatrix} (E+mc^2)u_1 \\ (E+mc^2)u_2 \\ Fu_1 \\ Fu_2 \end{Bmatrix} [4E(E+mc^2)]^{-\frac{1}{2}}, \tag{11}$$

where E is given by Eq. (7). The problem of constructing energy eigenfunctions is thereby reduced to solving Eq. (9). In the next section general solutions are given for a homogeneous magnetic field.

The matter density and spin density, calculated from Eq. (11), are

$$\psi^*\psi = (1/2)(|u_1|^2 + |u_2|^2), \tag{12}$$

$$\psi^*\sigma_z\psi = (1/2)(|u_1|^2 - |u_2|^2), \tag{13}$$

$$\psi^*(\sigma_x + i\sigma_y)\psi = u_1^* u_2. \tag{14}$$

The matter current, calculated from Eq. (11), is given by the right-hand sides of Eqs. (13) and (14) multiplied by cF/E and therefore differs from the spin density by a constant factor. For positive energy states the matter current and spin density are parallel for positive eigenvalues of I and are antiparallel for negative eigenvalues of I.

It is evident that if the eigenvalues of I occur in pairs $\pm F$, the energy is degenerate in the sign of F. A sufficient condition is the existence of an operator, R, which anticommutes with I. For, in this case,

$$I(R\psi_F) = -RI\psi_F = -F(R\psi_F), \tag{15}$$

so that $R\psi_F$ is an eigenfunction of I for the eigenvalue $-F$. It should be noted that, if $RI=-IR$, an R can always be found which commutes with ρ_i (since I commutes with ρ_i) and that $\rho_3 R$ then commutes with H and is another integral of motion. The commutator, $[I, \rho_3 R]$, according to a general theorem, is then a third integral. Here the third integral is $iI\rho_3 R$ which anticommutes with both I and $\rho_3 R$.

The condition on the magnetic field that

$$\mathbf{A}(xyz) = -\mathbf{A}(-x-y-z) \tag{16}$$

is also sufficient for degeneracy in the sign of F. For under this condition the eigenvalue equation

$$-c\boldsymbol{\sigma}\cdot[i\hbar\nabla + (e/c)\mathbf{A}(xyz)]\psi_{E,F}(xyz) = F\psi_{E,F}(xyz) \tag{17}$$

is transformed into itself by changing the independent variables from xyz to $-x-y-z$, except that F is replaced by $-F$. Thus $\psi_{E,F}(-x-y-z)$ is an eigenfunction of I for the eigenvalue $-F$. According to Eq. (10) $i\rho_3\psi_{E,F}(-x-y-z)$ is then an eigenfunction of H for the eigenvalue E and is therefore a simultaneous eigenfunction of H and I. As $i\rho_3\psi_{E,F}(-x-y-z)$ is the spinor obtained from $\psi_{E,F}(xyz)$ by a reflection in the origin and as the matter, spin and current densities are scalar, pseudovector and vector quantities respectively, the matter and spin density for the state F at the point (xyz) and for the state $-F$ at $(-x-y-z)$ are the same while the current for the state F at (xyz) is equal in magnitude and opposite in direction to the current for the state $-F$ at $(-x-y-z)$. In general the matter densities for the two degenerate states are very different.

III. HOMOGENEOUS MAGNETIC FIELD

With $\mathbf{H}$ along the z axis, π_z is a constant of motion. In a representation with π_z diagonal all eigenfunctions contain z in a factor $\exp(ipz/\hbar)$ where p is an eigenvalue of π_z. The field has the symmetry required by Eq. (16) and the energy is degenerate in the sign of F. The solution derived from $\psi_{E,F}$ by a reflection in the origin clearly has the eigenvalue $-p$ for π_z and represents motion in a reversed sense along the z axis. New solutions can also be derived from $\psi_{E,F}$ because of the high degree of symmetry in the field, by reflections in the xy plane or in the z axis. It is easily shown that the eigenvalue for π_z, x_0 or y_0 changes sign in the derived solutions which are therefore contained in a classification of states according to a complete set of constants chosen from π_z, x_0 and y_0.

The solutions in Eq. (54-A) show another degeneracy in which the matter density for the two degenerate states is very nearly the same in contrast to the pairs of states discussed. This spin degeneracy may be inferred by writing

$$I = c(\sigma_x\pi_x + \sigma_y\pi_y) + c\sigma_z p, \tag{18}$$

where the two terms on the right-hand side anticommute with each other and therefore their product anticommutes with I. Hence

$$T = i\rho_3(I - c\sigma_z p)\sigma_z = i\rho_3(I\sigma_z - cp) \tag{19}$$

anticommutes with I and commutes with H so that I, T and iIT are three anticommuting integrals of motion leading to degeneracy in the sign of F. As these integrals commute with π_z, x_0 and y_0, the degeneracy is in addition to any degeneracy in which the degenerate states have different eigenvalues of π_z, x_0 or y_0.

To determine the eigenvalues of I, T and iIT, it is sufficient to consider their squares as the commutation properties insure the eigenvalues occur in pairs of the form $\pm F$. For example

$$I^2 = c^2\pi^2 + 2e\hbar c\sigma_z\mathcal{H}. \tag{20}$$

The eigenvalues of π^2 have been determined, Eqs. (16-A) and (17-A), while those of σ_z are ± 1. We conclude

$$F = \pm[c^2p^2 + 2e\hbar c\mathcal{H}n]^{\frac{1}{2}}. \tag{21}$$

In a similar way the eigenvalues of T are $\pm[2e\hbar c\mathcal{H}n]^{\frac{1}{2}}$ and those of iIT are $\pm|F|[2e\hbar c\mathcal{H}n]^{\frac{1}{2}}$. While, according to Eq. (21) F is related to both transverse and longitudinal motion, the eigenvalues of T depend only on the motion in the transverse plane.

Another integral of motion which involves only the longitudinal motion can be constructed from I and iIt. Let L be defined by

$$I^2L=-mc^2(iIT)+c\pi_zHI, \tag{22a}$$

$$L=mc^2\rho_3\sigma_z+c\rho_1\pi_z. \tag{22b}$$

From its construction, Eq. (22a), L anticommutes with T. Its eigenvalues are $\pm[c^2p^2+m^2c^4]^{\frac{1}{2}}$. The Hamiltonian can be simply expressed in terms of L and T,

$$H=\sigma_z(L+\rho_2T), \tag{23a}$$

$$H^2=L^2+T^2. \tag{23b}$$

The eigenfunctions of I are easily derived from the ψ_n of Eqs. (36-A) and (37-A). In fact, setting $u_1=a'\psi_{n-1}$ and $u_2=b'\psi_n$, Eq. (9) is satisfied provided that

$$(cp-F)a'+(2e\hbar c\mathcal{H}n)^{\frac{1}{2}}b'=0, \tag{24a}$$

$$(2e\hbar c\mathcal{H}n)^{\frac{1}{2}}a'+(-cp-F)b'=0. \tag{24b}$$

Upon solving Eq. (24) and inserting the corresponding functions u_1 and u_2 into Eq. (11),

$$\psi_{E,F}=\begin{Bmatrix}(E+mc^2)(2e\hbar c\mathcal{H}n)^{\frac{1}{2}}\psi_{n-1}\\(E+mc^2)(F-cp)\psi_n\\F(2e\hbar c\mathcal{H}n)^{\frac{1}{2}}\psi_{n-1}\\F(F-cp)\psi_n\end{Bmatrix}\times[4E(E+mc^2)F(F-cp)]^{-\frac{1}{2}}, \tag{25}$$

where F is given by Eq. (21). According to Eq. (24) only the minus sign should be used in Eq. (21) for $n=0$, in which case Eq. (25) remains valid. The state $n=0$ is not degenerate in the spin variables.

The matter and spin densities, obtained by inserting the solution of Eq. (24) into Eqs. (12), (13) and (14), are

$$\psi^*\psi=p_1|\psi_{n-1}|^2+p_2|\psi_n|^2, \tag{26}$$

$$\psi^*\sigma_z\psi=q(p_1|\psi_{n-1}|^2-p_2|\psi_n|^2), \tag{27}$$

$$\psi^*(\sigma_x+i\sigma_y)\psi=2q'(p_1p_2)^{\frac{1}{2}}\psi^*_{n-1}\psi_n, \tag{28}$$

where

$$p_1=(2e\hbar c\mathcal{H}n)\div 2F(F-cp), \tag{29a}$$

$$p_2=(F-cp)/(2F), \tag{29b}$$

$$q=q'=1. \tag{29c}$$

The matter current is obtained by multiplying the spin density by cp/F. The quantities p_1 and p_2 can be interpreted as the probabilities for finding the NR "orbital states" ψ_{n-1} and ψ_n, respectively, in an eigenstate of H and I. Equation (27) then shows that ψ_{n-1} and ψ_n are associated with spin projections $+1$ and -1, respectively, along the z axis. In the two eigenstates $\pm|F|$, both probabilities p_1 and p_2 are different, the transverse spin densities are equal but opposite in sign and the transverse currents are the same. The expectation values of all transverse currents are zero while those of the longitudinal components are p/F and c^2p/E for spin and current, respectively.

Simultaneous eigenfunctions of H and T can now be found from Eq. (25) by projection operators. If the eigenvalues of T are $T'=\epsilon(2e\hbar c\mathcal{H}n)^{\frac{1}{2}}$, where $\epsilon=\pm1$, then $(T+T')\psi_{E,F}$ is an eigenfunction of T for the eigenvalue T'. We get

$$\psi_{E,T'}=\begin{Bmatrix}(E+mc^2)\psi_{n-1}\\i\epsilon(E+mc^2)\psi_n\\i(T'-icp)\psi_{n-1}\\\epsilon(T'-icp)\psi_n\end{Bmatrix}[4E(E+mc^2)]^{-\frac{1}{2}}. \tag{30}$$

The matter and spin density, calculated from Eq. (30), are given by Eqs. (26)–(28) with

$$p_1=p_2=\tfrac{1}{2}, \tag{31a}$$

$$q=1, \tag{31b}$$

$$q'=i\epsilon mc^2/E. \tag{31c}$$

The longitudinal current is given by the right-hand side of Eq. (27) with $q=c^2p/E$ and the sign before the second term changed to $+$ while the transverse current is given by the right-hand side of Eq. (28) with $q'=c|T'|/E$. In the two eigenstates $\pm|T'|$, the current and longitudinal spin densities are the same while the transverse spin densities are equal but opposite in sign. The expectation values of all transverse components is zero while those of the longitudinal components are 0 and c^2p/E for spin and current, respectively.

Simultaneous eigenfunctions for H and L are obtained in a similar way from the projection operator $L+L'$, where $L'=\pm[c^2p^2+m^2c^4]^{\frac{1}{2}}$.

$$\psi_{E,L'}=\begin{Bmatrix}(E+L')(L'+mc^2)\psi_{n-1}\\cp(2e\hbar c\mathcal{H}n)^{\frac{1}{2}}\psi_n\\(E+L')cp\psi_{n-1}\\(L'+mc^2)(2e\hbar c\mathcal{H}n)^{\frac{1}{2}}\psi_n\end{Bmatrix}\times[4E(E+L')L'(L'+mc^2)]^{-\frac{1}{2}}. \tag{32}$$

The matter and spin densities, calculated from Eq. (32), are given by Eqs. (26), (27) and (28) with

$$p_1=(E+L')/(2E), \tag{33a}$$

$$p_2=(2e\hbar c\mathcal{H}n)\div 2E(E+L'), \tag{33b}$$

$$q=1, \tag{33c}$$

$$q'=cp/L'. \tag{33d}$$

The longitudinal current is given by the right-hand side of Eq. (27) with $q=c^2p/L'$ while the transverse current is given by the right-hand side of Eq. (28) with $q'=c$. In the two eigenstates $\pm|L'|$, the probabilities p_1 and p_2 are different, the transverse spin densities are equal but opposite in sign while the transverse currents are the same. The expectation values of all transverse components is zero while those of the longitudinal components are L/E and c^2p/E for spin and current, respectively.

It is interesting to see how the addition of a Pauli term, $\mu\mathbf{H}\cdot\boldsymbol{\sigma}$, to the Hamiltonian removes the spin degeneracy. Since σ_z commutes with L, $\psi_{E,L'}$ are correct zeroth order eigenfunctions. The first-order correction to the energy[2] is given by the expectation value of σ_z.

$$\Delta E=\mu\mathcal{H}L'/E=\pm\mu\mathcal{H}[1-(2ehc\mathcal{H}n)/E^2]^{\frac{1}{2}}. \quad (34)$$

Since, in most cases, $2ehc\mathcal{H}n\ll E^2$, the additional energies are those of a dipole of strength μ, parallel or antiparallel to $\mathbf{H}$. The degenerate states are thus split by the energy $2\mu\mathcal{H}$.

III. FREE PARTICLE

In the absence of a field the kinetic momentum can be identified with the usual momentum, $\mathbf{p}$, whose eigenvalues we will designate by p_x', p_y' and p_z'. It is obvious that I is still an integral and it is easily verified that T is the z component of a vector all of whose components are integrals. The seven integrals, I, T and iIT may be expressed in terms of three independent integrals as can most easily be seen by taking the z axis along $\mathbf{p}'$ ($p_x'=p_y'=0$). In that case T_z and iIT_z are identically zero and the five remaining integrals reduce to σ_z, $\rho_3\sigma_y$ and $\rho_3\sigma_x$. It is evident that the component of $\boldsymbol{\sigma}$ along $\mathbf{p}$ is a constant of the motion and that no other component of $\boldsymbol{\sigma}$ has this property. The eigenstates of I are therefore the only possible states with exactly defined values of a spin projection.[3] The eigenstates of $\rho_3\sigma_y$ or $\rho_3\sigma_x$ are eigenstates of σ_y or σ_x only in the limit of small velocities ($\rho_3\to 1$).

From Eq. (21) the eigenvalues of I are $F=\pm cp' =\pm c[p_x'^2+p_y'^2+p_z'^2]^{\frac{1}{2}}$. Simultaneous eigenfunctions of H and I can again be found from Eqs. (9) and (11). Setting $u_1=a''\exp(i\mathbf{p}'\cdot\mathbf{r}/h)$ and $u_2=b''\exp(i\mathbf{p}'\cdot\mathbf{r}/h)$, Eq. (9) becomes algebraic and its solutions substituted into Eq. (11) yield

$$\psi_{E,F}=\exp[i\mathbf{p}'\cdot\mathbf{r}/h]\begin{Bmatrix}(E+mc^2)(F+cp_z')\\(E+mc^2)c(p_x'+ip_y')\\F(F+cp_z')\\Fc(p_x'+ip_y')\end{Bmatrix}\times[4E(E+mc^2)F(F+cp_z')]^{-\frac{1}{2}}. \quad (35)$$

[2] The exact eigenvalues of the Hamiltonian $H=c\rho_1\boldsymbol{\pi}\cdot\boldsymbol{\sigma}+\rho_3mc^2+\mu\mathbf{H}\cdot\boldsymbol{\sigma}$ are $E'=\pm[E^2+\mu^2\mathcal{H}^2\pm2\mu\mathcal{H}E(1-2ehc\mathcal{H}n/E^2)^{\frac{1}{2}}]^{\frac{1}{2}}$, where E is given by Eqs. (7) and (21).

[3] This has been discussed in detail from a different standpoint in A. Sommerfeld's, ***Wellenmechanik*** (Frederick Ungar Publishing Company), p. 330 ff.

The matter and spin densities, obtained by substituting the solutions of Eq. (9) into Eqs. (12)–(14), are

$$\psi^*\psi=1, \quad (36)$$

$$\psi^*\sigma_z\psi=cp_z'/F, \quad (37)$$

$$\psi^*(\sigma_x+i\sigma_y)\psi=c(p_x'+ip_y')/F. \quad (38)$$

Thus the spin densities are just the projections of a unit vector, parallel to $\mathbf{p}'$ for F positive and antiparallel to $\mathbf{p}'$ for F negative, on the coordinate axes. The matter current, obtained by multiplying the spin density by cF/E, is identical with the classical velocity vector, c^2p'/E for both eigenstates $\pm|F'|$.

Simultaneous eigenfunctions of H and other integrals of motion associated with spin degeneracy can now be obtained by projection operators. For example the eigenfunctions of L are

$$\psi_{E,L'}=\exp(i\mathbf{p}'\cdot\mathbf{r}/h)\begin{Bmatrix}(E+L')(L'+mc^2)\\cp_z'c(p_x'+ip_y')\\(E+L')cp_z'\\(L'+mc^2)c(p_x'+ip_y')\end{Bmatrix}\times[4E(E+L')L'(L'+mc^2)]^{-\frac{1}{2}}. \quad (39)$$

The matter and spin density, calculated from Eq. (39) are

$$\psi^*\psi=1, \quad (40)$$

$$\psi^*\sigma_z\psi=L'/E, \quad (41)$$

$$\psi^*(\sigma_x+i\sigma_y)\psi=cp_z'c(p_x'+ip_y')/(EL'). \quad (42)$$

The matter current is again identical with the classical velocity vector. In the two eigenstates $\pm|L'|$, the currents are the same and the spin densities are equal but opposite in sign.

Free particle solutions can also be obtained from solutions in a homogeneous field by passing to the limit $\mathcal{H}\to 0$. The ψ_n of Eq. (37-A) contain λ (Eq. (19-A)) as a scale factor for the transverse coordinates and λ is proportional to $\mathcal{H}^{-\frac{1}{2}}$. If n is held fixed, ψ_n approaches the value of ψ_n at $x=y=0$ for any finite value of x and y as $\mathcal{H}\to 0$. Thus the space parts of the wave functions approach plane waves along the z axis as should be expected for, in this limit, the energy of transverse motion is zero. The spinor parts of the eigenfunctions go over into the corresponding spinors for a free particle. Thus $\psi_{E,F}$ goes over into the solution of Eq. (35) and $\psi_{E,L'}$ into the solution of Eq. (39) with $p_x'=p_y'=0$ in both cases.