Field Dynamics. I. Classical

K. V. Roberts
King's College, Cambridge, and H. H. Wills Physical Laboratory, University of Bristol, England
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THE interaction representation has been derived[1] from the non-covariant Heisenberg-Pauli treatment via the Heisenberg representation, but it is evident from the lengthy deduction[2] and justification[3] required in meson theory that this method is unsatisfactory. In this note and the following one,[4] we derive the interaction Hamiltonian by a classical method and indicate how the S-matrix is to be constructed.

The treatment of both Bose and Fermi fields depends on classical field dynamics because the starting point is a Lagrange principle. Heisenberg and Pauli reduced the problem to one involving an infinite number of degrees of freedom, and one independent variable t, but used a particular Lorentz frame. In a relativistic treatment it still seems reasonable to adopt a classical approach and the mode of attack has been shown by Weiss.[5] We extend it and show the connection with the interaction representation.[6] It might be thought that this approach is unnecessarily highbrow for practical purposes. This is not so: it leads to shorter calculations, as shown by Matthews.[7]

The generalization from particle to field dynamics is based upon two analogies: (i) between equations in l and n independent variables, (ii) between hyperbolic equations in $m+1$ variables, all m. Essentially, the Lagrangian part of the theory uses the first analogy; the Hamiltonian part the second. Quantities transform according to two groups, the Lorentz group and the group of transformations of an arbitrary space-like surface σ.

σ is a *parametrized* space-like surface, u^r the three parameters. N_μ are the three-rowed minors of $(\mathfrak{d}u^r/\mathfrak{d}x^\mu)$. w is a locally normal coordinate satisfying $(\mathfrak{d}x^\mu/\mathfrak{d}w)N_\mu=1$. Field quantities are denoted collectively by $\varphi^\alpha(x)$, α standing for an unspecified tensor or spinor index. $\varphi_\mu{}^\alpha=\mathfrak{d}\varphi^\alpha/\varphi x^\mu$. $\mathfrak{L}$ is the Lagrangian.

In this theory, canonical coordinates and momenta are variables $\mathbf{q}$, $\mathbf{p}$ whose values are *functions* on any arbitrary σ with domain the u^r or any transform of them. The fundamental coordinate and momentum functions are $\varphi^\alpha(u)$ and $\pi_\alpha(u)=(\partial\mathfrak{L}/\partial\varphi_\nu{}^\alpha)N_\nu$. $U_\nu{}^\mu=\mathfrak{L}-\varphi_\nu{}^\alpha\pi_\alpha{}^\mu$. The Hamiltonian functions are

$$\mathcal{H}(u)=U_\nu{}^\mu N_\mu(\mathfrak{d}x^\nu/\mathfrak{d}w)\quad\text{and}\quad\mathfrak{h}_r(u)=U_\nu{}^\mu N_\mu(\mathfrak{d}x^\nu/\mathfrak{d}u^r)$$

expressed in terms of φ^α, $(\mathfrak{d}\varphi^\alpha/\mathfrak{d}u^r)$, and π_α. $H=\int_\sigma\mathcal{H}(u)du$ and $h_r=\int_\sigma h_r(u)du$ are the Hamiltonians; the canonical equations, determining the variation with w and u^r for fixed σ, are

$$\begin{aligned}\frac{\mathfrak{d}}{\mathfrak{d}w}\varphi^\alpha(u)&=-\frac{\delta H}{\delta\pi_\alpha(u)},\quad\frac{\mathfrak{d}}{\mathfrak{d}w}\pi_\alpha(u)=\frac{\delta H}{\delta\varphi^\alpha(u)},\\\frac{\mathfrak{d}}{\mathfrak{d}u^r}\varphi^\alpha(u)&=-\frac{\delta h_r}{\delta\pi_\alpha(u)},\quad\frac{\mathfrak{d}}{\mathfrak{d}u^r}\pi_\alpha(u)=\frac{\delta h_r}{\delta\varphi^\alpha(u)}.\end{aligned}\tag{1}$$

They are equivalent to the field equations. δ denotes functional differentiation. The energy-momentum vector is

$$P_\mu=\int_\sigma U_\nu{}^\mu(u)N_\mu(u)du.$$

Poisson brackets (P.B.) may be defined[5] and shown to be constants of the motion; (Eq. (1) may be restated in terms of them. A canonical transformation (c.t.) is any transformation from $\mathbf{p}_\beta$, $\mathbf{q}^\beta$ to $\mathbf{p}_\gamma{}^*$, $\mathbf{q}^{*\gamma}$ such that

$$\int_\sigma p_\gamma{}^*(w)\delta q^{*\gamma}(w)dw-\int_\sigma p^\beta(v)\delta q^\beta(v)dv$$

is the perfect differential of a functional W of $\mathbf{p}$, $\mathbf{q}$, and possibly σ. *Any functional of σ must satisfy the integrability conditions*, given by Dirac[8]

$$\begin{aligned}[W_r(u),W_s(u')]-\frac{\delta W_r(u)}{\delta u^s(u')}+\frac{\delta W_s(u')}{\delta u^r(u)}&=-W_r(u')\frac{\partial}{\partial u^s}\mathfrak{d}(u-u')\\&\quad-W_s(u)\frac{\partial}{\partial u^r}\mathfrak{d}(u-u'),\end{aligned}$$

$$[W_r(u),W_n(u')]-\frac{\delta W_r(u)}{\delta w(u')}+\frac{\delta W_n(u')}{\delta u^r(u)}=W_n(u)\frac{\partial}{\partial u^r}\delta(u-u'),\tag{2}$$

$$\begin{aligned}[W_n(u),W_n(u')]-\frac{\delta W_n(u)}{\delta w(u')}+\frac{\delta W_n(u')}{\delta w(u)}&=-\frac{\partial}{\partial u^s}\delta(u-u')\\&\quad\times\{\gamma_{rs}(u)W_r(u)+\gamma_{rs}(u')W_r(u')\},\end{aligned}$$

where $W_n(u)=\delta W/\delta w(u)$, $W_r(u)=\delta W/\delta u^r(u)$, and γ_{rs} is the metric tensor of the surface.

Any c.t. preserves the P.B. relations. Infinitesimal c.t.'s are generated by a functional $K[\mathbf{p},\mathbf{q},\sigma]$ which can depend on σ;

$$dq/d\theta=-\delta K/\delta p,\quad dp/d\theta=\delta K/\delta q.\tag{3}$$

$\int_\sigma\{\mathcal{H}(u)\delta w(u)+\mathfrak{h}_r(u)\delta u^r(u)\}du$ generates the c.t. induced by an arbitrary infinitesimal deformation of σ. H and h_r correspond to displacements along w, u^r, respectively, P_μ to displacements along the x^μ axis.

Any c.t. may be specified by $W[\mathbf{q},\mathbf{q}^*,\sigma]$; $p=-\delta W/\delta q$, $p^*=\delta W/\delta q^*$, $\mathcal{H}^*=\mathcal{H}+\delta W/\delta w$, $\mathfrak{h}_r{}^*=\mathfrak{h}_r+\delta W/\delta u^r$, and preserves the canonical form of the equations. The Hamilton-Jacobi equations define a W which transforms the original $\mathbf{p}$, $\mathbf{q}$ to constants;

$$\mathcal{H}+(\delta W/\delta w)=0,\quad\mathfrak{h}_r+(\delta W/\delta u^r)=0.\tag{4}$$

This appearance of the Hamiltonian functions as derivatives shows that they must satisfy the conditions (2). Their derivation from a Lagrange principle automatically ensures this.

[1] J. Schwinger, Phys. Rev. **74**, 1439 (1948). S. Tomonaga, Prog. Theor. Phys. **1**, 27 (1946).
[2] See, e.g., S. Kanesawa and S. Tomonaga, Prog. Theor. Phys. **3**, 1 (1948).
[3] F. J. Belinfante, Phys. Rev. **76**, 66 (1949).
[4] K. V. Roberts, Phys. Rev. **77**, 146 (1950).
[5] P. R. Weiss, Proc. Roy. Soc. **169**, 102 (1938).
[6] Also discussed by T. S. Chang, Phys. Rev. **75**, 967 (1949).
[7] P. T. Matthews, Phys. Rev. **75**, 1270 (1949).
[8] P. A. M. Dirac, Phys. Rev. **73**, 1092 (1948).

Field Dynamics. II. Quantum

K. V. Roberts
King's College, Cambridge, and H. H. Wills Physical Laboratory, University of Bristol, England
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THE state of a system is defined[1] on a *parametrized* σ by a state functional or vector Ψ; dynamical variables by operators defined with respect to σ. Here we employ canonical variables $\varphi^\alpha(u)$, $\pi_\alpha(u)$, corresponding to those of I^2, and functions or functionals of them.

The Heisenberg equations (equivalent to the field equations) are

$$\frac{\mathfrak{d}}{\mathfrak{d}w}\xi(u,\sigma)=[\xi,H],\quad\frac{\mathfrak{d}}{\mathfrak{d}u^r}\xi(u,\sigma)=[\xi,h_r].\tag{1}$$

A transformation of the surface defines a unitary transformation (u.t.);

$$\xi(u,\sigma)=U^{-1}[\sigma,\sigma_0]\xi(u,\sigma_0)U[\sigma,\sigma_0]\tag{2}$$

from the variables on σ_0 to those on ∂, with

$$\begin{aligned}i\hbar\frac{\delta}{\delta w(u)}U[\sigma,\sigma_0]&=\mathcal{H}(u,\sigma)U[\sigma,\sigma_0],\\i\hbar\frac{\delta}{\delta u^r(u)}U[\sigma,\sigma_0]&=\mathfrak{h}_r(u,\sigma)U[\sigma,\sigma_0].\end{aligned}\tag{3}$$

The Schrödinger equations are

$$\begin{aligned}i\hbar\frac{\delta}{\delta w(u)}\Psi[\sigma]&=\mathcal{H}(u)\Psi[\sigma],\\i\hbar\frac{\delta}{\delta u^r(u)}\Psi[\sigma]&=\mathfrak{h}_r(u)\Psi[\sigma].\end{aligned}\tag{4}$$