The foregoing method is not limited to head-on collisions only. For collisions with higher angular momenta, we assume the wave function to be of the form:

$$\psi_l(r)=f(r)(\pi kr/2)^{\frac{1}{2}}J_{l+\frac{1}{2}}(kr)+g(r)(\pi kr/2)^{\frac{1}{2}}J_{-l-\frac{1}{2}}(kr), \quad (8)$$

where $J_{\pm(l+\frac{1}{2})}(kr)$ denotes the Bessel function of order $\pm(l+\frac{1}{2})$.

Substituting Eq. (8) into the wave equation, we can derive after suitable transformations a variational integral of the type given by Eq. (5). Thus for $l=1$, we find

$$F\equiv[1+(1/k^2r^2)](f_1^2+g_1^2)+2k(gf_1-fg_1)+[1+(1/k^2r^2)]V(f^2+g^2). \quad (9)$$

From this we infer that the functions $f(r)$ and $g(r)$ for the case $l=1$ must be of order at least r^2 for $r\rightarrow 0$.

The foregoing method can also be generalized for the S scattering of electrons by a neutral hydrogen atom. We now assume

$$\psi=(e^{-r_1}/r_2)[f(r_1,r_2,r_{12})\sin kr_2+g(r_1,r_2,r_{12})\cos kr_2]. \quad (10)$$

Substituting this form for ψ in the appropriate wave equation in the variables r_1, r_2 and r_{12}, we find that the resulting pair of differential equations obtained by putting the coefficients of $\sin kr_2$ and $\cos kr_2$ equal to zero, are the Euler equations of the variational problem

$$\delta\int_0^\infty dr_1\int_0^\infty dr_2\int_{|r_1-r_2|}^{r_1+r_2} dr_{12}\cdot F(f_1,f_2,f_{12},g_1,g_2,g_{12},f,g;r_1,r_2,r_{12})=0, \quad (11)$$

where

$$\begin{aligned}F\equiv(e^{-2r_1}/r_2^2)[&r_1r_2r_{12}(f_1^2+f_2^2+2f_{12}^2+g_1^2+g_2^2+2g_{12}^2)\\&+r_2(r_1^2-r_2^2+r_{12}^2)(g_1g_{12}+f_1f_{12})+r_1(r_2^2-r_1^2+r_{12}^2)\\&\times(f_2f_{12}+g_2g_{12})+2kr_1r_2r_{12}(g_2f-f_2g)\\&+kr_1(r_2^2-r_1^2+r_{12}^2)(g_{12}f-gf_{12})-2r_1(r_{12}-r_2)(f^2+g^2)], \quad (12)\end{aligned}$$

and f_1, f_2 and f_{12} are derivatives of f with respect to r_1, r_2 and r_{12}, respectively.

The advantage of the variational principle in the form derived here is its simplicity for practical applications. Examples of the application of this formulation will be given in a later paper.

Finally I should like to express my sincere thanks to Professor S. Chandrasekhar for valuable discussions.

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Finite Relativistic Charge-Current Distributions

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HITHERTO, it has been supposed that Lorentz's theory of the non-point electron could not be made covariant without either: (a) making electrodynamics non-linear, or (b) taking into account the effects of non-electromagnetic forces which are assumed to hold the charge together. We wish to report here preliminary results concerning a method of making Lorentz's theory covariant, without resorting to either of these alternatives.

The principal difficulty is that one must describe the change of shape of the electron as it accelerates and simultaneously preserve conservation of charge-current without which Maxwell's equations become inconsistent. Let the world line of the center of the electron be parametrically described by the 4 vector $\xi_\mu(s)$, and let the shape be described by another 4 vector, $v_\mu(s)$, which may be, but is not necessarily the same as $d\xi_\mu/ds$. (A criterion for correct equations of motion is that the two should tend to be the same for steady motion.) The following charge-current distribution is found to be conserved identically.

$$j^\mu(x)=\int dsF(r_\alpha r^\alpha)g\left(\frac{r_\alpha v^\alpha}{(v_\sigma v^\sigma)^{\frac{1}{2}}}\right)\left\{\frac{d\xi^\mu}{ds}-r_\gamma\frac{v^\gamma\dot{v}^\mu-v^\mu\dot{v}^\gamma}{v_\alpha v^\alpha}\right\}, \quad (1)$$

where $r_\alpha=x_\alpha-\xi_\alpha$, F and g are arbitrary functions and the integral is over the whole world line. This expression contrasts in two ways with the ordinary expression[1]

$$j_\mu'(x)=\int ds\frac{d\xi_\mu}{ds}\Pi_\alpha\,\delta(x_\alpha-\xi_\alpha). \quad (2)$$

First, the δ-function kernel is replaced by $F(r_\alpha r^\alpha)g(r_\alpha v^\alpha/(v_\sigma v^\sigma)^{\frac{1}{2}})$ so that the charge distribution is spread out. Secondly, there is added to the current a spin-like term involving the anti-symmetric tensor, $\dot{v}^\mu v^\nu-\dot{v}^\nu v^\mu\equiv S^{\mu\nu}$. This latter turns out to be essential for conservation of charge. The present theory differs from that of MacManus and Peierls[2] in that now the charge associated with a given center can be localized in time as well as in space. The theory of MacManus and Peierls results in convergence factors which are functions of ω^2-k^2, where ω is the angular frequency of the electromagnetic wave and k is its wave number. Thus, for a light wave, it leads essentially to a point electron. It is able to yield finite self-energies in classical theory, but the effects of quantum field fluctuations would not in general be affected. The localization of the charge in time, as well as space, would remove all infinities.

To obtain the equations of motion, one adds to the usual free particle Lagrangian the term

$$\int j_\mu(x)A^\mu(x)d^4x, \quad (3)$$

where j_μ is evaluated from (1). Variation of this Lagrangian leads to covariant and finite classical equations of motion. We have not investigated these equations in detail nor have we succeeded as yet in quantizing them. We believe, however, that quantization will inolve methods closely related to those proposed by Yukawa.[3]

The "shape variables," v^μ, can be described in an alternative way which suggests a close relation to Dirac's equation. In three dimensions, a rigid body is described by three rotations, which may, for example, be specified by the three Euler parameters. In four dimensions, one needs three more complex rotations, representing the Lorentz transformations needed to transform the electron to the shape it has when at rest. These six rotations are conveniently represented in terms of the eight quantities corresponding to real and imaginary parts of a 4 component Dirac spinor, among which there are two covariant relations, leaving six independent quantities. Representing such a classical spinor by $\psi(s)$ one then obtains

$$v^\mu=\psi^+\gamma^\mu\psi \quad (4)$$

with $\psi^+\psi=\pm 1$, $\psi^+\gamma^5\psi=0$. The ψ's are now the basic coordinates replacing the v's. We are now investigating the possibility that the Dirac equation wave equation is an approximation to the lowest quantum state of this system. If this were so, then the "Zitterbewegung" could be related to quantum fluctuations in the shape of the electron.

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Note on the Variational Method for Finding Extrapolated Endpoints*

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A VARIATIONAL method which can be used to treat integral equations of the form

$$f(x)=\int_0^\infty f(x')G(|x-x'|)dx'\int_{-\infty}^{+\infty}G(|x|)dx=1 \quad (1)$$

has recently been described by R. E. Marshak.[1]