Finally, we mention that formula (17) may be simplified by always calculating with integrals from $-\delta'$ to δ' instead of dividing B into B^+ and B^-. Then only the first two terms of (17) with a factor 2 remain, in which now

$$\chi=\pi+\arctan[2(\delta/b)/(1-(\delta/b)^2+l_{\sigma\tau}^2)] \quad \text{and} \quad C=1/\pi;$$

the indices + or − are of course superfluous now.

Whether there is a considerable numerical difference between these two approximations, has not been investigated. We ourselves (reference 5) utilized Eq. (17), which, in principle, appears to be the better approximation: still further division of the integrals would give still better solutions which, however, could not be handled well.

IV. SUMMARY

The computation of "total absorption," as defined by Ladenburg and Reiche, is extended to the case of overlapping lines which all are equally broad, but which have different distances and different, not too large, intensities. The method is elaborated in detail for the overlapping of two lines in Section II; the general case is dealt with briefly in Section III. Final results: Eqs. (10) and (13) for two lines, Eq. (17) for n lines; restricting assumptions, see Section IIC. The total absorption is proportional to the square root of the path-length also with overlapping lines, as long as this "root-law" is valid for a single line. But there might be deviations from the $p^{\frac{1}{2}}$-law under certain circumstances explained at the end of Section II.

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The Equations of Codazzi and the Relations between Electromagnetism and Gravitation

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The aim of this paper is to show that, quite independently of any physical theory, the general equations of Codazzi on differential geometry lead to fundamental relations between the electromagnetic and the gravitational fields as soon as the external metric tensor of space-time is interpreted as an electromagnetic tensor. When the important special case of quasi static fields is considered, we get for a *rotating body with no permanent magnetization:* (1) The relation, previously studied by the author, between magnetic moment and angular momentum which explains the general features of stellar and terrestrial magnetism as well as the magnetic moment of the neutron; (2) a relation between gravitation and the electrostatic field, such that any massive body creates an electrostatic field by its own gravitation. *The mean electrostatic fields of celestial bodies, including the earth, can be ascribed to this effect.* When the gravitation produced by a given body is negligible (as in the laboratory) the equations of Codazzi show that the familiar Coulomb field is merely a consequence of the very rapid vibrations of the components g_{4i} ($i=1, 2, 3$) of the internal metric tensor. Finally, for an uncharged body with permanent magnetization it can be shown that the curl of the g_{4i} and the magnetic field are related as cause and effect.

We think that these results are a confirmation of a fundamental result of our unified field theory: *That the geometrization of electromagnetism must necessarily be achieved by the external metric of space-time.*

I. EQUATIONS OF GAUSS AND CODAZZI

ANY space (l_N) with an arbitrary number (N) of dimensions can be considered as a hypersurface of an enveloping space l_{N+1} of $N+1$ dimensions. The space l_N has then an external as well as an internal metric (to which the second and the first fundamental quadratic forms correspond respectively). These two metrics are well defined by the equation of l_N in l_{N+1} and by the internal metric which can be imposed on l_{N+1}. When l_N is a differentiable variety, the internal (g_{ik}) and the external (ω_{ik}) metric tensors (both symmetric) must necessarily satisfy the fundamental compatibility equations of Gauss and Codazzi.[1,2] Denoting by

R_{ijkl} the Riemann-Christoffel tensor of l_N;
$\bar{R}_{\alpha\beta\gamma\delta}$ the Riemann-Christoffel tensor of l_{N+1} *taken on* l_N;
n^α the contravariant components of the unit normal to l_N;
x^i general coordinates in l_N ($i=1, 2, \cdots, N$);
X^μ general coordinates in l_{N+1} ($\mu=1, 2, \cdots, N+1$),

and putting

$$X_{,i}{}^\mu \equiv \partial X^\mu/\partial x^i,$$

the equations of Gauss and Codazzi, for an l_N embedded in a l_{N+1} take the form

$$R_{ijkl}-(\omega_{ik}\omega_{jl}-\omega_{il}\omega_{jk})=\bar{R}_{\alpha\beta\gamma\delta}X_{,i}{}^\alpha X_{,j}{}^\beta X_{,k}{}^\gamma X_{,l}{}^\delta, \quad (1)$$

$$\omega_{ik,j}-\omega_{ij,k}=\bar{R}_{\alpha\beta\gamma\delta}n^\alpha X_{,i}{}^\beta X_{,k}{}^\gamma X_{,j}{}^\delta. \quad (2)$$

(The comma denotes a covariant differentiation.) When

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[1] A. Gião, Comptes Rendus **224**, 1813 (1947); **225**, 924 (1947); **226**, 645, 1298, 2126 (1948). *Gazeta de Mat.* (Lisbon), **34** (1947) and **35** (1948).

[2] A. Gião, Portugaliae Physica **2**, 1–98 (1946). Portugaliae Mathematica **5**, 145–192 (1946); **6**, 67–114 (1947); **7**, 1–43 (1948). Bull. Soc. Port. Math. (A), 29–40 (1947).

l_N is a space of class 1, that is to say a space which can be embedded in an Euclidean or pseudo-Euclidean l_{N+1}, the right members of Eqs. (1) and (2) vanish identically.

Here we are specially interested in Eq. (2) of Codazzi. Since

$$\omega_{ij,k}=\frac{\partial\omega_{ij}}{\partial x^k}-\omega_{jl}\begin{Bmatrix} l \\ ik \end{Bmatrix}-\omega_{il}\begin{Bmatrix} l \\ jk \end{Bmatrix},$$

the $\begin{Bmatrix} l \\ ik \end{Bmatrix}$ being the Christoffel symbols of second kind formed with the g_{ik}, the equations of Codazzi can be written as follows

$$\frac{\partial\omega_{ik}}{\partial x^j}-\frac{\partial\omega_{ij}}{\partial x^k}-\left(\omega_{kl}\begin{Bmatrix} l \\ ij \end{Bmatrix}-\omega_{jl}\begin{Bmatrix} l \\ ik \end{Bmatrix}\right)=\bar{R}_{\alpha\beta\gamma\delta}n^\alpha X_{,i}{}^\beta X_{,k}{}^\gamma X_{,j}{}^\delta.$$

Putting

$$2A_k{}^i\equiv g^{li}\omega_{lk}, \tag{3}$$

we can also write

$$\frac{\partial\omega_{ik}}{\partial x^j}-\frac{\partial\omega_{ij}}{\partial x^k}-\left[A_k{}^h\left(\frac{\partial g_{ih}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^h}+\frac{\partial g_{jh}}{\partial x^i}\right)-A_j{}^h\left(\frac{\partial g_{ih}}{\partial x^k}-\frac{\partial g_{ik}}{\partial x^h}+\frac{\partial g_{kh}}{\partial x^i}\right)\right]=\bar{R}_{\alpha\beta\gamma\delta}n^\alpha X_{,i}{}^\beta X_{,k}{}^\gamma X_{,j}{}^\delta,$$

which reduces to

$$\frac{\partial\omega_{ik}}{\partial x^j}-\frac{\partial\omega_{ij}}{\partial x^k}=A_k{}^h\left(\frac{\partial g_{ih}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^h}+\frac{\partial g_{jh}}{\partial x^i}\right)-A_j{}^h\left(\frac{\partial g_{ih}}{\partial x^k}-\frac{\partial g_{ik}}{\partial x^h}+\frac{\partial g_{kh}}{\partial x^i}\right), \tag{4}$$

for a space l_N of class 1.

2. FOUR-DIMENSIONAL SPACE OF CLASS 1

Let us consider a four-dimensional space of class 1 and calculate by the corresponding Codazzi equations (Eq. (4)) the quantities:

$$\frac{\partial\omega_{4l}}{\partial x^j}-\frac{\partial\omega_{4j}}{\partial x^l}\quad (j,l=1,2,3)$$

and

$$\frac{\partial\omega_{44}}{\partial x^i}-\frac{\partial\omega_{4i}}{\partial x^4}\quad (i=1,2,3).$$

i, j, l denoting a cyclic permutation of 1, 2, 3, we obtain on the one hand

$$\begin{aligned}\frac{\partial\omega_{4l}}{\partial x^j}-\frac{\partial\omega_{4j}}{\partial x^l}=&(A_j{}^j+A_l{}^l)\left(\frac{\partial g_{4l}}{\partial x^j}-\frac{\partial g_{4j}}{\partial x^l}\right)\\&+A_l{}^i\left(\frac{\partial g_{4i}}{\partial x^j}-\frac{\partial g_{4j}}{\partial x^i}\right)+A_j{}^i\left(\frac{\partial g_{4l}}{\partial x^i}-\frac{\partial g_{4i}}{\partial x^l}\right)\\&+\left(A_l{}^4\frac{\partial g_{44}}{\partial x^j}-A_j{}^4\frac{\partial g_{44}}{\partial x^l}\right)+(A_l{}^l-A_j{}^j)\frac{\partial g_{jl}}{\partial x^4}\\&+\left(A_l{}^i\frac{\partial g_{ij}}{\partial x^4}-A_j{}^i\frac{\partial g_{il}}{\partial x^4}\right)+\left(A_l{}^j\frac{\partial g_{jj}}{\partial x^4}-A_j{}^l\frac{\partial g_{ll}}{\partial x^4}\right),\end{aligned}$$

and on the other hand

$$\begin{aligned}\frac{\partial\omega_{44}}{\partial x^i}-\frac{\partial\omega_{4i}}{\partial x^4}=&A_4{}^j\left(\frac{\partial g_{4j}}{\partial x^i}-\frac{\partial g_{4i}}{\partial x^j}\right)+A_4{}^l\left(\frac{\partial g_{4l}}{\partial x^i}-\frac{\partial g_{4i}}{\partial x^l}\right)\\&+A_4{}^h\frac{\partial g_{ih}}{\partial x^4}+A_i{}^h\left(\frac{\partial g_{44}}{\partial x^h}-2\frac{\partial g_{4h}}{\partial x^4}\right)+A_4{}^4\left(\frac{\partial g_{44}}{\partial x^i}-\frac{\partial g_{4i}}{\partial x^4}\right).\end{aligned}$$

In the very important special case of *quasi static* metric fields, that is to say when all the derivatives of g_{ik} and ω_{ik} with respect to the "temporal" variable x^4 vanish, excepting the temporal derivatives of g_{4i} and ω_{4i} ($i=1, 2, 3$), the preceding relations become:

$$\begin{aligned}\frac{\partial\omega_{4l}}{\partial x^j}-\frac{\partial\omega_{4j}}{\partial x^l}=&(A_j{}^j+A_l{}^l)\left(\frac{\partial g_{4l}}{\partial x^j}-\frac{\partial g_{4j}}{\partial x^l}\right)\\&+A_l{}^i\left(\frac{\partial g_{4i}}{\partial x^j}-\frac{\partial g_{4j}}{\partial x^i}\right)+A_j{}^i\left(\frac{\partial g_{4l}}{\partial x^i}-\frac{\partial g_{4i}}{\partial x^l}\right)\\&+\left(A_l{}^4\frac{\partial g_{44}}{\partial x^j}-A_j{}^4\frac{\partial g_{44}}{\partial x^l}\right),\end{aligned} \tag{5}$$

and

$$\begin{aligned}\frac{\partial\omega_{44}}{\partial x^i}-\frac{\partial\omega_{4i}}{\partial x^4}=&A_4{}^j\left(\frac{\partial g_{4j}}{\partial x^i}-\frac{\partial g_{4i}}{\partial x^j}\right)+A_4{}^l\left(\frac{\partial g_{4l}}{\partial x^i}-\frac{\partial g_{4i}}{\partial x^l}\right)\\&+A_4{}^4\frac{\partial g_{44}}{\partial x^i}+\sum_1^3{}_h A_i{}^h\left(\frac{\partial g_{44}}{\partial x^h}-2\frac{\partial g_{4h}}{\partial x^4}\right).\end{aligned} \tag{6}$$

The mean curvature χ of the four-dimensional space is given by $4\chi\equiv g^{ik}\omega_{ik}$ and putting $\omega_{ik}\equiv\chi g_{ik}+\tilde{\omega}_{ik}$ we get from (3):

$$2A_k{}^i=\chi\delta_k{}^i+g^{li}\tilde{\omega}_{lk}. \tag{3a}$$

For a space such that the $\tilde{\omega}_{ik}$ $(i\neq k)$ are small quantities against χ (all spaces with slight variable mean curvature obey this condition), the components of the tensor $A_k{}^i$ (for $i\neq k$) are, according to (3a), small quantities against $A_i{}^i$. Equations (5) and (6) can then be simplified as follows:

$$\frac{\partial \omega_{4l}}{\partial x^j}-\frac{\partial \omega_{4j}}{\partial x^l}=(A_j{}^j+A_l{}^l)\left(\frac{\partial g_{4l}}{\partial x^j}-\frac{\partial g_{4j}}{\partial x^l}\right), \tag{7}$$

and

$$\frac{\partial \omega_{44}}{\partial x^i}-\frac{\partial \omega_{4i}}{\partial x^4}=(A_4{}^4+A_i{}^i)\frac{\partial g_{44}}{\partial x^i}-2A_i{}^i\frac{\partial g_{4i}}{\partial x^4}. \tag{8}$$

3. THE EXTERNAL METRIC TENSOR AND THE ELECTROMAGNETIC FIELD

The preceding sections are quite independent of any physical theory. In order to give them an essential physical meaning, one needs only to interpret physically the external metric tensor ω_{ik}, since it is well known that g_{ik} is the gravitational tensor. According to our unified field theory,[2] space-time is a four-dimensional hypersurface of class 1 and ω_{ik} is the tensor of the electromagnetic (electromotive) field. Moreover,[2] between the electric field intensity (E_i), the magnetic field intensity (H_i; $i=1, 2, 3$) and the components of the external metric tensor the following fundamental relations hold:

$$H_i=\frac{1}{\chi_0(-1)^{\frac{1}{2}}}\frac{(m_0)_ec^2}{e}\left(\frac{\partial \omega_{4l}}{\partial x^j}-\frac{\partial \omega_{4j}}{\partial x^l}+\frac{\partial \omega_{jl}}{\partial x^4}\right), \tag{9}$$

and

$$E_i=-\frac{1}{2\chi_0}\frac{(m_0)_ec^2}{e}\left(\frac{\partial \omega_{44}}{\partial x_i}-2\frac{\partial \omega_{4i}}{\partial x^4}\right), \tag{10}$$

where χ_0 is a positive constant (mean curvature of the isotropic and homogeneous space-time nearest to the real space-time), $(m_0)_e$ and e the rest mass and electric charge of the electron. It must be remarked that the field (9), (10) behaves as a tensor F_{jl} only for pure spatial coordinate changes, and this tensor is antisymmetric (as the tensor of the Maxwellian field) only when the metric is static.

For a quasi static field the relations (9) and (10) reduce to:

$$\left.\begin{aligned} H_i&=\frac{1}{\chi_0(-1)^{\frac{1}{2}}}\frac{(m_0)_ec^2}{e}\left(\frac{\partial \omega_{4l}}{\partial x^j}-\frac{\partial \omega_{4j}}{\partial x^l}\right),\\ E_i&=-\frac{1}{2\chi_0}\frac{(m_0)_ec^2}{e}\left(\frac{\partial \omega_{44}}{\partial x^i}-2\frac{\partial \omega_{4i}}{\partial x^4}\right),\end{aligned}\right\} \tag{11}$$

and Eqs. (7) and (8) give then:

$$H_i=\frac{1}{\chi_0(-1)^{\frac{1}{2}}}\frac{(m_0)_ec^2}{e}(A_j{}^j+A_l{}^l)\left(\frac{\partial g_{4l}}{\partial x^j}-\frac{\partial g_{4j}}{\partial x^l}\right), \tag{12a}$$

$$E_i=-\frac{1}{2\chi_0}\frac{(m_0)_ec^2}{e}\left[(A_4{}^4+A_i{}^i)\frac{\partial g_{44}}{\partial x^i}-2A_i{}^i\frac{\partial g_{4i}}{\partial x^4}-\frac{\partial \omega_{4i}}{\partial x^4}\right]. \tag{12b}$$

These equations will now be applied to the analysis of the relations between the electromagnetic and the gravitational fields. For this purpose, special cases will be considered which give a clear understanding of the main characteristics of these relations.

4. SPHERICAL ROTATING BODY WITH NO PERMANENT MAGNETIZATION

Let us consider a rotating sphere of mass M and constant angular momentum M_{rot}. There is no permanent magnetization and the angular velocity $\mathbf{\Omega}$ of each particle is parallel to the axis of rotation, but $|\mathbf{\Omega}|$ as well the density μ are arbitrarily distributed in the mass. Denoting by γ a numerical coefficient (≤ 1), by K the gravitational constant, by r the spatial distance from the center ($x^1=x^2=x^3=0$) of the sphere and by u^i the components of the unit vector of the axis of rotation, the well known Einstein equations for the internal metric field give[1] the following expression for the statical part of the g_{4i} ($i=1, 2, 3$) of the rotating sphere:

$$g_{4i}=2\frac{K}{c^3(-1)^{\frac{1}{2}}}\gamma M_{\text{rot}}\frac{1}{r^3}(u^jx^l-x^ju^l), \tag{13}$$

[$\gamma=1$ when $|\mathbf{\Omega}|$ and μ are spatially constant or have a spherical symmetry inside the sphere. It can be shown[3] that $\gamma>0.6$ for the earth]. Concerning the non-statical part $g_{4i}{}^*$ of the g_{4i} ($i=1, 2, 3$) we shall only assume that

$$g_{4i}{}^*=h(r, x^4)\partial r/\partial x^i. \tag{14}$$

Equations (12a) and (12b) then become

$$H_i=-\frac{2}{\chi_0}\frac{(m_0)_e}{ec}K\gamma M_{\text{rot}}\left\{\frac{\partial}{\partial x^l}\left[\frac{1}{r^3}(u^lx^i-u^ix^l)\right]-\frac{\partial}{\partial x^j}\left[\frac{1}{r^3}(u^ix^j-u^jx^i)\right]\right\}(A_j{}^j+A_l{}^l), \tag{15}$$

and

$$E_i=-\frac{1}{2\chi_0}\frac{(m_0)_ec^2}{e}\left[(A_4{}^4+A_i{}^i)\frac{\partial g_{44}}{\partial x^i}-2A_i{}^i\frac{\partial h}{\partial x^4}\frac{\partial r}{\partial x^i}-\frac{\partial \omega_{4i}}{\partial x^4}\right]. \tag{16}$$

(a) The Magnetic Field of the Rotating Sphere

In order to derive this static field from (15) we must take the large scale temporal mean of both sides of this equation. From (3a) we get $A_j{}^l+A_l{}^l\cong\chi$ when the $\bar{\omega}_{ik}$ are small quantities against χ. Writting $A_j{}^j+A_l{}^l\equiv\xi+\xi'$, where ξ' corresponds to the rapid changing part $g_{4i}{}^*$ of g_{4i}, we see that the large scale temporal mean $\langle[A_j{}^j+A_l{}^l]\rangle_{\text{Av}}$ of $A_j{}^j+A_l{}^l$ is simply

$$\langle[A_j{}^j+A_l{}^l]\rangle_{\text{Av}}=\xi.$$

[3] Comptes Rendus **226**, 1298 (1948).

Formula (15) then gives

$$\mathbf{H}=2\frac{\xi}{\chi_0}\frac{(m_0)_e}{ec}K\gamma M_{\text{rot}}\,\text{curl}\left[\mathbf{u}\times\nabla\left(\frac{1}{r}\right)\right]. \tag{17}$$

According to classical electromagnetism the magnetic moment M_{magn} which corresponds to the field (17) is related to $\mathbf{H}$ by

$$\mathbf{H}=M_{\text{magn}}\,\text{curl}[\mathbf{u}\times\nabla(1/r)]. \tag{18}$$

Comparing (17) and (18) we get finally

$$M_{\text{magn}}=2\frac{\xi}{\chi_0}\frac{(m_0)_e}{ec}K\gamma M_{\text{rot}}. \tag{19}$$

This is the very important formula explaining the general features of the magnetic fields of the stars and the earth as a fundamental consequence of their rotation. It explains also[4] the magnetic moments of the proton and the neutron. The recently discovered stars with periodic magnetic fields[5] can also be explained by (19), for it can be shown[6] that $\xi=\pm\lambda^2\chi_0$, and that in general λ is a periodic function of time. Formula (19) has been previously derived by the author from his unified field theory, quite independently of the Codazzi equations, and its consequences have been thoroughly studied.[1] For this reason we will only point out here the important fact that (19) is a mere consequence of the Codazzi equations when the external metric tensor is interpreted (by Eqs. (9) and (10)) as an electromagnetic tensor.

This result brings new evidence to our unified theory and we regard it as a good proof of the fundamental character of the creation of magnetism by rotation, even without any electrical charge.

(b) The Electrostatic Field

The force of gravitation G_i (per unity mass) is related to g_{44} by the well-known formula of the relativity theory:

$$G_i=-\frac{c^2}{2}\frac{\partial g_{44}}{\partial x^i}.$$

Introducing this in (16) we get

$$E_i=\frac{1}{\chi_0}\frac{(m_0)_e}{e}\left[(A_4{}^4+A_i{}^i)G_i+c^2A_i{}^i\frac{\partial h}{\partial x^4}\frac{\partial r}{\partial x^i}+\frac{c^2}{2}\frac{\partial\omega_{4i}}{\partial x^4}\right].$$

Since, according to (3a), $A_4{}^4+A_i{}^i\cong\chi=(\xi+\xi')$, and as ω_{4i} are bounded functions, we obtain for the large scale temporal mean value $\langle(E_{\text{st}})_i\rangle_{\text{Av}}$ of E_i, i.e., the electrostatic field:

$$\mathbf{E}_{\text{st}}=\frac{\xi}{\chi_0}\frac{(m_0)_e}{e}\left\{\mathbf{G}+\frac{c^2}{\xi}\left\langle\left[\xi'\frac{\partial h}{\partial x^4}\right]\right\rangle_{\text{Av}}\frac{\mathbf{r}}{r}\right\}. \tag{20}$$

The mean value of $\xi'\partial h/\partial x^4$ can be deduced from the equations of the metric fields. When, as in the present case, the T_{4i} components of the energy-momentum tensor correspond to the static part of g_{4i}, the non-static part $g_{4i}{}^*$ obeys the following linearized Einstein equation:

$$\Box g_{4i}{}^*=-(8\pi K/c^2)Tg_{4i}{}^*. \tag{21}$$

Taking (14) into account and putting

$$h(r,x^4)=f(x^4)\varphi(r)$$

the solution of (21) for a point outside the material sphere is:

$$(1/(-1)^{\frac{1}{2}})g_{4i}{}^*=\varphi(r)(\partial r/\partial x^i)\sin(c\sigma t),$$

with an appropriate time origin and σ denoting a constant. The electrostatic field will then be given by

$$\mathbf{E}_{\text{st}}=\frac{\xi}{\chi_0}\frac{(m_0)_e}{e}\left\{\mathbf{G}+\frac{c^2\sigma}{\xi}\langle[\varphi\xi'\cos(c\sigma t)]\rangle_{\text{Av}}\frac{\mathbf{r}}{r}\right\}. \tag{22}$$

The spatial variation of $\varphi\xi'$ is determined by the equation $\text{div}\,\mathbf{E}_{\text{st}}=0$ of the electrostatic field outside the charges. As $\text{div}\,\mathbf{G}=0$ outside the material sphere, we get the condition $\text{div}[(\mathbf{r}/r)\varphi(r)\xi'(r,t)]=0$, which leads to

$$(\partial(\varphi\xi')/\partial r)+(2/r)\varphi\xi'=0.$$

The solution is

$$\varphi\xi'=(a(t)/r^2),$$

and (22) becomes:

$$\mathbf{E}_{\text{st}}=\frac{\xi}{\chi_0}\frac{(m_0)_e}{e}\left\{\mathbf{G}+\frac{c^2\sigma}{\xi}\langle[a(t)\cos(c\sigma t)]\rangle_{\text{Av}}\frac{\mathbf{r}}{r^3}\right\}.$$

The function $a(t)$ is of course a periodic function with the same period as $g_{4i}{}^*$. Hence, by a Fourier expansion $\langle a(t)\cos(c\sigma t)\rangle_{\text{Av}}=\alpha$ with

$$\alpha=c\sigma\int_0^{2\pi/c\sigma}a(t)\cos(c\sigma t)dt.$$

Putting $c^2\sigma\alpha=\eta$ (constant) we get finally

$$\mathbf{E}_{\text{st}}=\frac{\xi}{\chi_0}\frac{(m_0)_e}{e}\left[\mathbf{G}+\frac{\eta}{\xi}\frac{\mathbf{r}}{r^3}\right]. \tag{23}$$

Since $\mathbf{G}=-KM\mathbf{r}/r^3$, (23) can still be written as follows:

$$\mathbf{E}_{\text{st}}=\frac{\xi}{\chi_0}\frac{(m_0)_e}{e}\left[1-\frac{\eta}{KM\xi}\right]\mathbf{G}. \tag{24}$$

This is the required relation between the electrostatic

[4] Portugaliae Mathematica **7**, 1–43 (1948).
[5] H. W. Babcock, Astronom. Soc. Pac. **59**, 260 (1947).
[6] Comptes Rendus **226**, 2126 (1948).

field and gravitation. We are inclined to think that (24), which is a consequence of the Codazzi equations, reveals *the real cause of the maintenance of the mean electrostatic field of the earth. This cause is merely the gravitation of the earth's mass*, and we conclude also from (24) that *any massive body must have an electric charge.*

According to Blackett,[7] who compared the magnetic moment M_{magn} and the angular momentum of the earth, the sun and the star 78Vir: $M_{\text{magn}}/M_{\text{rot}} = \beta(K)^{\frac{1}{2}}/2c$, β denoting a numerical coefficient ($\beta \sim 1$). This value of $M_{\text{magn}}/M_{\text{rot}}$ corresponds to

$$\xi/\chi_0 = \beta l/4(m_0)_e(K)^{\frac{1}{2}},$$

and it is quite natural to suppose that the ratio E/G is also nearly constant for the different celestial bodies. Denoting by R the radius of the bodies and by the subscript zero the values of the different quantities for the earth, Eq. (24) then gives the following relation for the electrostatic field at the surface of the stars:

$$\mathbf{E}_{\text{st}} = (\mathbf{E}_{\text{st}})_0 \frac{M}{M_0}\left(\frac{R_0}{R}\right)^2 = (\mathbf{E}_{\text{st}})_0 \frac{\bar{\mu}}{\bar{\mu}_0}\frac{R}{R_0} \tag{25}$$

the $\bar{\mu}$ denoting the mean densities. Hence, for instance: $(\mathbf{E}_{\text{st}})_{\text{sun}} = 28(\mathbf{E}_{\text{st}})_0$.

An alternative way of presenting (24) or (25) is to replace the electrostatic field intensity by the corresponding charges Q or potentials V. We obtain from (24)

$$Q = -\frac{2\xi}{\chi_0}\frac{(m_0)_e}{e}\left(1 - \frac{\eta}{KM\xi}\right)KM, \tag{26}$$

and (25) gives:

$$Q = Q_0(M/M_0), \tag{27}$$

and

$$V = V_0(\bar{\mu}/\bar{\mu}_0)(R/R_0)^2. \tag{28}$$

As this relation can be applied to the stars, we see that many stars must exist with the very strong electric potential required to give rise to the expulsion of cosmic rays.

When, as in the laboratory, the gravitation created by the body is quite negligible, Eq. (23) reduces to:

$$\mathbf{E}_{\text{st}} = \frac{\eta}{\chi_0}\frac{(m_0)_e}{e}\frac{\mathbf{r}}{r^3}. \tag{29}$$

This is the familiar Coulomb field of a sphere with the electric charge $Q = (\eta/\chi_0)(m_0)_e/e$. Remembering the nature of the constant η, we see that the Coulomb field is a direct consequence of the very rapid vibrations undergone by the g_{4i} ($i = 1, 2, 3$) components of the metric tensor corresponding to any charged body.

[7] P. S. Blackett, Nature **159**, 658–666 (1947).

5. STATIC FIELD OF A MAGNET

As a last special case illustrating the fundamental links of electromagnetism and gravitation, we consider a constant magnet whose magnetization may be of the permanent and of the induced types. For pure spatial coordinate transformations the g_{4i} ($i = 1, 2, 3$) and the T_{4i} ($i = 1, 2, 3$) components of the energy-momentum tensor are spatial vectors which will be denoted by $\mathbf{g}_4$ and $\mathbf{T}_4$. Clearly, Eq. (12a) can be applied to the case under examination. Taking into account that $A_j{}^j + A_l{}^l \cong \xi$ we get:

$$\mathbf{H} = \frac{\xi}{\chi_0(-1)^{\frac{1}{2}}}\frac{(m_0)_e c^2}{e}\,\text{curl}\,\mathbf{g}_4. \tag{30}$$

The equations of the internal metric field give

$$\Delta\mathbf{g}_4 = (8\pi K/c^2)(\mathbf{T}_4 - \tfrac{1}{2}T\mathbf{g}_4) \equiv [8\pi K/c^2]\mathbf{T}_4^*, \tag{31}$$

and reduce to

$$\Delta\mathbf{g}_4 = 0, \tag{32}$$

for a point outside the magnet. The solution of (32) must be compatible with the Maxwell equation $\text{curl}\,\mathbf{H} = 0$ for a magnetostatic field outside its sources. From (30) we then derive the conditions:

$$\text{curl curl}\,\mathbf{g}_4 = 0; \quad \nabla(\text{div}\,\mathbf{g}_4) = 0.$$

The appropriate solution of (32) is therefore

$$\mathbf{g}_4 = \mathbf{a} \times \nabla(1/r),$$

$\mathbf{a}$ being a constant vector and r the distance from the "center of gravity" of the magnet. Equation (30) then gives:

$$\mathbf{H} = [\xi/\chi_0(-1)^{\frac{1}{2}}][(m_0)_e c^2/e]\,\text{curl}[\mathbf{a} \times \nabla(1/r)]. \tag{33}$$

Comparing this result with the classical formula of the magnetostatic field for points outside the magnet and sufficiently far from it, we see at once that $\mathbf{a}$ is related to the magnetic moment $\mathbf{M}_{\text{magn}}$ of the magnet by:

$$\mathbf{a} = [\chi_0(-1)^{\frac{1}{2}}/\xi][e/(m_0)_e c^2]\mathbf{M}_{\text{magn}}. \tag{34}$$

Taking (31) into account (34) can be written as follows:

$$8\pi K[(m_0)_e/e]\xi/\chi_0 \int_v \mathbf{T}_4^* dv = \mathbf{M}_{\text{magn}}, \tag{35}$$

v denoting the volume of the magnet. This relation gives the material momentum $\int_v \mathbf{T}_4^* dv$ produced by the magnetization, but it must be remarked that this momentum is by no means associated with a macroscopic material current. It is the material counterpart of the well known electromagnetic momentum in the canonical equations of motion.

It would be rather easy to generalize the preceding results for points inside the magnet.