Non-Equilibrium States in Helium II

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It is shown that second sound is thermodynamically possible only in the absence of thermal equilibrium between the superfluid and normal component of He II. The tendency of the assembly to drop back irreversibly to the state of internal equilibrium within a certain time of relaxation provides a consistent account of the observed limitations of superfluidity and heat conductivity in liquid helium. As a consequence it is to be expected that the times of relaxation calculated from heat conductivity measurements should be identical with those which can be derived from Pellam's direct attenuation measurements in second sound. This expectation is confirmed within the accuracy of available data.

INTRODUCTION

MOST of the peculiar properties of helium II can be described in terms of the two-fluid model developed by Tisza.[1] According to this model helium II consists of two inter-penetrating fluids, one of which is the so-called superfluid (mass per unit volume ρ_s) and the other the normal fluid (mass per unit volume ρ_n). The sum $(\rho_s+\rho_n)$ is the total density ρ of the liquid. The superfluid can move with respect to the walls of the vessel up to a certain limiting critical speed without friction, while the normal fluid has a normal viscosity. The normal fluid carries almost the whole heat content of the liquid.

The extremely high heat conductivity of helium II[2] appears qualitatively in this model as caused by the flow of the normal fluid carrying the heat content down the temperature gradient, while this mass flow is compensated by an opposing stream of superfluid. The thermal resistance is in first approximation regarded as due to the viscosity of the normal fluid. This picture has been worked out quantitatively by F. London and P. Zilsel[3] for narrow slits and capillaries ($r\sim 10^{-4}$ cm) where the mobility of the normal fluid is rather strongly limited by viscosity. They obtained the correct order of magnitude for the heat current, at least for certain widths of the slits used.

The heat current calculated in this way for wider capillaries $r=(0.01$ to 0.1 cm) is greater than the observed currents by several orders of magnitude. This fact suggests that in these wider capillaries another type of resistance must be limiting the relative mobility of the two streams of fluid in the heat current. Another very significant peculiarity in the behavior of the heat currents in these wider capillaries is their almost exact proportionality with the cube root of the temperature gradient,[4] and this also suggests that the normal viscosity picture becomes inadequate.

It seems natural to consider this additional limitation on the heat current as due essentially to the same factors that limit the superfluidity of the superfluid at the critical velocity as observed in direct flow measurements of the "film" or through narrow slits.[5–7] This additional limitation, therefore, is inoperative so long as the viscosity of the normal fluid is able to limit the compensating flow of the superfluid to levels lower than its critical speed, as happens in narrow capillaries. In the wider capillaries the viscosity is unable to do this, and the new limitation becomes operative.

The velocity of flow of the normal fluid u_n in the heat current is easily calculated from the two fluid model.[8] If W is the heat current density and ϵ the thermal energy per unity mass of the normal fluid, u_n its mean velocity, we have

$$W=\epsilon\rho_n u_n. \tag{1a}$$

But $\epsilon\rho_n$ is to this approximation the heat content ρQ per unit volume, therefore

$$W=\rho Q u_n. \tag{1b}$$

The values of u_n calculated in this way from the measured values of Q and of W for temperature gradients of 0.001°/cm are not only approximately equal to the critical velocities observed in the superfluid flow of the "film" and through narrow slits (1μ or less), but depend on temperature in about the same manner (compare reference 8, Table I). Moreover the rate of flow of the "film" at velocities higher than the critical speed is not proportional to the pressure or temperature gradients producing it, but depends approximately in the same way as the heat currents on the cube root of the gradient.*

It has been suggested that this limitation of super-

[1] L. Tisza, Comtes Rendus **207**, 1035, 1186 (1938); J. Phys. Rad. 1, 165, 350 (1940); Phys. Rev. **72**, 838 (1947); Phys. Rev. **75**, 885 (1949).
[2] W. H. Keesom, *Helium* (Elzivier Amsterdam-New York, 1942), p. 282.
[3] F. London and P. R. Zilsel, Phys. Rev. **74**, 1148 (1948).
[4] Keesom, Saris, and Meyer, Physica **VII**, 817 (1940).
[5] J. G. Daunt and K. Mendelssohn, Proc. Roy. Soc. **A170**, 423 (1939).
[6] P. L. Kapitza, J. Phys. U.S.S.R. **5**, 59 (1941).
[7] L. Meyer and J. H. Mellink, Physica **13**, 197 (1947).
[8] W. Band and L. Meyer, Phys. Rev. **73**, 226 (1948).
* Atkins' observation (K. R. Atkins, Nature **185**, 925 (1948)) of anomalously high superfluid velocities appear to be essentially observations of super-critical speeds. The flow of superfluid from one chamber to the other was produced by supplying heat to the latter. As long as the flow velocity is below the critical speed the heat supplied is completely compensated by thermomechanical cooling due to the incoming superfluid. A temperature difference between the two chambers can appear only if the flow of superfluid is unable to compensate the supplied heat, i.e., when the critical velocity is exceeded. See Kapitza, reference 6 and Meyer and Mellink, reference 7.

fluidity is essentially a quantum effect, and that no understanding of the phenomenon will be possible until a rigorous quantum hydrodynamics has been set up.[9,10]

It has also been imagined that the limitation of superfluidity, and also the cube root law dependence of the heat current on the temperature gradient may be due to turbulence or some other higher order terms in the hydrodynamical equations for the two-fluid model, and that, therefore, any attempt to study the problem quantitatively was not likely to succeed.[1,3,11,**]

This paper aims to show that the limitation of superfluidity can be described in terms of perturbations of the thermodynamic equilibrium between the normal fluid and the superfluid whatever the eventual interpretation of the superfluid and of the "normal fluid" may be. The inevitable tendency of the mixture to return irreversibly to thermal equilibrium and the resulting dissipation of energy provide a consistent picture both of the attenuation of second sound and of the thermal resistance.

SECOND SOUND AND NON-EQUILIBRIUM STATES

Second sound was predicted by Tisza[1] from the two-fluid model on the basis of the following considerations: If the two-fluid components do not interact—in the sense that there is no internal viscosity associated with their relative motion—the mixture has an additional basic mode of oscillation in which the two components move in opposite phase with each other. The first basic mode of motion in which the two components move in phase with each other consists of compressional waves or "phonons": the concentration ρ_n/ρ of the mixture remains sensibly constant during these oscillations, and only the total density ρ varies. These are the acoustical modes of vibration and are identical with the longitudinal modes which play an important role in the theory of solids. In the second basic mode, "second sound" vibrations, the total density ρ remains fixed, (normal sound is absent) and only the concentration ρ_n/ρ undergoes periodic changes. The two parts of the fluid have opposite velocities at every point, so that this mode is possible only in the absence of momentum transfer between the two components.

However, there seems to be at first sight a serious difficulty with thermodynamics. Helium II is thermodynamically a one component system, and its state is completely determined by the two variables ρ and T, the total density and the temperature respectively. As the total density remains constant in second sound, there is only one free variable, the temperature. Therefore, the concentration ρ_n/ρ is uniquely defined by the temperature everywhere and local oscillations of this concentration in second sound would appear to be local oscillations of the temperature. In fact, Tisza originally called second sound "temperature waves."

The heat carried by the second sound waves, namely, by the excess concentration $\epsilon\Delta\rho_n$, is propagated with the wave. This heat would be traveling not only down but also up the temperature gradients without regard for the second law of thermodynamics. Consider an arrangement in which standing second sound waves are maintained between the source and a perfect heat reflector, and let there be at least one antinode A in the wave pattern. If it is correct to regard ρ_n and ρ_s as uniquely defined by the temperature, then T must be oscillating at A between two extremes, say T_0 and $T_1 > T_0$. The nodes meanwhile remain at a fixed temperature, say T. During one phase of the wave the temperature T at A is increasing from T towards T_1 and heat is flowing into the region around A through the nodes, and therefore, going up the temperature gradient. Meanwhile no work is being done on the liquid because the source of the second sound is simply an oscillatory heat supply and the density ρ remains ideally constant. If in fact T and ρ were the only independent variables this would be an obvious contradiction with the second law of thermodynamics. This complication cannot be avoided by ascribing infinite thermal conductivity to the liquid and regarding second sound as simply the limiting form of a Fourier heat wave: such a heat wave would have an infinite speed of propagation in the limit of infinite thermal conductivity, while if the speed were finite it would depend upon distance from the source in the well-known classical manner. (Compare reference 12.) Second sound definitely has a finite speed of propagation characteristic of the liquid at the temperature of interest, independent of the distance from the source.

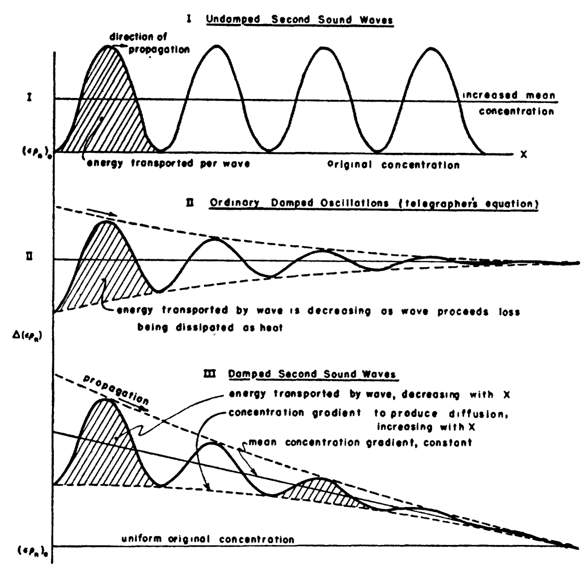

FIG. 1. Table of wave motion.

[9] L. Landau, J. Phys. U.S.S.R. **5**, 71 (1941); **8**, 1 (1944); Phys. Rev. **75**, 884 (1949).
[10] H. S. Green, Proc. Roy. Soc. (London) **A194**, 244 (1948).
[11] C. J. Gorter, Phys. Rev. **74**, 1544 (1948).
** Recent statistical work on turbulence by Heisenberg (Proc. Roy. Soc. **A195**, 402 (1948)) may permit progress along this line of attack.

Possibly in an unconscious effort to escape from this dilemma several writers have stated, either explicitly or implicitly, that second sound does not carry heat, but only the kinetic energy of the relative motion of the two-fluid components.[3,12] In practice, however, second sound is always excited by an oscillatory heat source that never cools the liquid below the initial temperature during any phase of the oscillation; the heat supply is unidirectional and oscillates only between a minimum (possibly zero) and a maximum input. If the mean rate of heat input is W ergs/cm^2 sec., this thermal energy must be carried by the wave at the speed of its phase velocity V_s, there is a concentration of heat in excess of normal W/V_s ergs/cm^2 per cm of path. The amplitude of the heat wave measured from minimum to maximum heat content is $2W/V_s$ ergs/cm^3. This statement is independent of any model, and simply follows from the experimental fact that heat supplied at the source is received at the detector with the phase velocity.

This point is so fundamental that we give in Appendix I a more detailed proof that the heat carried by second sound is large compared with the kinetic energy of the relative motion of the two-fluid components.

The answer to this apparent thermodynamic paradox is in fact almost trivial: namely, that at high enough frequencies T and ρ alone are not sufficient to specify the state of the liquid. The existence of second sound is possible only in the *absence* of thermal equilibrium between ρ_n and ρ_s; this provides the required extra degree of freedom and, in fact, renders the thermodynamical temperature T of the mixture as a whole essentially meaningless. This situation is already familiar in connection with Einstein's[12a] theory of dispersion of ordinary sound in a dissociating gas.[8] At low frequencies the dissociation equilibrium can follow the changes of pressure and temperature in the wave, but by increasing the frequency we must reach a situation in which the dissociation equilibrium cannot adjust itself rapidly enough to keep up with the sound vibrations. At still higher frequencies the two dissociating components behave as essentially independent fluids, their mutual thermal equilibrium never having time to readjust itself to the changes occurring in the wave. The concentration then becomes a third independent variable. Einstein showed that this mechanism would cause dispersion and attenuation of ordinary sound in the dissociating gas in a transition region at frequencies where the period of the wave nearly equals the time required for a perturbation from equilibrium to regress spontaneously, usually called the time of relaxation.

The two components of a dissociating gas do not lead to observable second sound even at high enough frequencies simply because internal viscosity hinders their relative motion: if such second sound existed it would probably be damped out too rapidly to be observed. But in He II the superfluidity of the low-energy component ensures the existence of second sound provided the frequency is high enough to inhibit the thermal equilibrium of the concentration.

If we can decrease the frequency so far that the wave period becomes comparable with or even longer than the relaxation time for the thermal equilibrium between the two parts of the fluid, then thermodynamics will enforce its restrictions and second sound will be damped out. At high frequency the concentration ρ_n/ρ varies in a manner that can be calculated from the equation of continuity applied to each fluid component separately, that is, assuming no transitions of particles between the two fluids and no coupling with temperature; at low frequency the concentration variations become uniquely determined by temperature variations and particles have to transfer from one component of the fluid to the other if they move relative to each other. The numerical evaluation of this effect is given in Appendix II.

This picture leads to the following wave equation, the "telegrapher's equation," for second sound applicable at all frequencies:[8]

$$\partial^2\rho_n/\partial t^2+\partial\rho_n/\partial t/\tau=V_s^{*2}\nabla^2\rho_n. \qquad (2)$$

In this equation the term $\dot{\rho}_n/\tau$ takes care of the dissipation of relative momentum due to transition of particles between the two fluids during their relative motion. The velocity V_s^* is a complex quantity[13] because of the irreversible relaxation towards equilibrium after perturbation, and its effect on the "elasticity constants" in the Hamiltonian. Essentially the same relaxation time τ appears*** in the detailed expression for V_s^*. If we were to neglect the relative momentum dissipation, and retain only the complex velocity, we would obtain the same effects as derived by Einstein for normal sound. Second sound waves would exist even at frequencies much lower than $1/\tau$ but with a different velocity of propagation. Attenuation would be present only in the transition region. The presence of the relative momentum dissipation, however, requires that when the maximum attenuation is reached, the wave equation degenerates into a diffusion equation. For still lower frequencies we then get only the classical Fourier type of heat wave (diffusion) without any true propagation of second sound, just as in the well-known solutions of the "telegrapher's equation."

HEAT TRANSFER AND ATTENUATION IN SECOND SOUND

Consider a second sound wave in which the density of normal fluid oscillates in the following manner: (See

[12] R. Dingle, Proc. Phys. Soc. (London) **61**, 9 (1948).

[12a] A. Einstein, Sitz. Berl. Akad. **1920**, 380.

[13] W. Band and L. Meyer, Phys. Rev. **74**, 386 (1948).

*** Dr. J. B. Garrison has drawn our attention to the desirability of a more generally rigorous treatment of the relaxation problem in a system that has three independent variables at high frequencies. Such a treatment would yield more than one relaxation time and possibly even modify the non-equilibrium elastic constants slightly.

TABLE I. Values of relaxation times.

T°K	$V_{s\infty}$ m/sec.	μ per cm	Second sound $\tau\times10^3$ sec. from μ	Steady heat flow $\tau\times10^{-3}$ sec. for $W=0.03$ cal./cm² sec.
2.18	4.25	3.0	0.4	—
2.15	8.05	1.0	0.6	—
2.10	12.25	0.3	1.35	1.1
2.0	16.69	0.15	2.0	2.4
1.9	18.68	0.10	2.7	2.7
1.8	19.90	0.07?	3.6?	2.2
1.7	20.35	0.05?	5.0?	1.5

Appendix I, Eqs. (6), (18).)

$$\Delta\rho_n=\Delta\rho_{n0}e^{-\mu x}\sin 2\pi\nu(t-x/V_s)+\Delta\rho_{n0}.$$

Then the rate of propagation of heat by this wave is

$$W_P=\epsilon_0\Delta\rho_{n0}e^{-\mu x}V_s. \tag{3}$$

Curve I of Fig. 1 illustrates the propagation when damping is neglected. The form (3) is familiar in electrical solutions of the "telegrapher's equation." In the electrical problem the dissipated energy is small compared with the thermal capacity of the conductor and we can safely neglect it (see curve II of Fig. 1); if it produces any effects on the wave, they are at most second order effects due to thermo-e.m.f.'s. Here, however, the attenuation of W_P appearing in (3) cannot be neglected. The primary wave itself is already essentially heat energy and the dissipated energy is neither changed in nature nor reduced by the "mechanical equivalent" factor, so a gradient must be set up to carry it away. Therefore, a concentration gradient of the normal fluid carrying the heat current has to be superposed on the concentration wave. The required arrangement is shown in curve III of Fig. 1. Here the lower envelope of the wave has a gradient which is everywhere proportional to the loss of energy from the wave, namely, to $W_{P0}(1-e^{-\mu x})$ where W_{P0} is the heat propagation at $x=0$. At great values of x the wave will have completely died out and the concentration gradient is such that it can carry the whole heat supplied to the system by a diffusion process in accordance with the equation:

$$W=-\tau V_s^2(d\epsilon\rho_n/dx). \tag{4}$$

The detailed proof of this is given in Appendix II.

A source of plane second sound waves propagated into an infinite medium must build up a mean temperature gradient large enough to take care of the entire heat flow. The wave may be regarded as a modulation of the resulting heat transport. The latter may be expressed either as excess heat content $\epsilon\Delta\rho_n$ propagated with the velocity of propagation of the wave, or as total heat content $\epsilon\rho_n$ transported with the mean speed $\bar{u}_n$ of the normal fluid down the temperature gradient. The two quantities are identical (see Appendix III).

However, if second sound is emitted from a finite (small) source into an infinite medium in a more-or-less parallel beam, the energy lost from the wave is dissipated more or less laterally, so that the liquid around the path of the beam is heated, avoiding the necessity of establishing a temperature gradient in the direction of propagation. In this situation we have to regard the propagation of heat as essentially that of excess heat content with speed V_s. It is in principle also possible to shoot second sound waves up a temperature gradient maintained against it. The thermal diffusion current is always down the effective temperature gradient, in whatever direction that may happen to be; the second sound merely superposes concentration waves on the concentration gradient. In none of these situation is there any difficulty with the laws of thermodynamics because the second sound wave essentially does not cause waves of thermodynamic equilibrium temperature.

Using this picture we were able to derive the approximate times of relaxation from the experimental evidence on second sound and thermal conductivities.[8, 13] Recently Pellam[14] published data on the direct measurements of the attenuation of second sound. If the picture developed here is correct. Pellam's results should lead to the same values of the relaxation time as we derived from the heat conductivity measurements because the thermal diffusion current appears as the limiting case of completely damped second sound of zero frequency.

COMPARISON WITH OBSERVATION

The equation for high frequency second sound derived from the picture of the absence of internal equilibrium (see reference 13, Eqs. (C26) and (C27)), is

$$\ddot{U}_n+\dot{U}_n/\tau_1=V_{s\infty}^2\{1+1/2\pi i\nu\tau_3\}^{-1}\nabla^2U_n \tag{5}$$

where ν is the frequency,

$$1/\tau_3=(1/\tau_2)\{3-2\rho_n'Q/\rho_nC-S(\rho-\rho_n)/V_s^2\rho_n'\}, \tag{6}$$

and τ_1 is the relaxation time for the momentum exchange between oppositely moving streams of normal and superfluid components, while τ_2 is the relaxation time for the readjustment of the population towards thermal equilibrium between the two components. From the data on the quantities appearing in Eq. (6) we find that

$$\tau_3 \doteqdot 4\tau_2 \tag{7}$$

to within about 5 percent at all temperatures of interest. It is, of course, not certain how τ_2 and τ_1 are related, but at least we can expect them to be the same order of magnitude and probably proportional to each other. The attenuation coefficient in cm^{-1} is now

$$\mu=1/\tau V_{s\infty} \tag{8}$$

where

$$1/\tau=\tfrac{1}{2}(1/4\tau_2+1/\tau_1). \tag{9}$$

[14] R. Pellam, Phys. Rev. **75**, 1183 (1949); **74**, 841 (1948).

Table I shows the results of using (8) to calculate τ from Pellam's data on the attenuation of second sound. The relation (8) is valid only if $2\pi\nu\tau \gg 1$, and this must be confirmed. The values of τ found are roughly between $4\cdot 10^{-4}$ and $5\cdot 10^{-3}$ sec. The frequency to be ascribed to Pellam's pulses is not very certain, but a safe minimum is obtained by assuming the time-length of each pulse to be a half period for the fundamental frequency in a Fourier analysis of the pulse form. This time length was reported as about 100 microsec., so it may be assumed that $\nu \geqslant 10^4$ sec.$^{-1}$. With these figures we find $2\pi\nu\tau > 10$ at least for all temperatures below 2.10°K where $\tau > 10^{-3}$. The accuracy nearer the lambda point may be doubtful.

Table I shows the times of relaxation derived for a heat current assumed to be $W=0.03$ cal./cm² sec. which seems to be a reasonable value for the energy density in Pellam's waves. The values of τ derived in this way from Pellam's attenuation data are seen to be almost identical with the times calculated before from steady thermal currents. Unfortunately, there is no information available concerning the heat current density in Pellam's second sound waves. It is not even known whether the same power input was used at all the different temperatures. From Table I of a previous paper[15] and the fact that τ is inversely proportional to the square of the heat current we can calculate τ for any stated heat current. An even better agreement could be obtained between the two sets of figures if it were assumed that the heat current was only 0.02 cal./cm² sec. at the two lower temperatures recorded. The fact that Pellam's results for μ show a relatively wide scatter might indicate that the power input was not quite constant even at a given temperature.†

Note added in proof: Since communicating this paper we learned about several papers by V. P. Peshkov, Journal Experimental and Theoretical Physics, U.S.S.R. **18**, 857, 867, and 951 (1948) on second sound and its attenuation. In these papers the discrepancy between the infinite heat conductivity in second sound and the finite heat resistance in steady state heat flow experiments representing effectively second sound of zero frequency is pointed out. This is exactly the problem the solution for which we have proposed above and in references 8 and 13. Furthermore Peshkov succeeded in producing second sound without superimposed heat flow by means of the "filtration" method. His result that in this case the attenuation is considerably smaller than when producing second sound in the usual manner by periodic heating must be considered as a qualitative confirmation of the above ideas.

ACKNOWLEDGMENTS

The writers are happy to thank Messrs. Zener, Long, Stout and Garrison for valuable discussions.

APPENDIX I

Heat Transfer and Kinetic Energy in Second Sound

At sufficiently high frequencies the wave equation for second sound is

$$\partial^2\mathbf{X}_n/\partial t^2+\partial\mathbf{X}_n/\partial t/\tau = V_s^{*2}\nabla\nabla\cdot\mathbf{X}_n \tag{I, 1}$$

where

$$V_s^{*2}=V_s^2(1+1/2\pi i\nu\tau)^{-1}$$

is the complex velocity, V_s being the real velocity. The term in $1/\tau$ is due to momentum dissipation and the complex factor due to equilibrium relaxation. $\mathbf{X}_n$ is the vector of displacement of the normal fluid. Differentiate this equation with respect to time and we obtain the equation for the instantaneous velocity $\mathbf{U}_n$ of the normal fluid:

$$\ddot{\mathbf{U}}_n+\dot{\mathbf{U}}_n/\tau = V_s^{*2}\nabla\nabla\cdot\mathbf{U}_n. \tag{I, 2}$$

Suppose the damping to be very small, then we can assume the normal fluid mass conserved:

$$\Delta\dot{\rho}_n/\rho_{n0} = -\nabla\cdot\mathbf{U}_n. \tag{I, 3}$$

There is a phase factor correction when damping is appreciable. Use (I, 3) in (I, 2) to derive the equation for $\Delta\rho_n$:

$$\Delta(\partial^2\rho_n/\partial t^2)+\Delta(\partial\rho_n/\partial t)/\tau = V_s^{*2}\nabla\cdot\nabla(\Delta\rho_n). \tag{I, 4}$$

Seeking a solution to (I, 2) which is consistent with (I, 3), we find to a first approximation

$$\mathbf{U}_n=\mathbf{U}_{n0}e^{-\mu x}\sin 2\pi\nu(t-x/V_s)+\mathbf{U}_{n1} \tag{I, 5}$$

and

$$\Delta\rho_n=\Delta\rho_{n0}e^{-\mu x}\sin\{2\pi\nu(t-x/V_s)-\varphi\}+\Delta\rho_{n1} \tag{I, 6}$$

where

$$\mu = 1/\tau V_s \tag{I, 7}$$

and

$$\Delta\rho_{n0} = U_{n0}\rho_{n0}/V_s. \tag{I, 8}$$

The approximation is involved in small corrections on the V_s factor in (I, 7) and (I, 8) and in the evaluation of the small phase factor ϕ. $\mathbf{U}_{n1}$ and $\Delta\rho_{n1}$ are arbitrary constants which may, however, depend linearly on x and still satisfy the differential equations.

The heat entering per cm² per sec. at the source is given as a function of the time by

$$\mathbf{R}=\epsilon\rho_n\mathbf{U}_n, \tag{I, 9}$$

where ϵ is the energy needed to excite a gram of superfluid arriving at the source into normal fluid leaving the

[15] L. Meyer and W. Band, Phys. Rev. **74**, 394 (1948).

† There is indirect evidence that μ depended on the power input in the same way as the heat conductivity because Pellam reported that he could "over-drive" the second sound with resulting excitation of ordinary sound and undoubtedly greatly increased attenuation of the second sound. Experiments on the dependence of the attenuation on power input would be decisive for the present theory, because according to this theory τ depends on the heat current density, as shown by the fact that the heat current depends on the cube root of the temperature gradient (see references 8 and 15).

source, $\mathbf{U}_n$ is given by (I, 5) at $x=0$, and ρ_n is the normal component density given by (I, 6) at $x=0$ and

$$\rho_n=\rho_{n0}+\Delta\rho_n \qquad \text{(I, 10)}$$

ρ_{n0} being the initial value of ρ_n.

The energy ϵ is also oscillating about a mean value ϵ_0, the amplitude of the oscillations being given by the ordered kinetic energy of the fluid components:

$$\epsilon=\epsilon_0+\Delta\epsilon \qquad \text{(I, 11)}$$

where

$$\begin{aligned}\rho_{n0}\Delta\epsilon&=\tfrac{1}{2}(\rho_{n0}U_n^2+\rho_{s0}U_s^2)\\&=\tfrac{1}{2}\rho(\rho_{n0}/\rho_{s0})U_n^2.\end{aligned} \qquad \text{(I, 12)}$$

First consider a wave in which $\mathbf{U}_{n1}=0$, so that the velocity oscillates about a zero mean value

$$\overline{\mathbf{U}}_n=0. \qquad \text{(I, 13)}$$

The mean rate of heat supply at the source is then

$$\begin{aligned}\overline{\mathbf{R}}=\langle(\epsilon_0+\Delta\epsilon)(\rho_{n0}+\Delta\rho_n)\mathbf{U}_n\rangle_{\mathrm{Av}}=\epsilon_0\rho_{n0}\overline{\mathbf{U}}_n+\rho_{n0}\langle\Delta\epsilon\mathbf{U}_n\rangle_{\mathrm{Av}}\\+\epsilon_0\langle\Delta\rho_n\mathbf{U}_n\rangle_{\mathrm{Av}}\text{ plus smaller terms.}\end{aligned} \qquad \text{(I, 14)}$$

The first two of these terms vanish identically and for the third, using (I, 5) and (I, 6) we find

$$\overline{\mathbf{R}}=\tfrac{1}{2}\epsilon_0\Delta\rho_{n0}\mathbf{U}_{n0}. \qquad \text{(I, 15)}$$

Because of (I, 8) this can be written in either of the forms

$$\bar{R}=\tfrac{1}{2}\epsilon_0\rho_{n0}U_{n0}^2/V_s$$

or

$$\bar{R}=\tfrac{1}{2}(\epsilon_0/\rho_{n0})(\Delta\rho_{n0})^2V_s. \qquad \text{(I, 16)}$$

This is a second order of small quantities. This kind of wave receives heat from the source during one-half of the period and returns heat to the source during the other half. The net difference is enough to transmit the ordered kinetic energy of the wave (ϵ_0 is of the same order of magnitude as V_s^2). Of course the result (I, 16) is not exact because of the approximations present in (I, 5) and (I, 6); if the calculation were carried out exactly the equality of (I, 16) with the kinetic energy transport would have been exact. These approximations are of no importance for the larger first order energy transport that will be discussed next.

Calculations of the energy carried by second sound waves previously published,[9, 12] have been confined to the above type of wave, in spite of the fact that actual second sound is produced by unilateral heat pulses.

In practice the source of second sound never receives heat from the liquid during any phase of the wave. It only supplies heat and maintains, therefore, a flow of heat into the liquid and the appropriate solutions of (I, 5) are those for which $\mathbf{U}_{n1}$, the arbitrary constant, is not zero. When $\mathbf{U}_{n1}$ is of the same order as $\mathbf{U}_{n0}$—or even equal to $\mathbf{U}_{n0}$, then the first term in the expression (I, 14) for the energy transport, becomes the dominant term and all the others are negligible, including the term responsible for the ordered kinetic energy. Thus we have now

$$\overline{\mathbf{R}}=\epsilon_0\rho_{n0}\mathbf{U}_{n1} \qquad \text{(I, 17)}$$

To prevent R from ever being negative, as required by the experimental boundary conditions at the source, we have to take

$$\mathbf{U}_{n1}=\mathbf{U}_{n0}\quad\text{and}\quad\Delta\rho_{n1}=\Delta\rho_{n0} \qquad \text{(I, 18)}$$

so that, using (I, 18) and (I, 8) we can write (I, 17) in the alternative form

$$\overline{\mathbf{R}}=\epsilon_0\Delta\rho_{n0}V_s. \qquad \text{(I, 19)}$$

Comparison between (I, 19) and the second form of (I, 16) shows that this heat transmission is large compared with the ordered kinetic energy transmission in the ratio $\rho_{n0}/\Delta\rho_{n0}$, and this, of course, may easily be as high as 100:1.

The two alternative forms (I, 17) and (I, 19) illustrate the fact that it is, formally at least, quite arbitrary whether we regard the heat transport as a convection of the *total* heat content $\epsilon_0\rho_{n0}$ with the mean *drift* speed $\mathbf{U}_{n0}$, or as a propagation of *excess* heat content $\epsilon_0\Delta\rho_{n0}$ with the *phase* speed V_s.

APPENDIX II

Numerical Population Adjustments Due to Relaxation

If the liquid were merely not in equilibrium with its surroundings, but internally in thermal equilibrium, the variation of ρ_n could be considered as equivalent to a variation of temperature:

$$\Delta\rho_n=(d\rho_n/dT)\Delta T. \qquad \text{(II, 1)}$$

Following Tisza, consider an adiabatic cylinder containing the liquid initially in equilibrium at temperature T. Let there be a porous piston in the cylinder that can be caused to move, and let the material of the piston be a perfect insulator. If the pistion is caused to oscillate *slowly* back and forth so that the superfluid passes freely from one side to the other through the pores, the liquid adjusts itself continuously on each side of the piston and we must find a temperature oscillation set up due to the thermomechanical effect.

Let the stroke of the piston be through a volume ΔV. The superfluid carries no heat—or rather we neglect it if it exists—and the energy balance on each side is very simple. A temperature difference ΔT develops and corresponding to this temperature difference there will be a pressure difference ΔP given by the fountain pressure relation

$$\Delta P=\rho S\Delta T. \qquad \text{(II, 2)}$$

The conservation of energy in the volume V on one side of the piston where the total energy is, say Q, gives

$$\Delta(Q/V)=\rho C\Delta T \qquad \text{(II, 3)}$$

where C is the specific heat. The amount of heat produced by the mechanical work done by the piston is the only source of heat in the volume V, so that

$$Q=-\tfrac{1}{2}\Delta P\Delta V. \qquad \text{(II, 4)}$$

This is of the second order of small quantities and may be neglected in comparison with the thermomechanical effect due to the flow of superfluid through the piston, represented by the other part of the variation in (II, 3). Thus (II, 3) gives

$$\Delta T=-(Q/\rho CV)(\Delta V/V). \qquad \text{(II, 5)}$$

As we are here considering slow oscillations the equilibrium variation of ρ_n is given by (1):

$$\Delta_e\rho_n=(d\rho_n/dT)\Delta T=-(Q/\rho CV)(d\rho_n/dT)(\Delta V/V). \qquad \text{(II, 6)}$$

We can write

$$(Q/V)=\epsilon\rho_n \quad \text{and} \quad \rho C=d(\epsilon\rho_n)/dT \qquad \text{(II, 7)}$$

so that (II, 6) becomes

$$\Delta_e\rho_n/\rho_n=-\{(\epsilon\rho_n')/(\epsilon\rho_n)'\}(\Delta V/V) \qquad \text{(II, 8)}$$

the primes indicating temperature derivatives.

On the other hand, if we assume rapid oscillations so that there is no time for population adjustments on either side of the piston we use the mass continuity equation for each fluid component, and at once obtain Tisza's equation

$$\Delta\rho_n/\rho_n=-\Delta V/V. \qquad \text{(II, 9)}$$

Comparison of (II, 8) and (II, 9) is very simple. We have roughly $\rho_n\propto T^6$ and $Q/V\propto T^7$ experimentally, so that

$$\Delta_e\rho_n/\rho_n=-(6/7)(\Delta V/V)=0.85(\Delta\rho_n/\rho_n). \qquad \text{(II, 10)}$$

APPENDIX III

Heat Transfer and Attenuation in Second Sound

The only part of (I, 9) that finally contributes to the mean heat transfer is

$$\mathbf{R}_t=\epsilon_0\rho_{n0}\mathbf{U}_n. \qquad \text{(III, 1)}$$

Using (I, 5) and (I, 18) in this we find

$$\mathbf{R}_t=\epsilon_0\rho_{n0}\mathbf{U}_{n0}\{1-e^{-\mu x}\sin 2\pi\nu(t-x/V_s)\} \qquad \text{(III, 2)}$$

and neglecting ϕ in (I, 6) and using (I, 8) again we find

$$\mathbf{R}_t=\epsilon_0\Delta\rho_nV_s. \qquad \text{(III, 3)}$$

We must, therefore, regard the heat propagated as having the same attenuation as $\Delta\rho_n$, and write for this rate of propagation the mean value of R_t at distance x, namely

$$\bar{R}_t=W_p=\epsilon_0\Delta\rho_{n0}e^{-\mu x}V_s \qquad \text{(III, 4)}$$

W_p is the propagation, and the loss from W_p with increasing x must be regarded as added to the diffusion current, so that at distance x the total diffusion current —considered also as traveling in the same direction as the wave—is

$$W_d=\epsilon_0\Delta\rho_{n0}V_s(1-e^{-\mu x}). \qquad \text{(III, 5)}$$

This diffusion current demands the presence of a slope in the heat content Q or $\epsilon_0\rho_{n0}$. The figure in the text showed the curve of Q superposed on a sloping axis determined by the condition that the curve tangent to the minima in the wave shall everywhere have a slope proportional to the value of W_d at the point.

Consider an infinite slab lying between the planes $x=0$ and $x=L$, with a plane source at $x=0$ and receiver at $x=L$. Let the axis of the wave be represented by the straight line in the ρ_n, x graph:

$$Q=\epsilon_0\rho_{n0}+\epsilon_0\Delta\rho_{n0}\{1+\mu^*(L-x)\}, \qquad \text{(III, 6)}$$

where μ^* is to be determined by the stated condition. The tangent to the minima of the wave $\epsilon_0\Delta\rho_n$ then has the equation

$$Q_{\min}=\epsilon_0\rho_{n0}+\epsilon_0\Delta\rho_{n0}\{1+\mu^*(L-x)-e^{-\mu x}\} \qquad \text{(III, 7)}$$

and this has the slope

$$dQ_{\min}/dx=-\mu\epsilon_0\Delta\rho_{n0}(\mu^*/\mu-e^{-\mu x}). \qquad \text{(III, 8)}$$

Comparing this with (III, 5) we see at once that the required condition can be met only by writing $\mu=\mu^*$. This yields the relation

$$W_d=-(V_s/\mu)(dQ_{\min}/dx)=-\tau V_s^2(dQ_{\min}/dx). \qquad \text{(III, 9)}$$

This result is identical with the diffusion equation derived previously from the low frequency limit of the wave equation, namely, the diffusion equation:

$$\dot{\rho}_n/\tau=V_{s0}^2\nabla\cdot\nabla\rho_n. \qquad \text{(III, 10)}$$

The only difference is in the value of V_s.