On the Treatment of Quantum Electrodynamics without Eliminating the Longitudinal Field

NING HU*
Cornell University, Ithaca, New York

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It is shown in the present paper that the supplementary condition in quantum electrodynamics can be satisfied identically without eliminating the longitudinal components of the electromagnetic field, provided certain rules on the summation over the polarizations of the photons are observed. The result is shown to be identical with that obtained by the usual method in which the longitudinal field is eliminated. This simplifies greatly the calculation since no separate treatment of longitudinal and transverse fields is needed.

RECENTLY Tomonaga,[1] Schwinger,[2] and others have succeeded in formulating quantum electrodynamics in a completely co-variant form. As in customary formulations, these authors showed that the longitudinal components of the electromagnetic field can be eliminated, and thus insured the identical satisfaction of the supplementary condition. As a consequence of this elimination, the application of the theory becomes very much involved, since one has to deal with the transverse field and the coulomb interaction separately for each virtual transition. On the other hand, it has been pointed out by Feynman[3] that one can treat the longitudinal and transverse fields on the same footing by dealing with four different polarizations of virtual photons and also obtain the correct results. The Coulomb interaction is thus replaced by virtual emission and re-absorption of longitudinal photons. This simplifies tremendously the calculations since no separate treatment of longitudinal and transverse field is needed. It is the aim of the present paper to derive this new method from the formalism of Tomonaga and Schwinger. Only the case of spin-$\frac{1}{2}$ particles interacting with the electromagnetic field is considered in the present paper.

It is perhaps of interest to mention here that as long as one is concerned only with the rigorous solution of the problem, the supplementary condition can be included in the initial condition[4] and will then be automatically preserved after the initial state of the system has been specified correctly. However, if the Schrödinger equation is solved by perturbation methods, a special treatment (such as the elimination of the longitudinal field) is needed to insure that the supplementary condition is satisfied by the approximate solution at all stages of approximation.

The first part of the present paper deals with the elimination of longitudinal field which is essentially not much different from that given by Koba, Tati, and Tomonaga.[1] The second part consists essentially of undoing the elimination, preserving, however, the identical satisfaction of the supplementary condition achieved in the first part. It seems that there should be a more direct way of obtaining the same results without following the round-about way of eliminating and then restoring the longitudinal field. The present derivation, however, shows the connection between Feynman's method and the conventional one in which the longtudinal field is eliminated.

In the following calculations the same notation as that of Schwinger's paper is used except that here in addition the unit $\hbar=c=1$ is in use. Therefore no detailed explanation of terms and concepts is given here that have been explained clearly in Tomonaga and Schwinger's papers. The state vector $\Psi[\sigma]$ of the system of electronic and electromagnetic fields satisfies the following generalized Schrödinger equation:

$$i\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=-j_\mu(x)A_\mu(x)\Psi[\sigma] \quad (1)$$

and the supplementary condition

$$\left[\frac{\partial A_\mu(x)}{\partial x_\mu}-\int_{\sigma(x)}D(x-x')j_\mu(x')d\sigma_\mu'\right]\Psi[\sigma]=0. \quad (2)$$

$A_\mu(x)$ can be expanded into the following Fourier integral

$$A_\mu(x)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int[A_\mu(k)e^{i(kx)}+A_\mu^*(k)e^{-i(kx)}]\frac{d^3k}{k} \quad (3)$$

where $(kx)=k_0x_0-k_1x_1-k_2x_2-k_3x_3$, $x_\mu=(t,\mathbf{r})$ being the coordinates of any point in the space-time, and $A_\mu(x)=(\phi(t,\mathbf{r}),\,\mathbf{A}(t,\mathbf{r}))$ is the four-vector potential of the electromagnetic field. $\sigma(x)$ denotes an arbitrary space like surface passing through the point x_μ, and $\Psi[\sigma]$ is understood as a functional of σ. The meaning of other notations can easily be found in Schwinger's paper. $A_\mu(k)$ and $A_\mu^*(k)$ are q-numbers satisfying the following

* Now at University of Wisconsin.
[1] S. Tomonaga, Proc. Theor. Phys. **1**, 27 (1946); Koba, Tati, and Tomonaga, Proc. Theor. Phys. **2**, 101, 198 (1947).
[2] J. Schwinger, Phys. Rev. **73**, 416 (1948); **74**, 1439 (1948).
[3] R. P. Feynman, Phys. Rev. **74**, 939, 1430 (1948); and several papers still in the course of preparation.
[4] This follows from the fact that the supplementary condition and the Schrödinger equation are represented by operators commuting with each other. For earlier non-covariant derivation of the result see E. Fermi, Rev. Mod. Phys. **4**, 131 (1932); W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1944) §6.

commutation relations ($[A,B]=AB-BA$)

$$[A_0(k),A_0^*(k')]=-\tfrac{1}{2}k\delta(\mathbf{k}-\mathbf{k}')$$
$$[\mathbf{A}(k),\mathbf{A}^*(k')]=\tfrac{1}{2}k\delta(\mathbf{k}-\mathbf{k}'). \qquad (4)$$

Introduce for each value of k_μ the following coordinate system: x_4 along any time-like direction, x_1 and x_2 perpendicular to each other and to both x_4 and k_μ and x_3 forms the fourth perpendicular direction which is parallel to $\mathbf{k}$, the space component of k_μ. Let $n_\mu^{(\alpha)}$ be unit vectors in the direction of $x_\alpha(\alpha=0,1,2,3)$ referred to any fixed coordinate system in the space-time. For different values of k_μ, the directions of $n_\mu^{(\alpha)}(k)$ are different. Introduce further the following two zero vectors:

$$n_\mu^{(\kappa)}(k)=1/\sqrt{2}[n_\mu^{(3)}(k)+n_\mu^{(0)}(k)],$$
$$n_\mu^{(l)}(k)=1/\sqrt{2}[-n_\mu^{(3)}(k)+n_\mu^{(0)}(k)]. \qquad (5)$$

It can easily be verified that $(n^{(l)}n^{(l)})=(n^{(\kappa)}n^{(\kappa)})=0$ and $(n^{(\kappa)}n^{(l)})=-1$. $A_\mu(k)$ can then be written

$$A_\mu(k)=n_\mu^{(1)}A^{(1)}(k)+n_\mu^{(2)}A^{(2)}(k)+A_\mu'(k), \qquad (6)$$

where

$$A_\mu'(k)=n_\mu^{(0)}A^{(0)}(k)+n_\mu^{(3)}A^{(3)}(k)$$
$$=n_\mu^{(\kappa)}A^{(\kappa)}(k)+n_\mu^{(l)}A^{(l)}(k), \qquad (7)$$

and in coordinate space

$$A_\mu(x)=A_\mu^{(1)}(x)+A_\mu^{(2)}(x)+A_\mu'(x), \qquad (6a)$$

and

$$A_\mu'(x)=A_\mu^{(3)}(x)+A_\mu^{(0)}(x)$$
$$=A_\mu^{(\kappa)}(x)+A_\mu^{(l)}(x), \qquad (7a)$$

where

$$A_\mu^{(\alpha)}(x)=\int n_\mu^{(\alpha)}(k)[A^{(\alpha)}(k)e^{i(kx)}+A^{*(\alpha)}(k)e^{-i(kx)}]\frac{d^3k}{k}\ (\alpha=0,1,2,3,\kappa,l). \qquad (8)$$

It can be shown from (4) that $A^{(\alpha)}(k)$ satisfy the following commutation relations

$$[A^{(\alpha)}(k),A^{*(\alpha)}(k')]=\tfrac{1}{2}g^{(\alpha)}k\delta(\mathbf{k}-\mathbf{k}')\ (\alpha=0,1,2,3),$$
$$[A^{(l)}(k),A^{*(\kappa)}(k')]=-\tfrac{1}{2}k\delta(\mathbf{k}-\mathbf{k}'),$$
$$[A^{*(l)}(k),A^{(\kappa)}(k')]=-\tfrac{1}{2}k\delta(\mathbf{k}-\mathbf{k}'), \qquad (8a)$$

(other commutators vanish) where $g^{(\alpha)}=-1$ when $\alpha=0$ and $g^{(\alpha)}=1$ when $\alpha=1,2,3$. This difference in sign of $g^{(0)}$ is important since it follows from (8a) that $A^{*(\alpha)}(k)$ should be considered as emission operator and $A^{(\alpha)}(k)$ as absorption operator when $\alpha\neq 0$, whereas $A^{(0)}(k)$ should be considered as emission operator and $A^{*(0)}(k)$ as absorption operator. This important fact should be borne in mind when applying Feynman's method.

The supplementary condition (2) will be satisfied if $\Psi[\sigma]$ is of the following form

$$\Psi[\sigma]=e^{i\chi[\sigma]}\Omega[\sigma], \qquad (9)$$

where $\Omega[\sigma]$ is independent of $A^{(\kappa)}(k)$ and $A^{*(\kappa)}(k)$, i.e.,

$$A^{(l)}(k)\Omega[\sigma]=0,\quad A^{*(l)}(k)\Omega[\sigma]=0, \qquad (9a)$$

and

$$\chi[\sigma]=\int_{-\infty}^{\sigma} j_\mu(x')A_\mu^{(\kappa)}(x')dx'. \qquad (10)$$

It should be noted that χ contains operators at point x' which is on the past side of σ, whereas $\Omega[\sigma]$ on which χ is operated depends only on σ. One way of specifying the operation is to integrate χ with respect to x'. One finds readily

$$\chi=\int_\sigma j_\mu(x)B(x)d\sigma_\mu, \qquad (11)$$

where

$$\frac{\partial}{\partial x_\mu}B(x)=A_\mu^{(\kappa)}(x), \qquad (12)$$

or

$$B(x)=1/(2\pi)^{\frac{3}{2}}\int i[A^{(\kappa)}(k)e^{i(kx)}-A^{*(\kappa)}(k)e^{-i(kx)}]d^3k/k^2. \qquad (12a)$$

Then χ contains only operators referring to σ. An alternative way to specify the meaning of the operation is to follow the procedure of Dirac[5] who defines

$$j_\mu(x')A_\mu^{(\kappa)}(x')\Omega[\sigma]=j_\mu(x')A_\mu^{(\kappa)}(x')S[\sigma;\sigma']\Omega[\sigma'] \qquad (13)$$

where σ' is a space like surface in the past side of σ and passing through the point x'. $S[\sigma;\sigma']$ is an unitary operator transforming a solution $\Omega[\sigma']$ of (1) at time σ' to the same solution $\Omega[\sigma]$ at time σ. $S[\sigma,\sigma']$ may contain other operators such as $j_\mu(x'')A_\mu^{(\alpha)}(x'')$, therefore care must be taken not to disturb the original order of different operators unless they commute. These two ways of specifying the meaning of operation are, of course, equivalent—at least in the following calculation only such properties of χ are used for which the equivalence of (10) with (11) is obvious.

Since from (6)–(8)

$$\partial A_\mu/\partial x_\mu=\partial A_\mu^{(l)}/\partial x_\mu=1/(2\pi)^{\frac{3}{2}}\int i[A^{(l)}(k)e^{i(kx)}-A^{*(l)}(k)e^{-i(kx)}]d^3k/k,$$

and

$$\chi=\int_{-\infty}^{\sigma} j_\mu(x')dx'\int n_\mu^{(\kappa)}(k)[A^{(\kappa)}(k)e^{i(kx')}+A^{*(\kappa)}(k)e^{-i(kx')}]d^3k/k.$$

It follows from Eq. (8)

$$\partial A_\mu/\partial x_\mu\chi=\int_{-\infty}^{\sigma} j_\mu(x')dx'\cdot\tfrac{1}{2}\int k_\mu[e^{i(k,x-x')}+e^{-i(k,x-x')}]d^3k/k$$
$$=\int_{-\infty}^{\sigma} j_\mu(x')\frac{\partial}{\partial x_\mu}D(x-x')dx'$$
$$=\int_\sigma j_\mu(x')D(x-x')d\sigma_\mu', \qquad (14)$$

[5] P. A. M. Dirac, Rev. Mod. Phys. 17, 195 (1945).

where $(k,x-x')=k_0(x_0-x_0')-\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')$. If $\Omega[\sigma]$ is independent of $A^{(\kappa)}(k)$, and $A^{*(\kappa)}(k)$, or in other words, independent of $A^{(\kappa)}(x)$ then using (9a)

$$(\partial A_\mu/\partial x_\mu)e^{i\chi}\Omega[\sigma]=\int_\sigma j_\mu(x')D(x-x')d\sigma_\mu' e^{i\chi}\Omega[\sigma]. \quad (15)$$

This verifies that the supplementary condition is satisfied identically if (9a) is satisfied, i.e., if $\Omega[\sigma]$ is independent of $A^{(\kappa)}(x)$. It must now be shown that an $\Omega[\sigma]$ independent of $A^{(\kappa)}(x)$ is allowed by the Schrödinger Eq. (1).

It follows from (10)

$$e^{-i\chi}i(\delta/\delta\sigma)e^{i\chi}\Omega[\sigma]=i(\delta/\delta\sigma)\Omega[\sigma] - j_\mu(x)A_\mu^{(\kappa)}(x)\Omega[\sigma]. \quad (15a)$$

Since

$$\int_\sigma [j_\mu(x'),\ j_\alpha(x)]d\sigma_\mu'=0, \quad (15b)$$

$j_\mu(x')d\sigma_\mu'$ may be considered as commuting with $j_\alpha(x)$. Then with the help of the commutation relations (8a), it follows

$$e^{-i\chi}j_\mu A_\mu^{(\alpha)}e^{i\chi}=j_\mu A_\mu^{(\alpha)} \quad (\alpha=1,2,\kappa). \quad (16)$$

The following identity can easily be verified to be true

$$e^{-i\chi}j_\mu A_\mu^{(l)}e^{i\chi}=j_\mu A_\mu^{(l)}-[\chi,j_\mu A_\mu^{(l)}] +\frac{1}{2!}[\chi,[\chi,j_\mu A_\mu^{(l)}]] -\frac{1}{3!}[\chi,[\chi,[\chi,j_\mu A_\mu^{(l)}]]]+\cdots \quad (17)$$

From Eqs. (4), (12), and (15b), it follows, after some calculation,

$$[\chi,j_\mu A_\mu^{(l)}]=-\int j_\mu(x)j_\alpha(x')\mathfrak{L}_\mu(x-x')d\sigma_\alpha',$$

$$\mathfrak{L}_\mu(x-x')=\int n_\mu^{(l)}(k)[e^{i(k,x-x')}+e^{-i(k,x-x')}]\frac{d^3k}{k}. \quad (18)$$

It also follows from Eqs. (15b) and (18) that

$$[\chi,[\chi,j_\mu A_\mu^{(l)}]]=0.$$

Therefore, all the terms except the first two on the right-hand side of Eq. (17) vanish identically. Application of Eq. (17) on $\Omega[\sigma]$ gives further

$$e^{-i\chi}j_\mu A_\mu^{(l)}e^{i\chi}\Omega[\sigma]=-j_\mu A_\mu^{(l)}\chi\Omega[\sigma], \quad (19)$$

on account of the condition (9a). Finally the Schrödinger equation for $\Omega[\sigma]$ obtained by inserting (9) into (1) and using (15a), (16) and (19) is given by

$$i\frac{\delta\Omega[\sigma]}{\delta\sigma(x)}=-\sum_{r=1,2} j_\mu A_\mu^{(r)}\Omega[\sigma]+j_\mu A_\mu^{(l)}\chi[\sigma]\Omega[\sigma], \quad (20)$$

or, on inserting Eq. (18),

$$i\frac{\delta\Omega[\sigma]}{\delta\sigma(x)}=-\sum_{r=1,2} j_\mu A_\mu^{(r)}\Omega[\sigma] +\int_\sigma j_\mu(x)j_\alpha(x')\mathfrak{L}_\mu(x-x')d\sigma_\alpha'\Omega[\sigma]. \quad (21)$$

It is seen that the operators on the right hand side of (21) are independent of $A^{(\kappa)}(k)$ and $A^{*(\kappa)}(k)$. This furnishes the proof that (9a) is always satisfied if it is satisfied initially.

Equation (20) can be written in the following form

$$i\frac{\delta\Omega[\sigma]}{\delta\sigma(x)}=H(x)\Omega[\sigma], \quad (22)$$

where

$$H(x)=H_1(x)+H_2(x), \quad (23)$$

$$H_1(x)=-\sum_{r=1,2} j_\mu(x)A_\mu^{(r)}(x), \quad (24)$$

$$H_2(x)=-(-i)^2 j_\mu(x)A_\mu^{(l)}(x)\int_\sigma j_\alpha(x')B(x')d\sigma_\alpha'. \quad (25)$$

Following Schwinger and Dyson,[6] the solution of (22) can be written down immediately:

$$\Omega[\sigma]=S[\sigma]\Omega_0, \quad (26)$$

$$S[\sigma]=1-i\int_{-\infty}^{\sigma} H_1(x_1)dx_1 +(-i)^2\int_{-\infty}^{\sigma} H(x_1)dx_1\int_{-\infty}^{\sigma_1} H(x_2)dx_2+\cdots. \quad (27)$$

The general term of Eq. (27) can be written:

$$(-i)^n\int^{\sigma} H(x_1)dx_1\int H(x_2)dx_2\cdots\int H(x_n)dx_n, \quad (28)$$

where the limits of the integrals have been omitted, it being understood that the integration of each factor $H(x)$ is from $-\infty$ to a space-like surface passing through the variable point x'' of the factor $H(x'')$ just in front of $H(x)$, with the order of the H's according to the order of time.

On inserting Eq. (23), the integrand of (28) can further be expanded into different combinations of products of factors H_1 and H_2. Consider first the terms of (28) which contain H_1 only and which contain no virtual emission and re-absorption of photons. (This does not exclude the processes in which a transverse virtual photon was emitted but has not been absorbed at time σ.) One of these terms can be written in the following form

$$I_n=(-i)^n\int^{\sigma} H_1(x_1)dx_1 \times\int H_1(x_2)dx_2\cdots\int H_1(x_n)dx_n. \quad (29)$$

[6] F. J. Dyson, Phys. Rev. **75**, 486 (1949).

Let $I_n(i)$ be another term of (27) constructed from (29) by adding a factor $\int H_2(x')dx'$ between $\int H_1(x_i)dx_i$ and $\int H_1(x_{i+1})dx_{i+1}$:

$$I_n(i)=(-i)^{n+2}\int^{\sigma} H_1(x_1)dx_1\int H_1(x_2)dx_2\cdots$$
$$\times\int H_1(x_i)dx_i\int j_\mu(x')A_\mu{}^{(l)}(x')dx'$$
$$\times\int_{\sigma(x')} j_\alpha(x'')B(x'')d\sigma_\alpha''\int H_1(x_{i+1})dx_{i+1}\cdots$$
$$\times\int H_1(x_n)dx_n. \quad (30)$$

Now since $[A_\mu{}^{(l)}(x'),H_1(x)]=0$ and because of Eq. (15b), the added factor commutes with all the other factors of Eq. (30). By using the following identity

$$\int_{\sigma_i} j_\mu(x')B(x')d\sigma_{i\mu}'=\int_{\sigma\, i+1}^{\sigma\, i} j_\mu(x')A_\mu{}^{(\kappa)}(x')dx'$$
$$-\int_{\sigma\, i+1} j_\mu(x')B(x')d\sigma'_{i+1,\,\mu}, \quad (31)$$

it can easily be seen that

$$\int_{\sigma(x')} j_\alpha(x')B(x')d\sigma'\int_{-\infty}^{\sigma(x')} H_1(x_{i+1})dx_{i+1}$$
$$=\int_{-\infty}^{\sigma(x')} j_\alpha(x'')A_\alpha{}^{(\kappa)}(x'')dx''\int_{-\infty}^{\sigma(x'')} H_1(x_{i+1})dx_{i+1}$$
$$+\int_{-\infty}^{\sigma(x)} H_1(x_{i+1})dx_{i+1}\int_{\sigma\, i+1} j_\alpha(x')B(x')d\sigma_\alpha'. \quad (32)$$

This relation can be used repeatedly to push the surface integral in Eq. (30) to the right side of the last factor $\int H_1(x_n)dx_n$ and then replace it by a volume integral Eq. (10) integrated from minus infinity to σ_n. The final expression of Eq. (30) will contain no surface integral and is given by

$$I_n(i)=\sum\int^{\sigma} H_1(x_1)dx_1\int H_1(x_2)dx_2\cdots$$
$$\times\int H_1(x_i)dx_i\int j_\mu(x')A^{(l)}(x')dx'$$
$$\times\int H_1(x_{i+1})dx_{i+1}\cdots\int j_\mu(x'')A^{(\kappa)}(x'')dx''\cdots$$
$$\times\int H_1(x_n)dx_n, \quad (33)$$

where $\sum$ means summation over all possible positions of $\int j_\mu(x'')A^{(\kappa)}(x'')dx''$ between $\int H_1(x_i)dx_i$ and $\int H_1(x_n)dx_n$.

It will now be shown that the contribution due to the factors

$$\int j_\mu(x')A_\mu{}^{(l)}(x')dx', \quad \int j_\mu(x'')A_\mu{}^{(\kappa)}(x'')dx'' \quad (34)$$

in Eq. (33) represent just the virtual exchange of longitudinal photons between σ' and σ''. Since from (7a)

$$A_\mu{}^{(l)}(x)+A_\mu{}^{(\kappa)}(x)=A_\mu{}^{(0)}(x)+A_\mu{}^{(3)}(x),$$

it follows from the commutation relation that

$$A_\mu{}^{(l)}(x')A_\alpha{}^{(\kappa)}(x'')+A_\mu{}^{(\kappa)}(x')A_\alpha{}^{(l)}(x'')$$
$$=A_\mu{}^{(0)}(x')A_\alpha{}^{(0)}(x'')+A_\mu{}^{(3)}(x')A_\alpha{}^{(3)}(x'')$$
$$=\sum_{s=0,3} A_\mu{}^{(s)}(x')\cdot\sum_{t=0,3} A_\alpha{}^{(t)}(x'').$$

$A_\mu{}^{(\kappa)}(x)A_\alpha{}^{(l)}(x'')$ vanishes identically because of the condition (9a). The above equation becomes

$$A_\mu{}^{(l)}(x')A_\alpha{}^{(\kappa)}(x'')=\sum_{s=0,3} A_\mu{}^{(s)}(x')\cdot\sum_{t=0,3} A_\alpha{}^{(t)}(x''). \quad (35)$$

Therefore, the two factors (34) can be replaced by

$$\int j_\mu(x')\sum_s{}' A_\mu{}^{(s)}(x')dx',$$
$$\int j_\mu(x'')\sum_s{}' A_\mu{}^{(s)}(x'')dx'' \quad (36)$$

respectively, where the summation $\sum'$ is from 3 to 4. Now from (27), new groups of terms can be picked out which contain only H_1-factors and represent the same processes as (29), but in addition there is a virtual exchange of transverse protons between $\sigma(x')$ and $\sigma(x'')$. It is easily seen that the contribution of this new group of terms is just to replace $\sum'$ by another summation over all polarizations from 0 to 3.

So far only terms involving one pair of virtual emission and reabsorption have been considered. Let $I_n(i,j)$ be another term of (27) constructed from (29) by adding two H_2-factors in the following way:

$$I_n(i,j)=(-i)^{n+4}\int^{\sigma} H_1(x_1)dx_1\cdots\int H_1(x_j)dx_j$$
$$\times\int j_\nu(x_b')A_\nu{}^{(l)}(x_b')dx_b'\int_{\sigma(x_b')} j_\beta(x_b'')B(x_b'')d\sigma_{b\beta}''\cdots$$
$$\times\int H_1(x_i)dx_i\int j_\mu(x_a')A_\mu{}^{(l)}(x_a')dx_a'$$
$$\times\int_{\sigma(x_a')} j_\alpha(x_a'')B(x_a'')d\sigma_{a\alpha}'\cdots\int H_1(x_n)dx_n. \quad (37)$$

It follows from Eq. (15b) that $j_\mu(x_a')dx_a'$ and $j_\nu(x_b')dx_b'$ commute with each other and also with all the other current vectors in Eq. (37). The operators $A_\mu^{(l)}(x_a')$, $B(x_a'')$, $A_\nu^{(l)}(x_b'')$ and $B(x_b'')$ commute with all the other operators in Eq. (37). It is seen from the foregoing result that the B's represent the emission operators of the longitudinal photon* and the $A_\mu^{(l)}$'s represent the corresponding absorption operator. The order of these operators in (37) shows that $A_\mu^{(l)}(x_a')$ absorbs only the photon emitted by $B(x_a'')$ and $A_\nu^{(l)}(x_b')$ absorbs only the photon emitted by $B(x_b'')$. If the order of $B(x_b'')$ and $A_\mu^{(l)}(x_a')$ is changed, new possibility arises that the photon emitted by $B(x_b'')$ may be absorbed by $A_\mu^{(l)}(x_a')$, which process is not included in (37). This situation is avoided if the photon emitted by $B(x_b'')$ is marked so that it cannot be absorbed by $A_\mu^{(l)}(x_a')$. This amounts formally to considering the pairs $A_\mu^{(l)}(x_a')$, $B(x_a'')$ and $A_\nu^{(l)}(x_b')$, $B(x_b'')$ as emission and absorption operators of two different kinds of particles a and b. This is the same as the rule introduced in Dyson's paper to group the operators into pairs with operators in different pairs commuting with each other. According to this rule the factor $N^{\frac{1}{2}}$ and $(N+1)^{\frac{1}{2}}$ of the matrix elements of the emission and absorption operators under customary convention should now be replaced by unity, where N is the number of photons in the same state. This reduction in the strength of the operators will be compensated exactly by the increase in the number of precesses due to different permutation of the order of the operators.

If the new rule is used, the argument that leads from (30) to (33) and (34) can be applied without any modification in the present case. Therefore Eq. (37) becomes

$$I_n(i,j)=(-i)^{n+4}\sum_p\int^\sigma H_1(x_1)dx_1\int H_1(x_2)dx_2\cdots$$
$$\times\int H'(x_a')dx_a'\cdots\int H'(x_a'')dx_a''\cdots$$
$$\times\int H'(x_b')dx_b'\cdots\int H'(x_b'')dx_b''\cdots\int H_1(x_n)dx_n,$$
$$H'(x)=\sum_{s=1}^{4}j_\mu(x)A_\mu^{(s)}(x),\quad(38)$$

where the summation $\sum_p$ is taken over all the positions of the four $H'(x)$ factors among all the other $H_1(x)$-factors. It should be understood that $H'(x_a')$ and $H'(x_a'')$ form a pair of virtual emission and absorption operators and $H'(x_b')$ and $H'(x_b'')$ form another pair whereas all the other H_1-factors contribute no virtual emission-reabsorption process.

* The vacuum has been considered as free from longitudinal photons. Actually from (9a), vacuum should be considered as having an indefinite number of longitudinal photons. However this makes no difference on final results as far as virtual processes are concerned.

The above result can easily be extended to the case when more than two pairs of virtual emission-absorption occur. It will not be difficult to see that terms constructed from Eq. (29) in this way will eventually include all the terms of Eq. (27). The general procedure of solving Eq. (22) can now be stated as follows:

First solve the following Schrödinger equation

$$\frac{\delta\Omega[\sigma]}{\delta\sigma(x)}=H(x)\Omega[\sigma],\quad H(x)=-\sum_s j_\mu A_\mu^{(s)},\quad(39)$$

with the summation unspecified and obtain the following formal solution

$$\Omega[\sigma]=S[\sigma]\Omega_0$$

$$S[\sigma]=1-i\int_{-\infty}^{\sigma}H(x_1)dx_1+(-i)^2\int_{-\infty}^{\sigma}H(x_1)dx_1$$
$$\times\int_{-\infty}^{\sigma_1}H(x_2)dx_2+\cdots+(-i)^n\int_{-\infty}^{\sigma}H(x_1)dx_1$$
$$\times\int_{-\infty}^{\sigma_1}H(x_2)dx_2\cdots\int_{-\infty}^{\sigma_{n-1}}H(x_n)dx_n+\cdots.\quad(40)$$

Then the summation is specified as taken over all transverse and longitudinal polarizations for all the processes in which a photon has been emitted and then reabsorbed, and taken only over the transverse polarizations for all the other processes, namely, the processes in which either a real photon is emitted or absorbed or a virtual photon has been emitted but has not yet been absorbed. Equation (40) satisfies the supplementary condition identically even when a finite number of terms of $S[\sigma]$ are used as in the usual perturbation theory.

It should be noted that although emission and reabsorption of longitudinal photons can be considered as actually taken place in the *past*, the explicit expression of $S[\sigma]$ does not depend on the variables of the virtual longitudinal photons, as should be the case according to the condition (9a). In other words, no virtual longitudinal photon should be considered as existing at the *present*. On the other hand since the processes in which a transverse photon has been emitted but has not yet been absorbed are also included in Eq. (40), the expression of $S[\sigma]$ should therefore also depend on the variables of these virtual transverse photons. If however σ is put equal to infinity in Eq. (40), this remaining distinction between longitudinal and transverse virtual photons will disappear completely. $S[\infty]$ is just the representation used by Feynman.

It should further be pointed out that, strictly speaking, Ω_0 in (40) should satisfy the following condition as required by Eq. (9a):

$$A^{(l)}(k)\Omega_0=0,\quad A^{*(l)}(k)\Omega_0=0.\quad(41)$$

Now Ω_0 can be put into the following form

$$\Omega_0=\Omega_0'\Omega_0''$$

where Ω_0' depends only on the variables of the transverse photons and Ω_0'' depends only on those of the longitudinal photons belonging to the vacuum. Equation (41) becomes

$$A^{(l)}(k)\Omega_0''=0,\quad A^{*(l)}(k)\Omega_0''=0. \tag{42}$$

Since, as pointed out in the footnote before, the solution Eq. (40) does not depend on the exact form of Ω_0'', the factor Ω_0'' can simply be omitted from Eq. (40). This means that instead of choosing Ω_0 to satisfy Eq. (41), one can simply consider it as independent of the variables of the longitudinal photons. In other words, vacuum is considered as containing no longitudinal photons at all. This is just the convention adopted in the usual treatment in which the longitudinal field has been replaced by the Coulomb interaction.

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The Bose-Einstein Condensation for Thin Films

M. F. M. OSBORNE
Naval Research Laboratory, Washington, D. C.
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The distinction is made between a condensation temperature in the thermodynamic sense, where the thermodynamic functions have a discontinuity, and an abrupt accumulation (in the statistical mechanical sense) of particles in the ground state of a system. The criterion for the latter is that an integral will not approximate one term of a sum. The problem of a thin film of gas is solved to illustrate this distinction. For films thicker than a critical thickness a condensation occurs at the same temperature as that for bulk material. The thinner films are effectively two dimensional and have an abrupt accumulation temperature lower than the bulk condensation temperature. Films of liquid helium are briefly considered.

INTRODUCTION

CONSIDERABLE interest has been provoked by London's proposal[1] that the λ-phenomenon of helium is a Bose-Einstein condensation, or accumulation of a finite fraction of all the particles in the ground state of a potential hole of macroscopic dimensions. Tisza[2] has extended this suggestion to discuss helium in terms of the hydrodynamics of a two-fluid system, and more recently Ogg[3] has proposed that phenomena which he has observed in metal-ammonia solutions and possibly also the phenomenon of superconductivity might be explained by the condensation of a gas of electron pairs formed from the free electrons of the material. It is the purpose of this paper to examine in some detail the conditions under which these condensations occur and to discuss some examples of physical interest. The particular examples we shall discuss are: (1) Three-dimensional gas of uncharged particles in a thin film, (2) Two-dimensional gas, and in a subsequent paper (3) Gas of charged particles in a magnetic field, neglecting electrostatic interactions.

Condensation and Accumulation Temperatures

In classical thermodynamics a condensation temperature implies a discontinuity in one of the thermodynamic functions which describe the system. Such discontinuities are also described by phase changes of first, second, third order, depending on whether the function or one of its higher derivatives becomes discontinuous.

From the standpoint of statistical mechanics we shall define an accumulation temperature T_0 as a temperature at which a finite fraction of all particles begin to accumulate in just one state or set of states with the same energy. By "finite" fraction we mean less than, but not a great deal less than unity, or very much greater than $1/N$, N being the total number of particles. We can speak of an "abrupt" accumulation if the fraction in a given state changes from "small" to "finite" in a small temperature range $\Delta T\ll T_0$. Such finite fractions can usually be expressed as $1-(T/T_0)^n$, $T<T_0$. We speak of a gradual accumulation if it occurs over a range $\Delta T\simeq T_0$. Only in the first case would it be appropriate to speak of a well-defined accumulation temperature, in the second case one might speak of an accumulation temperature band. Consider the expression for N, the total number of particles in a Bose-Einstein assembly.

$$N=\sum_{k,\,l,\,m}1/(\exp(E(k,l,m)/kT+\alpha)-1)$$
$$=\sum_{k,\,l,\,m}f(E(k,l,m),\alpha,T), \tag{1}$$

where k, l, m are quantum numbers specifying the level of energy $E(k, l, m)$ and α is defined implicitly as a function of T by this relation.

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[2] L. Tisza, Phys. Rev. **72**, 838 (1947).
[3] R. A. Ogg, Preliminary Report to ONR, "Electronic Processes in Liquid Dielectric Media. The Constitution of Metal Ammonia Solutions." April 11, 1948, p. 20.