PHYSICAL REVIEW VOLUME 76, NUMBER 3 AUGUST 1, 1949

Diode Space Charge for Any Initial Velocity and Current

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(Received April 4, 1949)

The space charge equation is solved for any initial velocity which is the same for all electrons and for any realizable current, for (a) the plane diode, (b) the cylindrical diode. For the plane diode it is shown that all cases can be obtained from a single pair of master curves. For the cylindrical diode a series valid for large distances from the axis is obtained first, the form of which shows that the complete group of solutions form a bounded set. Next three solutions valid for small distances from the axis are obtained by means of which the first set of solutions can be extended. In addition to master curves representing these solutions, curves are plotted for special values of the initial velocity and current.

IN a recent paper[1] the authors presented a new solution of the space charge equation for the cylindrical diode which converges better the larger the ratio of the distance r from the axis of the diode to the radius a of the cathode. The solution was limited, however, to the case of saturation current and negligible initial velocity.

The present paper extends the solution to the case of any realizable current and any radial initial velocity u which is the same for all the ions (electrons). This initial velocity may be that of emission from a hot cathode, or the velocity with which the electrons pass through a screen or grid coaxial with the cathode after having been accelerated. The analysis is restricted to the case where all the ions at any one point are moving in the same direction.

Previous consideration has been given to this problem by several authors. Of these Crank, Hartree, Ingham, and Sloane[2] have presented the most extensive treatment. They employed mechanical integration by means of a differential analyzer and obtained portions of some of the complete set of curves for which analytical formulas are developed in the present paper.

Although the solution of the space charge equation for the plane diode, consisting of two parallel plane electrodes from one of which ions are emitted, has been given by others[3] for the case of arbitrary initial velocity and arbitrary current, we shall preface our discussion of the cylindrical diode, composed of coaxial cylindrical electrodes from the inner of which ions are emitted, by a brief treatment of the plane diode. This is undertaken for the purpose of making the theory of the cylindrical diode more easily comprehensible by putting the simpler solution for the plane diode in a form analogous to that found most convenient for the cylindrical diode, and also of pointing out the fact, apparently overlooked by previous writers, that all the curves showing the potential as a function of distance from the emitting electrode, are obtainable from a single pair of master curves.

THE PLANE DIODE

Let x be the distance measured perpendicular to the parallel planes of the electrodes, u the initial velocity of the ions emerging normally from the electrode specified by the smaller value of x, V the excess of the potential at any point over that of the emitting electrode, ρ the charge density, j the current density and e/m the ratio of charge to mass of the ions. Then, in Heaviside-Lorentz units,

$$\dot{x}^2 = -(2e/m)V + u^2, \tag{1}$$

$$d^2V/dx^2 = -\rho, \tag{2}$$

$$j = \rho\dot{x}. \tag{3}$$

Eliminating ρ and $\dot{x}$, and putting

$$\chi \equiv (m/2ej)^{\frac{2}{3}}\{-(2e/m)V + u^2\}, \tag{4}$$

we find that χ satisfies the differential equation

$$d^2\chi/dx^2 = \chi^{-\frac{1}{2}}, \tag{5}$$

of which the first integral is

$$d\chi/dx = 2\{\chi^{\frac{1}{2}} + C\}^{\frac{1}{2}}. \tag{6}$$

Evidently $d\chi/dx$ is proportional to the accelerating field. Since $\chi^{\frac{1}{2}}$ is essentially positive, the electric field can become zero and change sign only when the constant C is zero or negative. The curve we obtain by making $C=0$ we shall call the *limiting curve.* It represents saturation current for the case of zero initial velocity.

Integrating a second time we have

$$\chi^{\frac{3}{2}} - 3C\chi + 4C^3 = (9/4)x^2,$$

where we have made the second constant of integration zero by suitably choosing the position of the origin of x. Finally we solve this cubic in $\chi^{\frac{1}{2}}$, putting

$$\left.\begin{aligned} &k \equiv 8C^3/9x^2, \\ &\Psi \equiv \frac{\chi^{\frac{1}{2}}}{(3x/2)^{\frac{2}{3}}} = (2m/9ejx^2)^{\frac{1}{3}}\{-(2e/m)V + u^2\}^{\frac{1}{2}}. \end{aligned}\right\} \tag{7}$$

[1] L. Page and N. I. Adams, Phys. Rev. **68**, 126 (1945).
[2] Crank, Hartree, Ingham, and Sloane, Proc. Phys. Soc. (London) **51**, 952 (1939).
[3] Plato, Kleen and Rothe, Zeits. f. Physik **101**, 509 (1936); W. Kleen and H. Rothe, Zeits. f. Physik **104**, 711 (1937); B. Salzberg and A. V. Haeff, RCA Rev. **II**, 336 (1937–8); Fry, Samuel and Shockley, Bell Sys. Tech. J. **17**, 49 (1938).

This gives

$$\left.\begin{aligned}\Psi&=\frac{1}{2^{\frac{1}{3}}}\big[k^{\frac{1}{3}}+\{1-k+(1-2k)^{\frac{1}{2}}\}^{\frac{1}{3}}\\&\qquad+\{1-k-(1-2k)^{\frac{1}{2}}\}^{\frac{1}{3}}\big],\quad k\leqslant\tfrac{1}{2};\\\Psi&=(k/2)^{\frac{1}{3}}[1+2\cos(\theta/3)],\quad \tan\theta\equiv\frac{(2k-1)^{\frac{1}{2}}}{1-k},\\&\qquad k\geqslant\tfrac{1}{2}.\end{aligned}\right\}\tag{8}$$

When $k\rightarrow 0$, due either to $C\rightarrow 0$ or $x\rightarrow\infty$, $\Psi\rightarrow 1$. Thus, for the limiting curve, $\Psi=1$. Note that both k and Ψ are dimensionless ratios.

Actually we are not interested in Ψ but in the effective potential

$$W\equiv V+\left(-\frac{mu^2}{2e}\right)=\left(\frac{9ej}{2m}\right)^{\frac{2}{3}}\left(-\frac{m}{2e}\right)x^{4/3}\Psi^2. \tag{9}$$

If we put $\pm|k|^{\frac{1}{3}}=g/x$, where we choose the upper sign when k is positive and the lower when k is negative so that g has the same sign as C,

$$W=\left(\frac{9ej}{2m}\right)^{\frac{2}{3}}\left(-\frac{m}{2e}\right)g^{4/3}\left(\frac{\Psi}{k^{\frac{1}{3}}}\right)^2. \tag{10}$$

Consequently, if we plot $(\Psi/k^{\frac{1}{3}})^2$ against $1/|k|^{\frac{1}{3}}=x/|g|$, we have a pair of *master curves*, one for positive k and the other for negative k, from which all possible potential curves can be obtained by multiplying the ordinates by

$$(9ej/2m)^{\frac{2}{3}}(-m/2e)g^{4/3}$$

and the abscissas by $|g|$.

The curves on Fig. 1 are specified in terms of the values of the constant g, the one marked 0 being the limiting curve. The master curve for positive k is that for which $g=1$ and the master curve for negative k that for which $g=-1$.

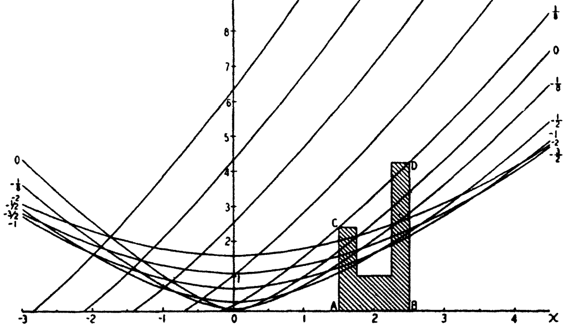

FIG. 1. Generalized potential curves for plane diodes.

Since

$$d/dx(x^{4/3}\Psi^2)=(2m/9ej)^{\frac{2}{3}}(-2e/m)dV/dx, \tag{11}$$

the electric field at any point is proportional to the slope of the curve.

Consider first the emission of ions with zero initial velocity. If ions are emitted in unlimited numbers, the potential as a function of x is given by the portion of the limiting curve 0 extending from the origin to the right. On the other hand, if saturation conditions do not exist, the potential is given by one of the curves with positive g, the distance from the emitting electrode being that from the intercept of the curve with the axis of abscissas. In the first case the field at the emitting electrode is zero, but in the latter case an accelerating field is present.

Next consider the general case in which the distance x_B-x_A between the emitting electrode A and the collecting electrode B and the effective potentials W_A and W_B at the two electrodes are given, as well as the current density j. From (9) we calculate the value of $x^{4/3}\Psi^2$ at each electrode and then cut a card (Fig. 1) so that the distance AB is equal to x_B-x_A and the ordinates AC and BD are equal to the values of $x^{4/3}\Psi^2$ at the electrodes A and B respectively. Sliding the card along the axis of abscissas, keeping AB on this axis, we find a curve (such as the curve $\frac{1}{8}$ in Fig. 1) which passes through the two points C and D. This curve specifies the distribution of potential between the two electrodes. If no curve passes through C and D, the current density j has been taken too large for realization. If C and D lie on the limiting curve 0 on opposite sides of the origin, a virtual cathode is present at the origin, whereas if they lie on a curve of negative k on opposite sides of the origin, a decelerating field exists in the neighborhood of the emitting electrode and the potential passes through a minimum in the region between the electrodes.

THE CYLINDRICAL DIODE

Here we have two coaxial cylindrical electrodes of radii a and b $(b>a)$, the inner of which emits ions (electrons) with the radial velocity u. The inner electrode may be a screen or grid through which electrons pass with velocity u, of course, instead of an actual electrode. Let r be the distance from the common axis of the electrodes and j_l the current per unit length. Then, if the remaining symbols have the same significance as in the previous case, we have, in Heaviside-Lorentz units,

$$\dot{r}^2=-(2e/m)V+u^2, \tag{12}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right)=-\rho, \tag{13}$$

$$j_l=2\pi\rho r\dot{r}. \tag{14}$$

Eliminating ρ and $\dot{r}$, and putting[4]

$$\left.\begin{aligned} \Psi^2 &\equiv (4\pi m/9ej_l r)^{\frac{2}{3}}\{-(2e/m)V+u^2\}, \\ z &\equiv \tfrac{2}{3}\log(r/a), \\ W &\equiv V+\left(-\frac{mu^2}{2e}\right)=\left(\frac{9ej_l a}{4\pi m}\right)^{\frac{2}{3}}\left(-\frac{m}{2e}\right)\left(\frac{r}{a}\right)^{\frac{2}{3}}\Psi^2, \end{aligned}\right\} \quad (15)$$

we find that Ψ satisfies the non-linear differential equation

$$\Psi[d^2/dz^2(\Psi^2)+2d/dz(\Psi^2)+\Psi^2]=1. \quad (16)$$

Since z does not appear explicitly in this equation, Ψ must be a function of $z+\alpha$, where α is an arbitrary constant.

First we look for solutions of (16) which are valid for large z, then for solutions which are valid for small z. The first we shall refer to as "far" formulas, the second as "near" formulas.

FAR FORMULA

Clearly $\Psi=1$ is a solution of (16). Hence we suspect a solution of the form

$$\Psi=1+(\xi+\eta)+(a_{20}\xi^2+a_{11}\xi\eta+a_{02}\eta^2) +(a_{30}\xi^3+a_{21}\xi^2\eta+a_{12}\xi\eta^2+a_{03}\eta^3)+\cdots.$$

By substituting in (16) we find that a real solution of this form exists provided

$$\xi=\exp\{-[(z+\alpha')+i(2)^{-\frac{1}{2}}(z+\alpha'')]\}$$

and η is the complex conjugate of ξ.

The coefficients of the first six terms (in addition to the term unity) of this solution were calculated. The first five of these were obtained in fractional form and verified by substituting back in the differential equation. While the sixth term was calculated only in decimal form, it was checked by independent recalculation. The solution can best be expressed in terms of

$$\phi\equiv(2)^{-\frac{1}{2}}(z+\alpha''), \quad c\equiv\exp[-(z+\alpha')]. \quad (17)$$

Then

$$\begin{aligned} \Psi=1&+c[2.00000\cos\phi]+c^2[-1.09091\cos 2\phi -0.51426\sin 2\phi] \\ &+c^3[1.00000\cos 3\phi+0.82496\sin 3\phi +0.22727\cos\phi+0.93210\sin\phi] \\ &+c^4[-1.06618\cos 4\phi-1.35505\sin 4\phi -0.41786\cos 2\phi-2.11568\sin 2\phi-0.00957] \\ &+c^5[1.14794\cos 5\phi+2.29087\sin 5\phi +0.33420\cos 3\phi+4.34538\sin 3\phi -0.02034\cos\phi+1.94913\sin\phi] \\ &+c^6[-1.07982\cos 6\phi-3.93115\sin 6\phi +0.60929\cos 4\phi-8.65663\sin 4\phi +0.89535\cos 2\phi-6.43121\sin 2\phi -0.03878] \\ &+\cdots. \end{aligned} \quad (18)$$

[4] The function Ψ used here is the cube root of the function g used in reference 1.

TABLE I. Values of the coefficients a_1, a_2, etc.

ϕ	a_1	a_2	a_3	a_4	a_5	a_6
$0, \pi$	± 2.00000	-1.09091	± 1.22727	-1.49361	± 1.46180	0.38605
$\pi/8,\ 9\pi/8$	± 1.84776	-1.13503	± 1.71152	-3.15611	± 6.54679	-14.62605
$\pi/4,\ 5\pi/4$	± 1.41421	-0.51426	± 0.69603	-1.05907	± 1.76859	-3.14813
$3\pi/8,\ 11\pi/8$	± 0.76537	0.40775	∓ 0.29146	0.14494	± 0.00518	-0.10611
$\pi/2,\ 3\pi/2$	0.00000	1.09091	± 0.10714	-0.65789	∓ 0.10538	0.75498
$5\pi/8,\ 13\pi/8$	∓ 0.76537	1.13503	± 1.38235	0.42687	∓ 1.48285	-2.76477
$3\pi/4,\ 7\pi/4$	∓ 1.41421	0.51426	± 1.78883	3.17229	± 3.89341	1.85198
$7\pi/8,\ 15\pi/8$	∓ 1.84776	-0.40775	± 0.52620	2.54602	± 7.20718	17.34181

This is a complete solution since it contains the two independent constants α' and α''. Since it is periodic in ϕ, we see that the solutions form a bounded set. We have computed the function Ψ for values of ϕ equal to the first sixteen positive integral multiples of $\pi/8$. If we designate the coefficient of c by a_1, that of c^2 by a_2, and so on, we obtain Table I, where, in each row, the upper sign goes with the first value of ϕ and the lower sign with the second.

Since the position of the origin of z is immaterial, we may, without any loss of generality, keep the constant α' in (17) fixed in plotting curves showing Ψ^2 as a function of z, increasing α'' by $(2)^{\frac{1}{2}}\pi/8=0.55536$ in passing from one curve to the next of higher ordinal number, so that when ϕ has increased by 2π we will have come back to the curve with which we started. In Fig. 2 we have made $\alpha'=1.11072$ and numbered the curves so that $\alpha''=N(2)^{\frac{1}{2}}\pi/8$, where N is the ordinal number of the curve. Under these circumstances, even though the series converge less rapidly the smaller z, the convergence is sufficiently good to give an accuracy of one percent for most of the curves at $z=0$. Hence, if we can obtain a near formula which converges satisfactorily for negative values of z, we can use it to extend into this region the curves given for positive values of z by the far formula. The near formula, must, of course, be a complete solution of (16), containing the two independent arbitrary constants necessary to enable us to fit both the function and its derivative. Actually, it has not been found possible to obtain a single near formula which converges sufficiently well to enable us to extend by means of it all of the sixteen curves obtained from the far formula. On the contrary, it has been found necessary to develop three different near formulas. Before discussing them, however, we shall give the formula for the limiting curve which represents the saturation current for zero initial velocity.

LIMITING CURVE

The limiting curve representing saturation current for zero initial velocity has been discussed in reference 1. In terms of our present more convenient notation, it is specified for small and moderate values of z by the near formula

$$\Psi=(1.5z)^{\frac{2}{3}}[1-0.400{,}000z+0.097{,}500z^2 -0.015{,}212z^3+0.001{,}799z^4-0.000{,}184z^5 +0.000{,}014z^6+\cdots]. \quad (19)$$

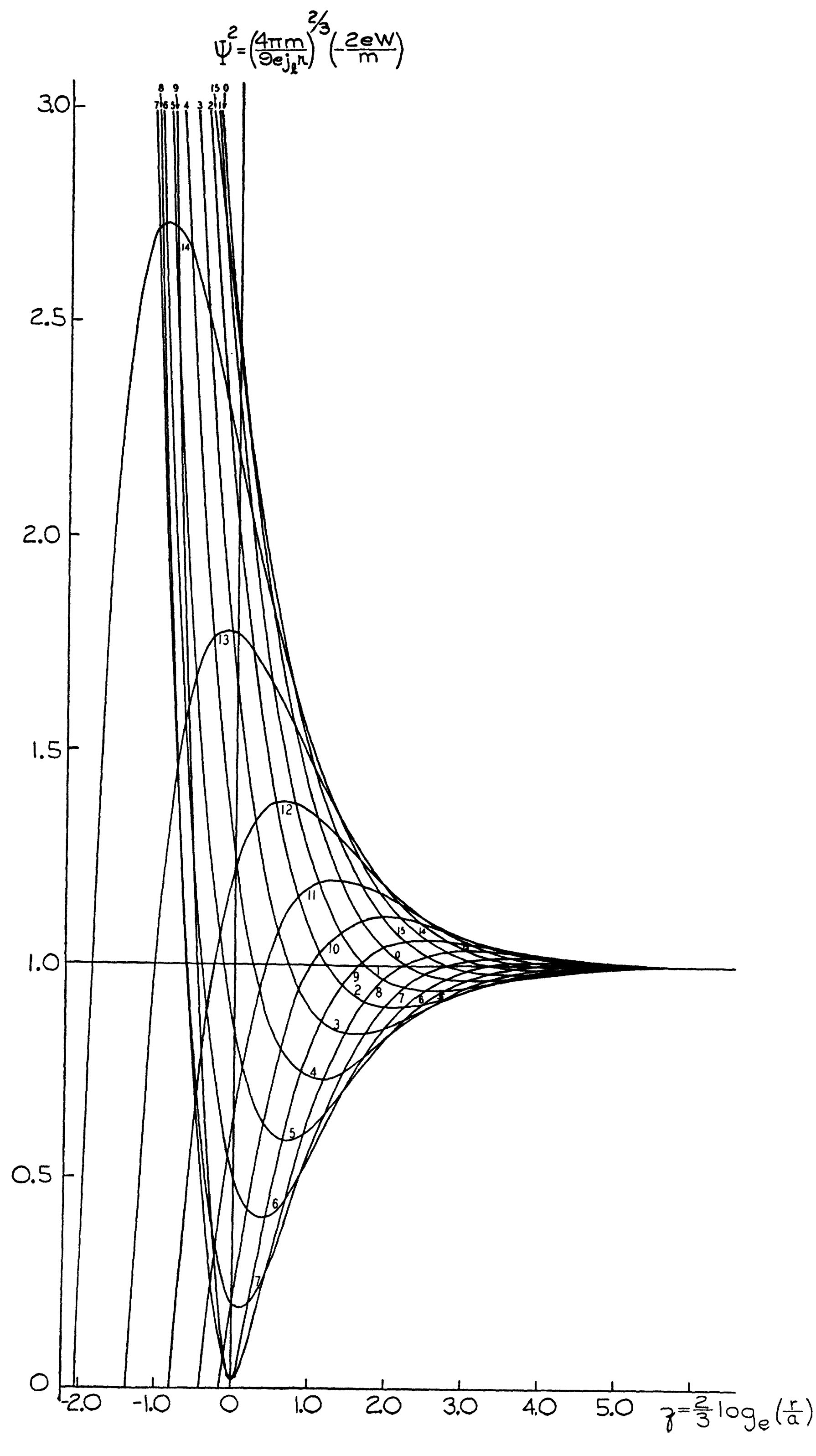

FIG. 2. Complete solution of space charge equation for cylindrical diode.

To fit this curve the constants in the far formula must be given the values

$$\phi = 40°.5142z + 202°.83, \quad \alpha' = 1.12086, \tag{20}$$

so that

$$\begin{aligned}\Psi = 1 &+ e^{-z}[0\cdot 65200 \cos\phi] - e^{-2z}[0.11594 \cos 2\phi \\ &\qquad + 0.05465 \sin 2\phi] \\ &+ e^{-3z}[0.03465 \cos 3\phi + 0.02858 \sin 3\phi \\ &\qquad + 0.00787 \cos\phi + 0.03229 \sin\phi] \\ &- e^{-4z}[0.01204 \cos 4\phi + 0.01530 \sin 4\phi \\ &\qquad + 0.00472 \cos 2\phi + 0.02390 \sin 2\phi + 0.00011] \\ &+ e^{-5z}[0.00423 \cos 5\phi + 0.00844 \sin 5\phi \\ &\qquad + 0.00123 \cos 3\phi + 0.01600 \sin 3\phi \\ &\qquad - 0.00007 \cos\phi + 0.00718 \sin\phi] \\ &- e^{-6z}[0.00130 \cos 6\phi + 0.00472 \sin 6\phi \\ &\qquad - 0.00073 \cos 4\phi + 0.01039 \sin 4\phi \\ &\qquad - 0.00107 \cos 2\phi + 0.00772 \sin 2\phi + 0.00005] \\ &+ \cdots. \end{aligned} \tag{21}$$

The following values show how good the fit is and how well the series is converging:

z	From Near Formula (19) Ψ	$d\Psi/dz$	From Far Formula (21) Ψ	$d\Psi/dz$
0.55536	0.7131		0.7132	
1.11072	0.9242	0.2299	0.9242	0.2299
1.66608	1.0049		1.0048	

NEAR FORMULA I

The most obvious near formula for curves other than the limiting curve is a simple power series in z of the form

$$\Psi = A[1 + a_1(z-\alpha) + a_2(z-\alpha)^2 + a_3(z-\alpha)^3 + \cdots], \tag{22}$$

where α is the value of z at which the curve described by this formula is to be fitted to the specified curve described by the far formula. Evidently A is the value of the function and Aa_1 the value of its derivative at $z = \alpha$.

As might be expected, a power series formula does not converge satisfactorily in regions where Ψ is small compared with unity, but is very satisfactory where Ψ is the order of magnitude of unity or greater. It was used to extend curves 15, 0, 1, 2, 3, 4, 5, 6. In some cases the fit was secured from the values of the function and its derivative at one point, in others from the values of the function at two nearby points, and in the case of curve 6 it was necessary to fit twice, using two different values of α. The coefficients for the curves specified above are given in Table II.

TABLE II. Values of the coefficients for the curves specified above.

Curve	α	A	a_1	a_2	a_3	a_4	a_5	a_6
15	0.5554	1.3458	−0.2492	0.0708	−0.0002	−0.0023		
0	0.5554	1.3171	−0.2854	0.1041	−0.0089	−0.0012		
1	0.5554	1.2525	−0.3083	0.1381	−0.0167	−0.0015		
2	0.5554	1.1574	−0.3139	0.1759	−0.0257	−0.0039		
3	0.0000	1.2875	−0.4651	0.2241	−0.0216	−0.0001		
4	0.0000	1.115	−0.5014	0.3016	−0.0207	−0.0078		
5	0.0000	0.916	−0.5240	0.4620	−0.0133	−0.0461	−0.0429	0.0041
6	0.0000	0.709	−0.5925	0.8684	0.0559	−0.2621	−0.2756	0.0912
6	−0.5554	1.119	−0.9160	0.4249	0.0334	0.0599		

The accuracy is least, of course, for curve 6. To give an indication of the goodness of the fit, and also of the convergence of the series used, we note the following values for curve 2:

z	Ψ from (22) Near Formula	Ψ from (18) Far Formula
0.0000	1.427	1.425
0.5554	1.1574	1.1574
1.1107	1.0130	1.0129
1.6661	0.9575	0.9571

NEAR FORMULA II

For curves near the limiting curve representing saturation for zero initial velocity, it should be possible to develop a solution in the form of a power series in a parameter γ, which reduces to (19) when γ becomes zero. After some labor the following near solution was found. Let

$$\beta \equiv \frac{9}{4}\frac{(z-\alpha)^2}{\gamma^3}, \quad w \equiv (\beta - 2)\{(\beta-2)^2 - 3\}. \tag{23}$$

Then

$$\begin{aligned}\Psi = \gamma[&\{w^{1/9} + 1 + w^{-1/9} - (1/9)w^{-17/9} \\ &\qquad + (1/9)w^{-19/9} + \cdots\} \\ &- 1.5(z-\alpha)\{0.26667w^{1/9} + 0.53333 \\ &\qquad + 0.80000w^{-1/9} - 0.53333w^{-5/9} + 0.53333w^{-7/9} \\ &\qquad - 0.53333w^{-11/9} + 0.53333w^{-13/9} + \cdots\} \\ &+ 2.25(z-\alpha)^2\{0.04333w^{1/9} + 0.11651 \\ &\qquad + 0.22222w^{-1/9} + 0.01873w^{-2/9} + 0.17302w^{-4/9} \\ &\qquad - 0.40029w^{-5/9} + \cdots\} \\ &- 3.375(z-\alpha)^3\{0.00451w^{1/9} + 0.01469 \\ &\qquad + 0.03371w^{-1/9} - 0.00853w^{-2/9} \\ &\qquad + 0.00748w^{-3/9} + \cdots\} \\ &+ 5.0625(z-\alpha)^4\{0.00036w^{1/9} + 0.00139 \\ &\qquad + 0.00375w^{-1/9} + \cdots\} \\ &- 7.5938(z-\alpha)^5\{0.000{,}024w^{1/9} \\ &\qquad + 0.000{,}110 + \cdots\} \\ &+ 11.391(z-\alpha)^6\{0.000{,}001w^{1/9} + \cdots\} \\ &+ \cdots]. \end{aligned} \tag{24}$$

This formula was used to extend curves 7, 8, 9. The fitting of the near formula to a specified curve described by the far formula was extremely laborious, since the only feasible method was to fit the one curve to the other at two points by trial and error. The following are the values found for the constants α and γ:

Curve	α	γ
7	0.018	−0.450
8	0.010	−0.250
9	0.000	−0.0085

TABLE III. Values of the coefficients in Eq. (30).

y	B	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
0.05	0.1357	0.6279	−0.3450	0.3095	−0.3316	0.3905	−0.4879		
0.10	0.2154	0.5926	−0.3062	0.2588	−0.2616	0.2903	−0.3417	0.4193	−0.5298
0.20	0.3420	0.5255	−0.2416	0.1814	−0.1626	0.1603	−0.1676	0.1827	−0.2050
0.30	0.4481	0.4674	−0.1911	0.1276	−0.1017	0.0891	−0.0829	0.0804	−0.0802
0.40	0.5429	0.4161	−0.1514	0.0900	−0.0639	0.0498	−0.0412	0.0356	−0.0317
0.50	0.6300	0.3707	−0.1201	0.0636	−0.0402	0.0279	−0.0206	0.0159	−0.0126
0.60	0.7113	0.3311	−0.0957	0.0453	−0.0255	0.0158	−0.0104	0.0073	−0.0051
0.70	0.7884	0.2959	−0.0764	0.0323	−0.0163	0.0090	−0.0053	0.0033	−0.0021
0.80	0.8618	0.2648	−0.0612	0.0231	−0.0104	0.0052	−0.0027	0.0015	−0.0009
0.90	0.9322	0.2373	−0.0491	0.0166	−0.0067	0.0029	−0.0014	0.0007	−0.0004
1.00	1.0000	0.2129	−0.0395	0.0120	−0.0043	0.0017	−0.0007	0.0003	−0.0002
1.10	1.0657	0.1914	−0.0318	0.0087	−0.0028	0.0010	−0.0004	0.0002	−0.0001
1.30	1.1912	0.1554	−0.0209	0.0046	−0.0012	0.0004	−0.0001		
1.40	1.2515	0.1406	−0.0171	0.0034	−0.0008	0.0002	−0.0001		
1.80	1.4797	0.0947	−0.0077	0.0010	−0.0002				
2.00	1.5874	0.0786	−0.0053	0.0006	−0.0001				

For curve 9, the nearest to the limiting curve, the fit is as follows:

z	Ψ from (24) Near Formula	Ψ from (18) Far Formula
0.5554	0.7078	0.7077
1.1107	0.921	0.922
1.6661	1.003	1.004

This is very satisfactory, but the fit is not as close for curves 7 and 8.

An unfortunate defect of the solution (24) is that the series does not converge for very small values of w.

Curve 9 lies so close to the limiting curve that it is impossible to distinguish the one from the other on the graph of Fig. 2 except in the immediate neighborhood

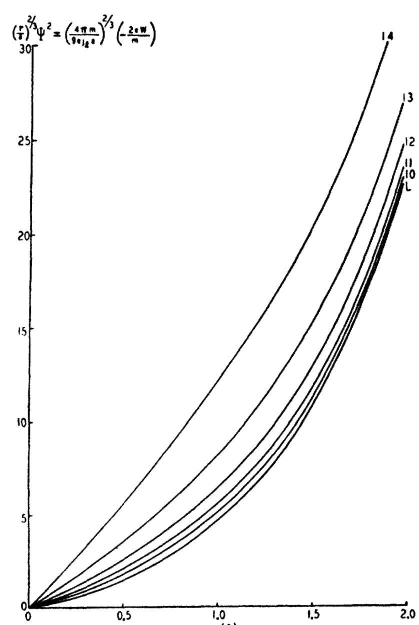

FIG. 3. Generalized potential curves for cylindrical diode with zero emission velocity.

of the origin, where the limiting curve has been indicated by a broken line. At the origin the limiting curve makes tangential contact with the axis of abscissas.

NEAR FORMULA III

It follows from the definition (15) of Ψ that this quantity becomes infinite when there is no emission of ions. In such a case (16) reduces to

$$d^2/dz^2(\Psi^2)+2d/dz(\Psi^2)+\Psi^2=0, \tag{25}$$

of which the complete solution is

$$\Psi^2=(A_0+B_0z)e^{-z}=\{A_0+\tfrac{2}{3}B_0\log(r/a)\}(a/r)^{\frac{2}{3}}, \tag{26}$$

giving for the effective potential

$$W=A+B\log(r/a)$$

in agreement with the elementary expression for the potential between the plates of a cylindrical condenser.

It would be expected, therefore, that for small emission currents, a series solution should exist the first term of which is the equivalent of (26). If, for convenience in writing this solution, we put $y\equiv z-\alpha$, we find such a solution in the form

$$\begin{aligned}\Psi=&cy^{\frac{1}{2}}e^{-y/2}+\frac{1}{c^2}ye^{y}[0.66667p]\\&+\frac{1}{c^5}y^{\frac{3}{2}}e^{5y/2}[-0.38889p^2s_2]+\frac{1}{c^8}y^2e^{4y}[0.37037p^3s_3]\\&+\frac{1}{c^{11}}y^{5/2}e^{11y/2}[-0.42130p^4s_4]+\frac{1}{c^{14}}y^3e^{7y}[0.52675p^5s_5]\\&+\frac{1}{c^{17}}y^{7/2}e^{17y/2}[-0.69882p^6s_6]+\frac{1}{c^{20}}y^4e^{10y}[0.96571p^7s_7]\\&+\frac{1}{c^{23}}y^{9/2}e^{23y/2}[-1.37499p^8s_8]+\cdots\end{aligned} \tag{27}$$

where

$$\begin{aligned}p\equiv&1-\frac{2}{5}(3y)+\frac{3}{5\cdot7}(3y)^2-\frac{4}{5\cdot7\cdot9}(3y)^3+\cdots\\&+(-1)^n\frac{n+1}{5\cdot7\cdot9\cdots(2n+3)}(3y)^n+\cdots,\end{aligned} \tag{28}$$

$$s_2\equiv1-0.003{,}675y^2-0.001{,}471y^3-0.000{,}226y^4+0.000{,}027y^5+0.000{,}021y^6+\cdots,$$

$$s_3\equiv1-0.004{,}286y^2-0.001{,}590y^3-0.000{,}182y^4+0.000{,}049y^5+\cdots,$$

$$s_4\equiv1-0.006{,}940y^2-0.002{,}421y^3-0.000{,}206y^4+\cdots,$$

$s_5 \equiv 1-0.010{,}205y^2-0.003{,}438y^3+\cdots,$

$s_6 \equiv 1-0.013{,}118y^2+\cdots,$

$s_7 \equiv 1+\cdots,$

$s_8 \equiv 1+\cdots.$

Putting

$$t\equiv(y^{\frac{1}{2}}e^{3y/2}/c^3) \tag{29}$$

we can write (27) in the form

$$\Psi=(B/t^{\frac{1}{3}})[1+b_1t+b_2t^2+b_3t^3+\cdots] \tag{30}$$

where the coefficients of t are, of course, functions of y although not of c. In Table III we list the values of the coefficients for various values of y.

We have used (27) to extend the curves 10, 11, 12, 13, 14. Since the fitting of the function and its derivative had to be done by trial and error it was very laborious. It should be noted that, after a hump which becomes higher the greater the ordinal number of the curve, the curve drops abruptly to the axis of abscissas and ends there. While the accuracy of the fit is diminished as the ordinal number of the curve decreases, this lessened accuracy is compensated by the fact that the extended portion of the curve is shortened. The values of α and c found for these curves are as follows:

Curve	α	c
10	-0.14	1.031
11	-0.41	1.295
12	-0.81	1.568
13	-1.38	1.955
14	-2.06	2.584

DISCUSSION OF RESULTS

The solutions obtained are shown graphically by the curves in Fig. 2. Just as in the case of the plane diode, a sliding card (Fig. 1) can be used to pick out the potential curve for any assigned effective cathode potential W_a and anode potential W_b, any assigned anode to cathode ratio b/a, and any realizable current j_l per unit length. In this case, however, since the abscissa is $z=(\frac{2}{3})\log(r/a)$, the distance AB on the card must be made equal to

$$z_b-z_a=\tfrac{2}{3}\log(b/a) \tag{31}$$

and, since the ordinate is Ψ^2, the heights AC and BD must be made equal to the values of

$$(4\pi m/9ej_lr)^{\frac{2}{3}}\{-(2e/m)W\} \tag{32}$$

at each of the two electrodes, in accord with (15). It should be noted that the factor in the first parenthesis of (32) is of the dimensions of the reciprocal of the cube of a velocity and that in the second of the dimensions of the square of a velocity, so that the whole expression is a pure number.

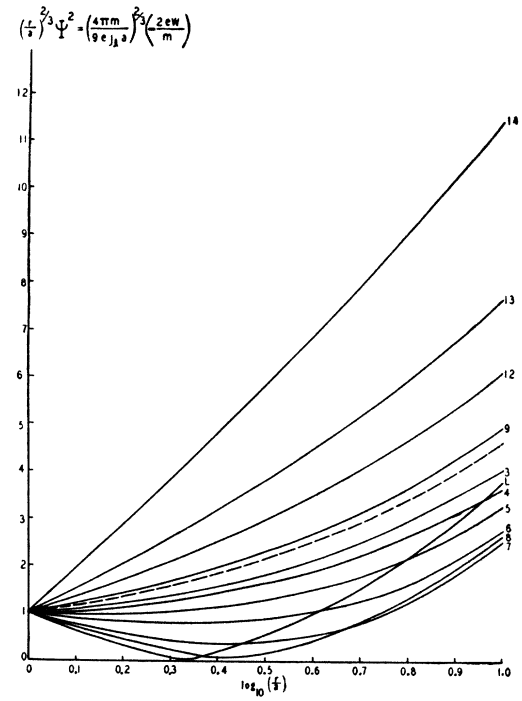

FIG. 4. Generalized potential curves for cylindrical diode with finite emission velocity.

If, when AB is slid along the axis of abscissas, the points C and D lie on the limiting curve on opposite sides of the origin, a virtual cathode is present. If no curve on which they both lie can be found, the value assumed for the current is too large.

Since

$$\frac{dV}{dr}=(2/3r)\left(\frac{9ej_lr}{4\pi m}\right)^{\frac{2}{3}}\left(-\frac{m}{2e}\right)\left[\frac{d}{dz}(\Psi^2)+\Psi^2\right], \tag{33}$$

the electric field is proportional to the slope of the curve only where $\Psi^2=0$. The limiting curve is the only one which meets the axis of abscissas with zero slope.

Any desired information regarding the cylindrical diode can be obtained from the master curves in Fig. 2. If, for instance, it is desired to study the characteristics of the cylindrical diode for the emission of ions with zero velocity but in varying numbers, we draw a new graph from the curves numbered 14 to 9 inclusive, taking as origin in each case the point where the curve intersects the axis of abscissas, and using $\log_{10}(r/a)$ rather than z as abscissa and $(r/a)^{\frac{2}{3}}\Psi^2$ rather than Ψ^2 as ordinate. The new ordinates, then, are directly proportional to the effective potential W, and the slope of the curve is proportional to the electric field at any point. Such curves have been drawn in Fig. 3, the numbering of the curves corresponding to that of the master curves of Fig. 2. The limiting curve is indicated by L. If, now, it is desired to apply the graph to the specific case of a tube in which the ratio b/a is, say, one hundred, the figure must be cut off on the right by a vertical line of abscissa two. Then the ordinates of the points where the successive curves cut this limiting line are, for a given emission current, proportional to the potentials on the anode, each curve giving the distribution of potential between the electrodes for the assigned potential difference and assigned emission current.

Figure 4 shows a set of curves differing from those of Fig. 3 only in that an initial velocity corresponding to

$$W_a = (9ej_l a/4\pi m)^{\frac{2}{3}}(-m/2e)$$

is assumed. The broken curve representing $(r/a)^{\frac{2}{3}}$ has been included for the sake of comparison. Now we have cases in which the field near the cathode retards the ions, causing a dip in the potential curve, and leading, in the extreme case, to the formation of a virtual cathode. While each curve represents a possible potential distribution, the theory developed here does not, of course, specify the experimental conditions under which a specified curve may be realized or the sequence in which different curves are reached when one of the parameters, such as the potential on the anode, is altered.

PHYSICAL REVIEW VOLUME 76, NUMBER 3 AUGUST 1, 1949

The Work Function of Copper

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(Received April 7, 1949)

The work functions and aging characteristics of fourteen copper surfaces have been determined by measurement of their contact differences of potential with respect to barium reference surfaces of known work function. Measurement was by the retarding potential method in tubes sealed from the pumps and gettered with vaporized barium. The copper surfaces were prepared by subjecting Hilger's "spectroscopic standard" copper to forty vacuum fusions followed by fractional distillation, redistillation of the fractions, and condensation of the vapor on glass. The barium films were prepared by a similar, standardized technique which yields surfaces reproducible and constant to 0.01 ev or better. The time interval between the deposition and initial measurement of the metal films was of the order of ten seconds, obtained conveniently with a tube in which the target is returned to the measuring position automatically after deposition of a metal film upon it.

The copper surfaces, formed from successive fractions of the distillate, showed marked variability of the initial work function and marked drift toward lower work function in films 1–4, progressive improvement in films 5–8 and, finally, good reproducibility and negligible drift in films 9–14 inclusive. The range of variation in films 1–8 coincides almost exactly with the range of divergent values reported in the literature. Copper evidently retains dissolved gases, in free or combined form, with extraordinary tenacity and these contaminants, evaporated with the copper and reabsorbed by the condensing film, appear to be responsible for the variations observed in films 1–8. The work function value given by films 9–14 is 4.46±0.03 ev, and this value is to be taken as representative of the present work.

THIS report describes further results in a series of studies of the external work functions of the pure metals. The more significant features of the experimental method, described in detail in previous reports,[1] may be summarized briefly as follows: (1) The surfaces to be measured are prepared by condensation of the metal vapor on glass after an exhaustive outgassing which includes repetitive vacuum fusion, fractional distillation, and, when necessary, redistillation of the selected fractions. (2) Each work function determination is based on the measurement of an extended series of different surfaces, a procedure which either establishes the reproducibility of the measurements or reveals inadequacies in the outgassing technique. (3) The first measurement on each surface is obtained immediately after its deposition and within a time interval which is short compared to the time required to form an equilibrium layer of contaminating gas on the surface. This procedure supplies a definite means for estimating the degree of contamination of the surface at the time of measurement.[2] (4) The work function of each surface is determined by measuring its difference of potential with respect to a barium reference surface prepared by a technique which has been found[1] to give surfaces consistently reproducible and constant to 0.01 ev or better.

A large number of unpublished measurements on copper, taken in this laboratory in 1939–41 with tubes of several different types, gave work function values which varied in random fashion over a range of about 0.5 ev without showing any trend toward better reproducibility. These results were obtained after outgassing treatments which were comparable in all respects to those which have given[1] excellent results on silver, zinc, magnesium, and lithium. In the present work, outgassing of the copper has been carried to an extreme and our earlier tube designs so modified as to simplify the manipulations required for obtaining short deposition-

* Assisted by the ONR under Contract M6-ori-167, Task I.
[1] P. A. Anderson, Phys. Rev. 59, 1034 (1941); 57, 122 (1940) and earlier reports cited therein.

[2] Reference 1, 1940 paper.