On a Generalization of the Lambda-Limiting Process and the Non-Uniqueness in the Removal of Divergence Difficulties in the Quantum Theory of Elementary Particles*

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A generalized process of the λ-limiting type is established, leading to the elimination of the divergences in the integrals of both the classical and quantum field theories. As a result, however, the process is not unique, and by a suitable choice of the convergence factor the observable quantities, such as the proper mass, may be given any finite value. This indicates either that limiting processes are not capable of solving the basic difficulties in the theory of fundamental particles, or that the necessity for additional postulates of the nature of selection rules.

As is well known, Wentzel[1] discovered the existence of a certain limiting process which leads to classical field equations without divergence difficulties. Dirac[2,3] gave the quantum version of this process by substituting the usual commutation relations for the Fourier components of the field

$$[a(k), a^+(l)] = \delta_{kl}, \tag{1}$$

by the relations

$$\begin{gathered}[a(k), a^+(l)] = \delta_{kl} \cos\bar{k}\bar{\lambda}, \\ \bar{\lambda}^2 = |\boldsymbol{\lambda}|^2 - \lambda_4^2 < 0.\end{gathered} \tag{2}$$

The divergence troubles[4] which arise as a result of quantization of the field are not removed by the Wentzel-Dirac λ-process. Dirac[5] therefore suggested that negative probabilities be considered, but these also lead to difficulties in the interpretation of the theory.[3]

It will be shown in the present note that the λ-process can be generalized in such a manner that the divergences in the quantum, as well as in the classical integrals may be removed. A very distinct non-uniqueness then appears, however, which also affects the classical integrals. As in the case of the usual λ-process (2) we shall modify the commutation relationships (1) to read

$$[a(k), a^+(l)] = \delta_{kl} D(k).$$

In order to determine the general expression for D we use the requirement that the function Δ_2[6–8] should not be introduced into the commutation relations for the field operators. The function D must therefore have the form:

$$D = \sum_n A(\bar{\lambda}_n) \cos\bar{k}\bar{\lambda}_n. \tag{3}$$

The $\bar{\lambda}_n$ are arbitrary four-dimensional vectors. If they satisfy the condition $\bar{\lambda}_n^2 = |\boldsymbol{\lambda}_n|^2 - \lambda_{4n}^2 < 0$ the classical divergencies will vanish (see Eq. (9) below). The quantities $A(\bar{\lambda}_n)$ are subjected to the condition $\sum_n A(\bar{\lambda}_n) = 1$. In order to eliminate the uneven[5] divergences we shall attempt to choose the quantities $A(\lambda_n)$ in such a manner that the various terms in the sum (3) lead to results which ultimately cancel out mutually. As an illustration, consider the following example. Let us determine the factor D which removes all even and the first uneven divergence. For this purpose we consider the expression

$$D = (4/3) \cos\bar{k}\bar{\lambda} - \tfrac{1}{3} \cos\tfrac{1}{2}\bar{k}\bar{\lambda}. \tag{4}$$

We change to a coordinate system in which only $\lambda_4 \neq 0$. Further assume (for the sake of simplicity) that the rest mass μ of the field quanta equals zero.

$$\int_0^\infty Dkdk = (4/3)\int_0^\infty kdk \cos k\lambda_4 - \tfrac{1}{3}\int_0^\infty kdk \cos\tfrac{1}{2}k\lambda_4 = -(4/3)\lambda_4^{-2} + \tfrac{1}{3}(2/\lambda_4)^2 = 0.$$

(See Pauli,[3] page 202.)

In the general form, if we wish to remove the first p uneven divergences, A must satisfy the equalities (if all $\bar{\lambda}_n$ have only λ_4 different from zero):

$$\sum_n A(\bar{\lambda}_n)/\lambda_{4n}^{2m} = 0. \quad M = 1, 2, \cdots, \tfrac{1}{2}(p+1). \tag{5}$$

* This paper, received more than two years ago, is now published although, due to circumstances beyond our control, it has not been possible for the author to have read proof.

1 B. Wentzel, Zeits. f. Physik **86**, 479, 635 (1933); **87**, 726 (1934).

2 P. A. M. Dirac, Ann. de l'Inst. Poincaré **9**, 13 (1939).

3 W. Pauli, Rev. Mod. Phys. **15**, 175 (1943).

4 I. Waller, Zeits. f. Physik **62**, 673 (1930).

5 P. A. M. Dirac, Proc. Roy. Soc. **A180**, 1 (1942).

6 W. Pauli, Rev. Mod. Phys. **13**, 203 (1941).

7 F. Bloch, Physik. Zeits. Sowjetunion **5**, 301 (1934).

8 M. Markov, J. Phys. **10**, 333 (1946).

Thus without introducing negative probabilities and by a simple generalization of the λ-process the uneven divergences can be eliminated from the theory.

However, a very distinct non-uniqueness of the theory immediately appears. The condition that the theory be free of uneven divergences can be satisfied also in case (5) is substituted by the condition

$$\lim_{\lambda_{4n}\to 0} \sum_n A(\bar{\lambda}_n)/\lambda_{4n}^{2m} = B_m \tag{6}$$

where the B_m's are arbitrary finite numbers.

For, if (4) is replaced by

$$(4/3)\cos\bar{k}\bar{\lambda} - \tfrac{1}{3}[1-\tfrac{3}{4}B\bar{\lambda}^2]\cos\tfrac{1}{2}\bar{k}\bar{\lambda},$$

(B being an arbitrary number), we obtain

$$\int_0^\infty kdk[(4/3)\cos k\lambda_4 - \tfrac{1}{3}(1+\tfrac{3}{4}B\lambda_4^2)\cos\tfrac{1}{2}k\lambda_4] = B.$$

This non-uniqueness exists also if the classical (even) divergences are considered. $D(K)$ can be chosen in such a manner that the integrals $\int_0^\infty D(K)K^{2m}dk$ take any finite value. For this purpose, along with the time-like vectors $\bar{\lambda}_n$, space-like vectors $\bar{\mu}_n$ must also be considered:

$$\bar{\mu}_n^2 = |\boldsymbol{\mu}_n|^2 - \mu_{4n}^2 > 0.$$

Thus, consider the case $m=0$.

$$\int_0^\infty \frac{d\mathbf{k}}{k^2}\cos(\mathbf{k}\boldsymbol{\mu} - k\mu_4) = \frac{2\pi}{\mu}\int_0^\infty \frac{dk}{k}[\sin(\mu-\mu_4)k + \sin(\mu+\mu_4)k].$$

Hence the D-factor which gives the integral $\int_0^\infty Ddk$ any assigned value has the form

$$D = \cos\bar{k}\bar{\lambda} + B/2\pi^2(\mu^2-\mu_4^2)^{\frac{1}{2}}\cos\bar{k}\bar{\lambda}. \tag{7}$$

(B being an arbitrary number), and obtain

$$\int_0^\infty (D/k^2)d\mathbf{k} = B(1-\mu_4^2/\mu^2)^{\frac{1}{2}}.$$

With $\bar{\mu}$ tending to zero any number between 0 and B can be obtained.

In the general case instead of (3) the following expression for D may be considered

$$D = \textstyle\sum_n \{A(\bar{\lambda}_n)\cos\bar{k}\bar{\lambda}_n + F(\bar{\mu}_n)\cos\bar{k}\bar{\mu}_n\}$$

$$\times \textstyle\sum_n \{A_n(0)+F_n(0)\} = 1.\ \ \lambda_n^2<0.\ \ \mu_n^2>0. \tag{8}$$

If we wish to obtain any finite value for the even (classical) integrals, D must be subjected to the condition

$$\lim_{|\boldsymbol{\mu}_n|\to 0} \sum_n \frac{F(\bar{\mu}_n)}{|\boldsymbol{\mu}_n|^{2m+1}} = B_{2m+1}. \tag{9}$$

In the case of uneven (quantum) divergence we arrive at the condition

$$\lim \sum_n \left\{ \frac{A(\bar{\lambda}_n)(2m+1)!}{\lambda_{4n}^{2m+2}} - \frac{\pi}{2}\frac{(2m)!F(\mu_n)}{\mu_n^{2m+2}} \right\} = B_{2m+2}. \tag{10}$$

In conclusion it should also be noted that the generalized λ-process of Eqs. (3) and (8) permits one also to sum logarithmically divergent integrals of the type[9]

$$\int_0^\infty dk/(k^2+m)^{\frac{1}{2}}$$

(Pauli,[3] p. 203). Using the relation

$$\int_0^\infty [\cos k\lambda/(k^2+m^2)^{\frac{1}{2}}]dk = -\ln\tfrac{1}{2}m\lambda - \gamma,$$

where γ is Euler's constant, one may, for example, consider the following expression as the D-factor eliminating the logarithmic divergences:

$$D = 2\cos\bar{k}\bar{\lambda} - \cos[a\bar{k}\bar{\lambda}(-\bar{\lambda}^2)^{\frac{1}{2}}]. \tag{11}$$

In the system of coordinates in which $\boldsymbol{\lambda}=0$:

$$D = 2\cos k\lambda_4 - \cos ak\lambda_4^2;$$

$$\int_0^\infty D/(k^2+m^2)^{\frac{1}{2}} = -\ln\tfrac{1}{2}m - \gamma + \ln a.$$

However, it is easy to choose such a D that, besides (9) and (10), the integral $\int_0^\infty D/(k^2+m^2)^{\frac{1}{2}}$ also takes any finite value.

Thus although the generalized λ-process (3) and (8) permits one to remove all divergences in the theory of fundamental particles, the theory thus obtained turns out to be non-unique; by varying the D-factor various values for the observable quantities (e.g., the proper mass) may be obtained. This non-uniqueness either indicates the invalidity of limiting processes for solving the basic difficulties in the theory of fundamental particles or else points to the necessity of additional postulates or "selection rules."

[9] V. Weisskopf, Zeits. f. Physik **89**, 90 (1934); Phys. Rev. **56**, 72 (1939).