Wave Equations in the de Sitter Space

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The second-order wave equation $\partial^2 u/\partial x_i \partial x^i = 0$ in five-dimensional flat space can reproduce approximately the field hitherto considered in four-dimensional space, if a de Sitter space with the radius R is imbedded in the five-dimensional flat space and the derivative of the wave function u in the direction normal to this de Sitter space is assumed to be equal to imu, where m is a constant proper to the field. Conversely, the ordinary field equations in four-dimensional space can be regarded as valid only for a portion of de Sitter space with the linear dimension, which is small compared with the radius R of the universe. Extensions of these arguments to electromagnetic field and spinor field are also considered.

QUANTUM mechanics of fields and general theory of relativity can be harmonized without interfering with the current principles of either theory as long as the gravitational force is not represented by the metric tensor, i.e., if the latter theory is applied to an empty, homogeneous space. The maximum unification of these two systems without a drastic alteration of their basic methods may therefore be attained by establishing a quantum theory of fields in the de Sitter space. Dirac has indeed shown that the electron wave equation can be generalized to the de Sitter space,[1] however his method of deduction is imbued with many arbitrary supplementary assumptions. The aim of the present paper is to introduce a consistent method which permits a systematic reformulation of all the known field equations in the de Sitter frame of reference.

We take a 5-dimensional flat space,

$$r^2 = g_{ij}x^i x^j = x_i x^i = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^0)^2 + (x^5)^2, \quad (1)$$

in which the de Sitter space:

$$r^2 = R^2 \quad (2)$$

is imbedded, R being the "radius" of the universe.

We make two fundamental assumptions, from which all the field equations will be automatically deduced:

First, any field quantity should satisfy the second-order wave equation:

$$\partial^2/\partial x_i \partial x^i u = 0. \quad (3)$$

Second, a field quantity differentiated in the direction normal to the de Sitter space should give itself multiplied by i-times a constant m proper to the field ($c = \hbar = 1$):

$$\partial u/\partial r = imu. \quad (4)$$

Alternative assumptions which can equally be used in place of (4) will be discussed later.

The author of this paper introduced some years ago a 5-dimensional formalism in which the fifth coordinate x^5 was regarded as the canonical conjugate of the mass constant.[2] He also discussed some practical consequences which can be drawn from this theory.[3] The following theory, which will be based on assumptions (3) and (4), will become identical with this previous formalism if a small 5-dimensional portion of the space near a point in the de Sitter space is taken into consideration, or in other words if the radius R is considered to be very large.

The differentiation operator $\partial/\partial x_i$ can be expressed in terms of the rotation operators:

$$\pi^{ij} = x^i(\partial/\partial x_j) - x^j(\partial/\partial x_i), \quad (5)$$

which do not go outside the de Sitter space and the operator of differentiation in the normal direction:

$$\pi = r(\partial/\partial r) = x^i(\partial/\partial x^i). \quad (6)$$

Among the ten operators π^{ij}, there are six identical relations of the type

$$x^k \pi^{ij} + x^i \pi^{jk} + x^j \pi^{ki} = 0, \quad (7)$$

so that (5) and (6) make just five independent operators and we have

$$\partial/\partial x_i = 1/r^2(-x_j \pi^{ij} + x^i \pi). \quad (8)$$

The field equations in the de Sitter space will be obtained by substituting (8) into (3) and rewriting π by the assumption (4). Since we have

$$\partial^2/\partial x_i \partial x^i = 1/r^2(\tfrac{1}{2}\pi_{ij}\pi^{ij} + \pi^2 + 3\pi), \quad (9)$$

the second-order wave equation in the de Sitter space becomes

$$(\tfrac{1}{2}\pi_{ij}\pi^{ij} - m^2R^2 + 4imR)u = 0. \quad (10)$$

The linearization of the second-order wave equation in terms of a tensor, as is usual for the electromagnetic or mesonic field, can be done in the following manner:

The field strengths defined by

$$f^{ij} = 1/R^2[x_k(\pi^{jk}u^i - \pi^{ik}u^j) + imR(x^i u^j - x^j u^i)] \quad (11)$$

satisfy the integrability condition,

$$x_l(\pi^{kl}f^{ij} + \pi^{il}f^{jk} + \pi^{jl}f^{ki}) - imR(x^k f^{ij} + x^i f^{jk} + x^j f^{ki}) = 0. \quad (12)$$

[1] P. A. M. Dirac, Ann. Math. **36**, 657 (1935).

[2] Satosi Watanabe, Sci. Pap. I.P.C.R. **39**, 157 (1941). Satosi Watanabe, Sci. Pap. I.P.C.R. **42**, 1 (1944).

[3] Satosi Watanabe, Phys. Rev. **74**, 1864 (1948).

In virtue of the Lorentz condition:

$$(-x^j\pi_{ij}+imRx_i)u^i=0, \tag{13}$$

the field Eq. (10) then becomes

$$(-x^k\pi_{jk}+imRx_j)f^{ij}=0. \tag{14}$$

This set of equations allows for the gauge transformation

$$u'^i=u^i+1/R^2(-x_k\pi^{ik}+imRx^i)\chi, \tag{15}$$

if χ satisfies an equation of the type (10).

The passage to the ordinary Minkowski space is possible for the small region:

$$\left.\begin{array}{l} x^5\approx R \\ |x^\mu|\ll R \quad (\mu=1,2,3,0) \end{array}\right\}. \tag{16}$$

The usual 4-dimensional mesonic equations can then be derived from the equations from (11) to (15) by identifying $f^{5\mu}$ with the mesonic potential v^μ multiplied with im:

$$f^{5\mu}=imv^\mu. \tag{17}$$

The gauge transformation (15) becomes in the region (16)

$$\left.\begin{array}{l} u'^\mu=u^\mu+\partial\chi/\partial x_\mu \\ u'^5=u^5+im\chi, \end{array}\right\} \tag{18}$$

with

$$(\Box-m^2)\chi=0. \tag{19}$$

This entitles us to make u'^5 vanish, so that we may have

$$v^\mu=u'^\mu. \tag{20}$$

This last equality, however, may fail in presence of interaction, whereas (17) is a general relation. Since the gauge transformation (18) leaves the mesonic potential v^μ unchanged, there is in general no gauge transformation for the latter.

The electromagnetic field equations can be obtained, under the condition (16), by simply putting $m=0$ in the equations from (10) to (15). The electromagnetic potential can be identified with the first four components of the 5-dimensional potential and the gauge transformation (18) acquires a 4-dimensional validity. u^5 and $f^{5\mu}$ are completely separated from the electromagnetic field and correspond to a massless c meson.[3]

The linearization of the second-order Eq. (10) for a spinor field can be done, with the help of the Dirac matrices,

$$\tfrac{1}{2}(E_iE_j+E_jE_i)=g_{ij}, \tag{21}$$

by writing

$$E_i(\partial/\partial x_i)\psi=0, \tag{22}$$

which becomes, in consideration of (8) and (4),

$$(-E_ix_j\pi^{ij}+imRE_ix^i)\psi=0. \tag{23}$$

The last equation reduces to the ordinary electron equation under the condition (16).

It will be clear from the above deduction that, in order to obtain the usual 4-dimensional wave equations under the condition (16), we need not necessarily assume (4) but only assume

$$r(\partial u/\partial r)=imR[1+0(1/Rm)]u \text{ for } r=R. \tag{24}$$

Condition (24) is also sufficient for the above theory to become, in the limiting case (16), identical with the 5-dimensional formalism which the author introduced previously.[2]

The quantities which play the role of momentum-energy vector in the above theory are

$$p^\mu=(i/R)\pi^{\mu 5}, \tag{25}$$

which satisfy the commutation relation

$$p^\mu x_\mu-x_\mu p^\mu=-i(x^5/R), \tag{26}$$

which tends to $-i$ as we approach the limiting case (16). The first three components of p^μ are quantized, but the quantum intervals are of the order of $1/R$, which is negligibly small compared with the values of momenta usually intervening in the atomic physics if R is taken from the cosmological determination.

The generalization of the above theory to fields interacting with each other can easily be carried out without introducing any new notion or assumption.