On the Formal Extension of Dirac's Equation under Continuous Transformation Groups

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The theory of equations of the Dirac type which are form-invariant under general finite continuous groups of point transformations in space-time is developed. The possible equations invariant under a given group depend on the structure of the group, but in each case the required invariance imposes a set of linear homogeneous relations on the β-operators with which any assumed algebra must be consistent. It is shown that the algebra appropriate to the general equation possessing form-invariance under the inhomogeneous Lorentz group is necessarily of infinite order.

1. INTRODUCTION

THE success of the Dirac equation for the electron and the positron and, to a lesser extent, that of the Proca equation for the meson[1,2] in providing a basis for the theory of elementary particles has led to an interest in more general equations of the same type which possess form-invariance under the Lorentz group, L_4, of the special theory of relativity.[3-5] A similar concern has arisen from the related study of possible quantum-mechanical equations in cosmological spaces.[6-8] Apart from the importance of these studies in the formulation of a theory of spin forces, it seems that considerable mathematical exploratory work of this nature will need to be carried out if effective progress is to be made in reconciling the divergent aims of quantum mechanics and of the general theory of relativity in the pursuit of a unified theory of matter.

The work to be detailed here had its origin in the writer's interpretation of the conformal group of transformations in space-time,[9] and in the desire to extend the consequences of that discussion into the domain of quantum mechanics. We shall be concerned with the generalization of equations of the Dirac-Proca type under finite continuous transformation groups in space-time, and shall show that such a generalization exists for an arbitrary group G. The form of the equation which is proposed is not specified uniquely by the group, but the invariance under G imposes on the β-operators a set of relations with which any assumed algebra must be consistent. It is found, for example, that the algebra for the general equation having form-invariance under the inhomogeneous Lorentz group is necessarily of infinite order. This must then also be the case for the conformal group as well as for the more extended groups which contain L_4 as a subgroup. The infinite algebras appropriate to these equations are as yet undeveloped, so that for the present we are forced to leave our discussion incomplete on the side of actual physical applications.

Earlier treatments of possible quantum-mechanical equations having form-invariance under the conformal group have been given by Dirac[10] and by Bhabha.[11] In this work the number of independent variables was extended in such a manner that the transformations of the conformal group become linear and homogeneous, a procedure which is equivalent to the use of the so-called hexaspherical coordinates. We do not follow this method in the present work, but base our procedures on the use of the basis operators of a finite continuous group as a linear vector space under the transformations of the group. In this way we are enabled to achieve a precise and manageable mathematical technique, and at the

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same time to leave the way open for a possible later treatment of more general types of canonical transformation groups which are of such great importance even in the elementary theory of classical and quantum mechanics, but which at present find no place in the general theory of relativity.

Another feature of the present approach which is of some interest is the manner in which it provides for the introduction of the "kinematical forces" associated with the transformations of reference systems between equivalent observers.

2. THEORY

We consider a finite continuous group, G, of point transformations on the space-time variables ($x^1=x$, $x^2=y$, $x^3=z$, $x^4=i\tau=ict$), having n essential parameters $a^1, \cdots, a^n$. The general transformation of the group will be designated as S_a; the parameters will be supposed to take the values $(a_0{}^\alpha)$ for the identity transformation, S_0, and the values $(\bar{a}^\alpha)$ for the inverse transformation, $S_a{}^{-1}$. We shall consider the group to be defined in terms of a set of basis operators[12]

$$X_\alpha \equiv \xi_\alpha{}^k(x)p_k, \tag{1}$$

where Greek indices take values from 1 to n, and Latin indices from 1 to 4, and where we employ the usual dummy index summation notation for both types of indices. The fundamental differential operators are taken in the quantum-mechanical form

$$p_k = -i\hbar\partial/\partial x^k, \tag{2}$$

and the functions $\xi_\alpha{}^k(x)$ will be considered to be *real* functions of the variables (x^k).

The general infinitesimal transformation of the group is then given by the formula

$$\delta x^k = \delta a^\beta [X_\beta, x^k] = -i\hbar\xi_\alpha{}^k(x)\delta a^\alpha, \tag{3}$$

where $[A,B]=AB-BA$ is the usual commutator symbol. Corresponding to Eq. (3), we define an infinitesimal transformation of the basis operators by the formula

$$\delta X_\alpha = \delta a^\beta [X_\beta, X_\alpha]. \tag{4}$$

It follows from the fact that we are dealing with a *group* of transformations that

$$[X_\beta, X_\alpha] = -i\hbar c_{\beta\alpha}{}^\gamma X_\gamma, \tag{5}$$

where the quantities $c_{\beta\alpha}{}^\gamma = -c_{\alpha\beta}{}^\gamma$ are the *structure constants* of the group. Our definition of Eq. (4) thus reduces to the form

$$\delta X_\alpha = i\hbar c_{\alpha\beta}{}^\gamma X_\gamma \delta a^\beta. \tag{4'}$$

It is readily verified that this definition is equivalent to that to be found directly from Eq. (3). For, if we write the infinitesimal transformation corresponding to Eq. (3) as

$$x'^k = x^k - i\hbar\xi_\alpha{}^k(x)\delta a^\alpha, \tag{6}$$

we obtain for the differential operators

$$\begin{aligned} p_k' &\equiv -i\hbar\partial/\partial x'^k \\ &= -i\hbar[\partial/\partial x^k + i\hbar\delta a^\alpha(\partial\xi_\alpha{}^m/\partial x^k)\partial/\partial x^m] \\ &= p_k + i\hbar(\partial\xi_\alpha{}^m/\partial x^k)p_m\delta a^\alpha, \end{aligned}$$

to first orders, and so have, correspondingly,

$$\delta p_k \equiv p_k' - p_k = i\hbar(\partial\xi_\alpha{}^m/\partial x^k)p_m\delta a^\alpha.$$

Similarly, we have

$$\delta\xi_\alpha{}^k \equiv \xi_\alpha{}^k(x') - \xi_\alpha{}^k(x) = -i\hbar(\partial\xi_\alpha{}^k/\partial x^m)\xi_\beta{}^m\delta a^\beta.$$

By combining these expressions we find

$$\begin{aligned} \delta X_\alpha \equiv \delta(\xi_\alpha{}^k p_k) &= \delta\xi_\alpha{}^k\cdot p_k + \xi_\alpha{}^k\cdot\delta p_k \\ &= +i\hbar[\xi_\alpha{}^m(\partial\xi_\beta{}^k/\partial x^m) \\ &\qquad - \xi_\beta{}^m(\partial\xi_\alpha{}^k/\partial x^m)]p_k\delta a^\beta. \end{aligned} \tag{7}$$

But it follows from the group properties of G that

$$\xi_\alpha{}^m(\partial\xi_\beta{}^k/\partial x^m) - \xi_\beta{}^m(\partial\xi_\alpha{}^k/\partial x^m) = c_{\alpha\beta}{}^\gamma\xi_\gamma{}^k,$$

and on substitution of this condition into Eq. (7) we again obtain the result of Eq. (4′).

We now consider the formal operator

$$\mathfrak{M} \equiv \beta^\alpha\xi_\alpha{}^k(x)[p_k - \epsilon A_k/c] + im_0c, \tag{8}$$

in which ϵ, c, and m_0 are the usual absolute constants. We require of the four functions $A_k(x)$ only that they transform as the covariant components of a 4-vector. The properties of the β-operators are to be determined in such a manner that the operator $\mathfrak{M}$ is formally invariant under the group G.

[12] L. P. Eisenhart, *Continuous Groups of Transformations*, (Princeton University Press, Princeton, 1933).

We have for the transformation of the 4-vector A_k,

$$\delta A_k \equiv A_k'(x') - A_k(x) = i\hbar(\partial \xi_\alpha{}^m/\partial x^k) A_m \delta a^\alpha.$$

If we let

$$\Pi_k \equiv p_k - \epsilon A_k/c,$$

we obtain

$$\delta \Pi_k = i\hbar(\partial \xi_\beta{}^m/\partial x^k)\Pi_m \delta a^\beta,$$

so that, finally,

$$\delta(\xi_\alpha{}^k \Pi_k) = i\hbar c_{\alpha\theta}{}^\gamma(\xi_\gamma{}^m \Pi_m)\delta a^\theta.$$

For the variational equation of $\mathfrak{M}$ we now find

$$0 = \delta\mathfrak{M} = \delta\beta^\alpha \cdot (\xi_\alpha{}^k \Pi_k) + \beta^\alpha \cdot \delta(\xi_\alpha{}^k \Pi_k)$$
$$= [\delta\beta^\gamma + i\hbar c_{\alpha\theta}{}^\gamma \beta^\alpha \delta a^\theta]\xi_\gamma{}^k \Pi_k. \quad (9)$$

By virtue of the independence of the quantities $\xi_\gamma{}^k\Pi_k$ we require that

$$\delta\beta^\gamma = -i\hbar c_{\alpha\theta}{}^\gamma \beta^\alpha \delta a^\theta = i\hbar c_{\theta\alpha}{}^\gamma \beta^\alpha \delta a^\theta. \quad (10)$$

Equations (10) constitute the necessary and sufficient conditions which we must impose on the β-operators in order to secure formal invariance of the operator $\mathfrak{M}$ under the group G. In view of the fact that we have considered only the infinitesimal transformations of G in the neighborhood of the identity transformation, we interpret Eqs. (8) as differential equations to be integrated along some definite path in the space of the parameters $a^1, \cdots, a^n$. If we let

$$a^\alpha = \rho^\alpha a,$$

where $\rho^1, \cdots, \rho^n$ are specified, but arbitrarily chosen, constants, and a is the running group parameter, we have the differential equations.

$$d\beta^\gamma/da = i\hbar c_{\theta\alpha}{}^\gamma \rho^\theta \beta^\alpha. \quad (11)$$

Since Eqs. (11) are linear and homogeneous in the β's, with constant coefficients, they can be integrated without specification of the algebraic properties of the β's.

An alternative discussion of this mathematical theory, which yields the integrated form of Eqs. (11), is given in Appendix A.

Our extension of the Dirac equation is now given by

$$\mathfrak{M}\Psi \equiv \{\beta^\alpha \xi_\alpha{}^k(x)\Pi_k + i m_0 c\}\Psi = 0. \quad (12)$$

In order that Eq. (12) shall constitute a true system of differential equations for the components of the wave function, we must subject the β's to a set of algebraic relations compatible with Eqs. (11), and must then find a suitable representation of this algebra. Furthermore, to give assurance that the equations so obtained will be completely form-invariant under G, the assumed algebra must lead to the condition that Eq. (11) is equivalent to a similarity transformation, that is, a non-singular operator (matrix) V_a, dependent on the transformation S_a, must exist such that

$$\beta^\alpha(a) = V_a \beta^\alpha(a_0) V_a^{-1}. \quad (13)$$

If we let

$$V_a = 1 + v_\alpha \delta a^\alpha, \quad V_a^{-1} = 1 - v_\alpha \delta a^\alpha, \quad (14)$$

for infinitesimal transformations, then we have from Eqs. (10) and (13) the conditions

$$[v_\alpha, \beta^\gamma] = i\hbar c_{\alpha\theta}{}^\gamma \beta^\theta, \quad (15)$$

as the determining relations for the n operators v_α.

The reality conditions to be imposed for a general group of transformations are somewhat obscure. However, it is expected on the basis of present theory that the representation of the algebra of the β-operators will be by Hermitian matrices, and the nomenclature of this paper has been chosen to conform to this premise. This property is lost, of course, under the transformations of Eqs. (11) and (13), but is restored on making a transformation on the representation space of the wave function, according to the relation

$$\Psi = V_a \Psi',$$

exactly as is the case for the Dirac equation for which the transformation matrix for the specifically Lorentz transformations, as distinguished from the three-dimensional rotations, is Hermitian rather than unitary.

We note that Eq. (12) is also form-invariant under a gauge transformation of the 4-vector, for if the latter is subjected to the transformation

$$A_k = A_k' + \partial\omega/\partial x^k,$$

Eq. (12) preserves its form if we make the replacement

$$\Psi = \exp(i\epsilon\omega/\hbar c)\Psi'.$$

On the other hand, if we interpret the functions A_k as the components of the electromagnetic 4-vector, and require also the form-invariance of the Maxwell field equations, then the largest group under which both Eq. (12) *and* the Maxwell field equations have form-invariance is the conformal group, C_4.

It is of importance to observe that if the group G is not simple, but contains an invariant subgroup of order $m<n$, then, with a proper ordering of the basis operators, one can set $\beta_{m+1}=\cdots=\beta_n=0$, and so can obtain an equation in place of (12) containing only m operators, but one which still is form-invariant under the full group. The consistency of this procedure with Eqs. (10) follows from the properties of the table of structure constants.[13] This circumstance is of paramount importance in the case of the Lorentz group, within which the translations form an invariant subgroup, and is responsible for the fact that the Dirac and Proca equations can have form-invariance under the full inhomogeneous Lorentz group. Under more general transformation groups this simplifying property may be lost.[14]

3. THE LORENTZ GROUP

The basis operators for the inhomogeneous Lorentz group may be taken conveniently in the form

$$X_1\equiv p_1,\quad X_2\equiv p_2,\quad X_3\equiv p_3,\quad X_4\equiv p_4, \tag{16a}$$

$$X_5\equiv yp_3-zp_2,\quad X_6\equiv zp_1-xp_3,\quad X_7\equiv xp_2-yp_1, \tag{16b}$$

$$X_8\equiv -i\tau p_1+xp_4,\quad X_9\equiv -i\tau p_2+yp_4,\quad X_{10}\equiv -i\tau p_3+zp_4. \tag{16c}$$

In writing out the expansion of Eq. (12) we shall drop the summation notation, lower the indices on the β-operators, and introduce the new symbols

$$\gamma_1=\tfrac{1}{2}(\beta_5+\beta_8),\quad \gamma_2=\tfrac{1}{2}(\beta_6+\beta_9),\quad \gamma_3=\tfrac{1}{2}(\beta_7+\beta_{10}), \tag{17a}$$

$$\omega_1=\tfrac{1}{2}(\beta_5-\beta_8),\quad \omega_2=\tfrac{1}{2}(\beta_6-\beta_9),\quad \omega_3=\tfrac{1}{2}(\beta_7-\beta_{10}). \tag{17b}$$

[13] Reference 12, Section 35.

[14] The translations do not form an invariant subgroup of the conformal group, C_4,

The latter quantities can be written formally as two vector operators $\boldsymbol{\gamma}$ and $\boldsymbol{\omega}$. We then obtain from Eq. (12) the result

$$\left\{\sum_{k=1}^{4}\beta_k\Pi_k+im_0c+\boldsymbol{\gamma}\cdot\left[\mathbf{M}-\frac{\epsilon}{c}(\mathbf{r}\times\mathbf{A})-\mathbf{r}\left(\frac{\hbar\partial}{\partial\tau}+\frac{i\epsilon}{c}\phi\right)-i\tau\boldsymbol{\Pi}\right]+\boldsymbol{\omega}\cdot\left[\mathbf{M}-\frac{\epsilon}{c}(\mathbf{r}\times\mathbf{A})+\mathbf{r}\left(\frac{\hbar\partial}{\partial\tau}+\frac{i\epsilon}{c}\phi\right)+i\tau\boldsymbol{\Pi}\right]\right\}\Psi=0, \tag{18}$$

in which we have introduced the usual notation

$$\mathbf{M}=\mathbf{r}\times\mathbf{p},\quad \boldsymbol{\Pi}=\mathbf{p}-\epsilon\mathbf{A}/c,\quad A_4=i\phi.$$

It will be convenient at this point to change the definition of the parameters of the group from that implied in Eq. (6) in such a manner that all of the parameters shall be real. The required transformation is given by the table

$$\mu_1=i\hbar a_1,\quad \mu_2=i\hbar a_2,\quad \mu_3=i\hbar a_3,\quad \mu_4=\hbar a_4,$$
$$\mu_5=i\hbar a_5,\quad \mu_6=i\hbar a_6,\quad \mu_7=i\hbar a_7,\quad \mu_8=\hbar a_8,$$
$$\mu_9=\hbar a_9,\quad \mu_{10}=\hbar a_{10}.$$

If we now let $\rho_1, \cdots, \rho_{10}$ be a set of arbitrary real constants, the differential equations (11) take the form

$$d\beta_1/d\mu=-(\gamma_3+\omega_3)\rho_2+(\gamma_2+\omega_2)\rho_3-i(\gamma_1-\omega_1)\rho_4-\beta_3\rho_6+\beta_2\rho_7+i\beta_4\rho_8, \tag{19a}$$

$$d\beta_2/d\mu=(\gamma_3+\omega_3)\rho_1-(\gamma_1+\omega_1)\rho_3-i(\gamma_2-\omega_2)\rho_4+\beta_3\rho_5-\beta_1\rho_7+i\beta_4\rho_9, \tag{19b}$$

$$d\beta_3/d\mu=-(\gamma_2+\omega_2)\rho_1+(\gamma_1+\omega_1)\rho_2-i(\gamma_3-\omega_3)\rho_4-\beta_2\rho_5+\beta_1\rho_6+i\beta_4\rho_{10}, \tag{19c}$$

$$d\beta_4/d\mu=(\gamma_1-\omega_1)\rho_1+(\gamma_2-\omega_2)\rho_2+(\gamma_3-\omega_3)\rho_3-i\beta_1\rho_8-i\beta_2\rho_9-i\beta_3\rho_{10}, \tag{19d}$$

$$d\gamma_1/d\mu=-\gamma_3\rho_6+\gamma_2\rho_7-i\gamma_3\rho_9+i\gamma_2\rho_{10}, \tag{20a}$$

$$d\gamma_2/d\mu=\gamma_3\rho_5-\gamma_1\rho_7+i\gamma_3\rho_8-i\gamma_1\rho_{10}, \tag{20b}$$

$$d\gamma_3/d\mu=-\gamma_2\rho_5+\gamma_1\rho_6-i\gamma_2\rho_8+i\gamma_1\rho_9, \tag{20c}$$

$$d\omega_1/d\mu=-\omega_3\rho_6+\omega_2\rho_7+i\omega_3\rho_9-i\omega_2\rho_{10}, \tag{21a}$$

$$d\omega_2/d\mu=\omega_3\rho_5-\omega_1\rho_7-i\omega_3\rho_8+i\omega_1\rho_{10}, \tag{21b}$$

$$d\omega_3/d\mu=-\omega_2\rho_5+\omega_1\rho_6+i\omega_2\rho_8-i\omega_1\rho_9. \tag{21c}$$

It can be verified directly from Eqs. (20a-21c) that the γ's and ω's can be subjected separately to the Pauli algebra

$$\gamma_1^2=\gamma_2^2=\gamma_3^2=1, \quad \gamma_1\gamma_2=-\gamma_2\gamma_1=i\gamma_3, \text{ etc.,} \tag{22a}$$

$$\omega_1^2=\omega_2^2=\omega_3^2=1, \quad \omega_1\omega_2=-\omega_2\omega_1=i\omega_3, \text{ etc.} \tag{22b}$$

A study of Eqs. (19a–d) shows, however, that unless we set

$$\gamma_1=\gamma_2=\gamma_3=\omega_1=\omega_2=\omega_3=0, \tag{23}$$

we cannot find a *finite* algebra which is compatible with all of these conditions under the full group L_4. This simplification is just that ordinarily made, and depends on the fact that the translations form an invariant subgroup of L_4, as pointed out in Section 2.

In order to show that the operator algebra defined by Eqs. (19a–21c) must be of infinite order, unless Eqs. (23) are separately imposed, we consider the effects of translations alone, and so set

$$\rho_5=\rho_6=\rho_7=\rho_8=\rho_9=\rho_{10}=0.$$

Under translations the γ's and ω's are invariant, while the equations for the β-operators are immediately integrable in the form

$$\beta_1(\mu)=\beta_1(\mu_0)+\{-(\gamma_3+\omega_3)\rho_2+(\gamma_2+\omega_2)\rho_3 -i(\gamma_1-\omega_1)\rho_4\}(\mu-\mu_0), \tag{24}$$

with similar relations for the remaining β's. Now, since the transformation (24) must be a similarity transformation, the eigenvalue spectrum of the matrix representations of the β's is preserved. But since the ρ's have indeterminate numerical values, while the group parameter μ can take all real values, both positive and negative, the eigenvalues of the matrix representations of $\beta_1, \cdots, \beta_4$ must vary continuously, and furthermore must be unbounded numerically, both above and below. It is clear that these conditions cannot be satisfied by representations with finite matrices. The writer has not yet found it possible to establish an algebra which is consistent with Eqs. (19a–21c) when Eqs. (23) are not satisfied.

When Eqs. (23) are imposed, the reduced forms of Eqs. (19a–d) furnish a relatively easy starting point for the generalization of the Dirac and the Duffin-Kemmer algebras. However, since this question has been examined in detail by Bhabha[4] and by Harish-Chandra[5] it will not be considered further here.

The writer is indebted to Professors C. Critchfield and J. Weinberg for helpful discussions of this and related topics.

APPENDIX A

In the interest of clarity it is of value to restate the invariance property of Eq. (12) by an interpretation of Eq. (6) as a change of reference coordinates in space-time. Under a general change of reference system, the basis operators X_α are interpreted as formal invariants,[12] so that the functions $\xi_\alpha{}^k(x)$ transform as the contravariant components of a set of 4-vectors, the index α specifying the vector, while the index k specifies the components. Designating the new coordinates by the transformation formula

$$y^k=Tx^k=f^k(x),$$

the transformations of the group, expressed in terms of the new coordinates will be given by the expressions

$$TS_aT^{-1}.$$

The basis operators, expressed in terms of the new coordinates, take the form

$$Y_\alpha\equiv\eta_\alpha{}^k(y)(-i\hbar\partial/\partial y^k),$$

with

$$\eta_\alpha{}^k(y)=\xi_\alpha{}^m(x)(\partial y^k/\partial x^m). \tag{25}$$

If this change of coordinate system is effected by means of a transformation $T=S_b$ of the group G itself, Eq. (25) takes the simpler form

$$\eta_\alpha{}^k(y)=\xi_\gamma{}^k(y)D_\alpha{}^\gamma(b),$$

where the n^2 functions $D_\alpha{}^\gamma(b)$ depend on the parameters defining the transformation S_b, but not on the space-time coordinates.

The transformation of the basis operators under the group is therefore such that

$$Y_\alpha=D_\alpha{}^\gamma(b)X_\gamma',$$

where X_γ' has exactly the same functional form as does X_γ, but is expressed in terms of the y-variables. If we now perform the transformation S_b on the space-time coordinates, we have

$$\beta^\alpha X_\alpha\rightarrow\beta^\alpha Y_\alpha=\beta^\alpha D_\alpha{}^\theta(b)X_\theta'.$$

In order to return this to the original form, but expressed in terms of the new variables, we have only to define the transformation on the β-operators

$$\beta'^{\theta} = D_{\alpha}{}^{\theta}(b)\beta^{\alpha}. \qquad (26)$$

As was discussed in Section 2, this transformation must be equivalent to a similarity transformation in order that the algebraic properties of the operators may be preserved, and the form-invariance of the equations for the components of the wave function assured.

Equations (26) give the integrated form of Eqs. (11), from which the latter can be obtained by consideration of the infinitesimal transformations in the neighborhood of the identity of the group.

Although Eqs. (11) are more useful for practical computations than are Eqs. (26), it is of interest to know the formulation of the coefficients $D_{\alpha}{}^{\beta}(b)$ in terms of the properties of the group. If S_a, S_b, and S_c are three transformations of the group such that

$$S_c = S_b S_a,$$

the corresponding functional relations among the parameters are expressed by the *group functions*

$$c^{\alpha} = \varphi^{\alpha}(a,b).$$

Considering these as functions of the $2n$ independent variables, we define the derived functions

$$\lambda_{\beta}{}^{\alpha}(a,b) = \partial\varphi^{\alpha}/\partial a^{\beta}, \quad \Lambda_{\beta}{}^{\alpha}(a,b) = \partial\varphi^{\alpha}/\partial b^{\beta}.$$

From these, making use of the general notation of Section 2, we define the sets of functions

$$A_{\beta}{}^{\alpha}(a) = \lambda_{\beta}{}^{\alpha}(a_0,a), \quad B_{\beta}{}^{\alpha}(a) = \Lambda(a,a_0),$$
$$\bar{A}_{\beta}{}^{\alpha}(a) = \lambda_{\beta}{}^{\alpha}(a,\bar{a}), \quad \bar{B}_{\beta}{}^{\alpha}(a) = \Lambda(\bar{a},a).$$

It can be shown from these relations that

$$D_{\beta}{}^{\alpha}(a) = \bar{B}_{\gamma}{}^{\alpha}(a) A_{\beta}{}^{\gamma}(a),$$

and

$$\bar{D}_{\beta}{}^{\alpha}(a) = \bar{A}_{\gamma}{}^{\alpha}(a) B_{\beta}{}^{\gamma}(a),$$

where

$$D_{\alpha}{}^{\gamma}(a)\bar{D}_{\gamma}{}^{\beta}(a) = D_{\gamma}{}^{\beta}(a)\bar{D}_{\alpha}{}^{\gamma}(a) = \delta_{\alpha}{}^{\beta}.$$

For transformations in the neighborhood of the identity we have the series expansions

$$D_{\beta}{}^{\alpha} = \delta_{\beta}{}^{\alpha} - i\hbar c_{\beta\gamma}{}^{\alpha}\cdot(a^{\gamma} - a_0{}^{\gamma}) + \cdots,$$
$$\bar{D}_{\beta}{}^{\alpha} = \delta_{\beta}{}^{\alpha} + i\hbar c_{\beta\gamma}{}^{\alpha}\cdot(a^{\gamma} - a_0{}^{\gamma}) + \cdots.$$

The relation with the method of Section 2 now follows on making the definition

$$\delta X_{\alpha} \equiv X_{\alpha}' - X_{\alpha},$$

for an infinitesimal transformation.