On Explicit Solutions of the Equations for Steady Compressible Flow

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IN classical aerodynamics the steady flow of incompressible, inviscid fluids can often be described explicitly in terms of functions of a complex variable. There are, however, few accurate solutions of the equations for compressible flow that can be obtained in closed form. In view of the importance of obtaining more exact solutions, especially in the transonic region, the following outline of a new method of approach is submitted (see related work of Munk[1]). The flows considered are *adiabatic* but not necessarily isentropic and iso-energetic.

For uniplanar irrotational flow, the compressible flow equations can be written

$$\partial N/\partial\psi=\lambda_1(N)\partial\vartheta/\partial\varphi, \tag{1}$$

$$\partial N/\partial\varphi=-[\lambda_2(N)]^{-1}\partial\vartheta/\partial\psi, \tag{2}$$

where the velocity vector $\mathbf{V}=[g(N)RT]^{\frac{1}{2}}\mathbf{N}$; $|\mathbf{N}|=N=|\mathbf{V}|/[g(N)RT]^{\frac{1}{2}}$; ϑ is the angle of inclination of the vector $\mathbf{V}$ to the x axis; and φ and ψ are potential and stream functions for the vector $\mathbf{N}$, defined by the equations $\mathbf{N}=\nabla\varphi=\nabla\times\mathbf{k}\psi$. To a given choice of the function $g(N)$, there correspond flows with a given physical characteristic, namely, a relation between N and the pressure p on curves normal to the streamlines. Choice of $g(N)$ also fixes the form of the functions λ_1, λ_2 and all other functions of N or $\mathfrak{N}$ which will be introduced subsequently.

Except when $g(N)\propto(1-N^2)^{-1}$, irrotationality of $\mathbf{N}$ requires,[2] if the equation of motion is to be integrable, that (i) the stagnation pressure p_t vary according to the equations

$$\log p_t=\Psi(\psi)=\mathfrak{N}(N)-\Phi(\varphi) \tag{3}$$

(where Φ and Ψ are arbitrary), and is constant (between shocks) on streamlines; (ii) $\nabla^2\varphi\propto\mathfrak{L}(N)\Phi'(\varphi)$. If $g(N)\propto(1-N^2)^{-1}$, the integrability condition yields, instead of (3), the simple result $\nabla p_t\equiv 0$. The second of Eqs. (3) can, however, be assumed here also, in order that explicit integration of the flow equations can be achieved for this important case. It is possible, with other choices of $g(N)$, that the detailed analysis may be considerably simplified as it is known to be for diabatic flow.[2]

We now assume that (1) and (2) are integrable; then if $N(\varphi,\psi)$ can be found by some means, ϑ is calculable from

$$\vartheta=-\int_{\varphi,\psi_0}^{\varphi,\psi}\lambda_2(N)\frac{\partial N}{\partial\varphi}d\psi+\int_{\varphi_0,\psi_0}^{\varphi,\psi_0}[\lambda_1(N)]^{-1}\frac{\partial N}{\partial\psi}d\varphi. \tag{4}$$

Equation (3) together with assumed integrability of (1) and (2), demands that the functional-differential equation, involving the functions $\Phi(\varphi)$ and $\Psi(\psi)$, must be satisfied.

$$\mu_2(\mathfrak{N})F'(\Phi)+2\mu_2'(\mathfrak{N})F(\Phi)+\mu_1(\mathfrak{N})P'(\Psi)+2\mu_1'(\mathfrak{N})P(\Psi)=0, \tag{5}$$

where

$$F(\Phi)=(d\Phi/d\varphi)^{-\frac{1}{2}};\quad P(\Psi)=(d\Psi/d\psi)^{-\frac{1}{2}}.$$

After repeated integration with respect to Φ and Ψ, using (3), Eq. (5) can be reduced to a functional equation in $\Phi(\varphi)$ and $\Psi(\psi)$, which we shall schematically represent by

$$\mathfrak{F}[\Phi,\Psi,c_i(\varphi),k_j(\varphi)]=0, \tag{6}$$

in which $c_i(\varphi)$ and $k_j(\varphi)$ are functions introduced in the partial integrations.

If *any* functions $\Phi(\varphi)$ and $\Psi(\psi)$ can be found that reduce (6) to an identity, then (3) and (4) immediately yield explicit, closed expressions for N and ϑ. Further integration would yield x and y as functions of φ and ψ. Simple radial and vortex flows correspond to choosing Ψ and Φ, in turn, to be constant. No other exact solutions of the functional Eqs. (5) and (6) are known at present, although all previously known compressible flows have not been tested for their possible conformity to these equations.

The utility of the method, as compared to others, will, of course, depend on whether exact or approximate solutions of (5) and (6) can be found that yield flow patterns of some physical interest. There are indications that such solutions can be found in the transonic region. It is noted that the description $\mathbf{N}=\nabla\varphi$ applies to fields other than those of constant entropy and stagnation pressure, and may, therefore, make possible investigations of compressible flows down-stream from regions of heating and viscous dissipation.

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[1] M. M. Munk, Phys. Rev. **72**, 176 (A) (1947).
[2] B. L. Hicks, Phys. Rev. **71**, 476 (A) (1947); B.R.L. Report No. 633, 1947. Also Hicks, Guenther, and Wasserman, Quart. App. Math. (October, 1947).

The Radial Density Variation of Gases and Vapors in a Centrifugal Field*

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IF the equation of state of a gas or vapor is expressed by the well-known relation

$$p=RT\left(\frac{d}{M}+B\frac{d^2}{M^2}+C\frac{d^3}{M^3}+\cdots\right)$$

where p is the pressure, T the absolute temperature, d the density, M the molecular weight, R the gas constant and where B, C, etc., are functions of T and are characteristic of the gas or vapor, then it can be shown theoretically that, under equilibrium conditions in a hollow centrifuge rotor with inside radius r spinning N r.p.s.,

$$\frac{2\pi^2MN^2r^2}{RT}=\log_e\frac{d_r}{d_0}+\frac{2B}{M}(d_r-d_0)+\frac{3C}{2M^2}(d_r^2-d_0^2)+\cdots, \tag{1}$$

where d_r and d_0 are the densities at the radius r and the axis, respectively. Since this latter equation may be used for determining the molecular weight of a gas or vapor, a precise experimental test of the relation has been undertaken. The apparatus consists of an air driven vacuum-type centrifuge which is similar to those previously described.[1] The rotor is a short accurately machined hollow cylinder