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The Hydrogen Atom in a Generalized Classical Electrodynamics

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The author recently proposed a generalized classical electrodynamics in which the usual assumption that the field of a moving electron is given by its retarded field is abandoned, and a combination of retarded and advanced fields is considered. This paper considers the application of the generalized equations of motion to investigate the motion of the electron in the hydrogen atom. It is shown that, unlike the Lorentz-Dirac equations, these generalized equations permit the electron to spiral round and fall into the nucleus.

I. INTRODUCTION

In some recent papers I have considered the application of Dirac's classical theory of radiating electrons[1] to various familiar problems; and the evidence gathered from these applications is that the equations of motion do not always have physically understandable solutions. There are, for instance, the self-accelerating motions that were considered by Dirac. Again, if an electron is moving along the x axis in an electric field of potential $\mu|x|$, that is, in the field due to a thin infinite charged plate in the yz plane, then the only motion allowed by the equations of motion is one in which the electron ultimately accelerates away from the plate with rapidly increasing speed, despite the attraction of the plate.[2] Further, when one investigates the motion of the electron in the hydrogen atom, it is seen that the electron does not spiral round and fall into the nucleus owing to loss of energy by radiation, as one would expect, but spirals outwards and escapes to infinity with increasing speed. The attraction of the nucleus has the effect of accelerating the electron away from the nucleus.[3] All these results point to the suggestion that the signs of certain terms in the Lorentz-Dirac equations do not fit in appropriately.

With a view to bring about the necessary alterations in sign, I have developed a generalized form of classical electrodynamics,[4] by abandoning the usual assumption that the field of a moving electron is given by its retarded field, and considering, instead, a combination of retarded and advanced fields. If F_{el} denotes the field due to an electron, and F_{ret} and F_{adv} the retarded and advanced fields, respectively, then we take

$$F_{\text{el}}=F_{\text{ret}}+k(F_{\text{ret}}-F_{\text{adv}}), \tag{1}$$

where k is a constant, the value of which is left arbitrary for the present. The equations of motion corresponding to the assumption (1) are then obtained in the usual way from the principles of conservation of energy, momentum, and angular momentum.

Suppose there are a number of electrons interacting with each other and with a given external field that is described by the field quantities $F_{\text{ext}}{}^{\mu\nu}$. If $z_{\mu n}(s_n)$, $(\mu=0, 1, 2, 3)$, denote the time and space coordinates of the nth electron of charge e_n and rest mass m_n, and dots denote differentiation with respect to the proper time s_n, and the metric tensor $g^{\mu\nu}$ is such that $g^{00}=1$, $g^{11}=g^{22}=g^{33}=-1$ while all the other components vanish, then the equations of motion of this electron are

$$\left.\begin{aligned} &m_n\frac{dv_{\mu n}}{ds}-\tfrac{2}{3}e_n{}^2(2k+1)\left[\frac{d^2v_{\mu n}}{ds^2}+\left(\frac{dv}{ds}\right)^2 v_{\mu n}\right] \\ &\quad =e_n v_{\sigma n}\Big[F^{\sigma}{}_{\mu,\,\text{ext}} \\ &\quad +\sum_{m\neq n}\{F^{\sigma}{}_{\mu m,\,\text{ret}}+k(F^{\sigma}{}_{\mu m,\,\text{ret}}-F^{\sigma}{}_{\mu m,\,\text{adv}})\}\Big], \\ &v_{\mu n}{}^2=1. \end{aligned}\right\} \tag{2}$$

As mentioned earlier the constant k is arbitrary. When $k=0$, Eqs. (2) reduce to the Lorentz-Dirac equations. When $k=-\frac{1}{2}$, then

$$F_{\text{el}}=\tfrac{1}{2}(F_{\text{ret}}+F_{\text{adv}}), \tag{3}$$

[1] P. A. M. Dirac, Proc. Roy. Soc. (A) 167, 148 (1938).
[2] C. J. Eliezer, Bull. Calcutta Math. Soc. 37, 125 (1945).
[3] C. J. Eliezer, Proc. Camb. Phil. Soc. 39, 173 (1943).
[4] C. J. Eliezer, Nat. Acad. Sci., India (1946).

and the equations of motion (2) become

$$\left.\begin{aligned} m dv_{\mu n}/ds = e v_{\sigma n}[F^{\sigma}{}_{\mu,\mathrm{ext}} \\ +\tfrac{1}{2}\sum_{m\neq n}(F^{\sigma}{}_{\mu m,\mathrm{ret}}+F^{\sigma}{}_{\mu m,\mathrm{adv}})], \\ v_{\mu n}{}^2 = 1. \end{aligned}\right\} \quad (4)$$

Equations (4) do not contain the usual damping terms. If $k<-\frac{1}{2}$, then the sign of the radiation damping term is opposite to that of the corresponding terms in the Lorentz-Dirac equations, and in the particular case when $k=-1$, the equations are the same, except for this difference of sign.

II. APPLICATION TO THE HYDROGEN ATOM

If we take the nucleus of the atom to be fixed (by considering its mass to be infinite), then with reference to axes whose origin is at the position of the nucleus, the equations of motion of the electron are

$$m\frac{d^2\mathbf{r}}{ds^2}-\tfrac{2}{3}e^2(2k+1)\left\{\frac{d^3\mathbf{r}}{ds^3}+\left[\left(\frac{d^2t}{ds^2}\right)^2-\left(\frac{d^2\mathbf{r}}{ds^2}\right)^2\right]\frac{d\mathbf{r}}{ds}\right\} = -e^2\frac{dt}{ds}\frac{\mathbf{r}}{r^3},$$
$$\left(\frac{dt}{ds}\right)^2-\left(\frac{dr}{ds}\right)^2=1, \quad (5)$$

where the (t, x, y, z) notation is now used. When $k>-\frac{1}{2}$, the behavior of the solution of Eqs. (5) will be similar to that of the solution of the Lorentz-Dirac equations, which have already been considered and which do not allow the electron to fall into the nucleus. When $k=-\frac{1}{2}$, Eqs. (5) reduce to the usual equations of motion without the radiation damping terms, and the solutions of these equations are well known.

In this paper we shall consider the behavior of the solution of Eqs. (5) when $k<-\frac{1}{2}$. It will be seen that the equations then allow the electron to spiral round and approach the nucleus closer and closer and ultimately fall into it.

III. RECTILINEAR MOTION

We shall first consider the simpler case of rectilinear motion of an electron moving along the x axis, the proton being situated at the origin of coordinates. The equations of motion (5) then give

$$a\frac{d^2x}{ds^2}-(2k+1)\times\left\{\frac{d^3x}{ds^3}-\frac{dx}{ds}\left(\frac{d^2x}{ds^2}\right)^2\left[1+\left(\frac{dx}{ds}\right)^2\right]^{-1}\right\} = -3\left[1+\left(\frac{dx}{ds}\right)^2\right]^{\frac{1}{2}}\Big/2x^2, \quad (6)$$

where $a=3m/2e^2$. It is unfortunately not possible to obtain an explicit solution of this equation. But our purpose here is to know roughly the nature of the motion, and this can be done without solving the equation. We write $dx/ds=u$ and transform the independent variable from s to x. Then $d^2x/ds^2=uu'$, $d^3x/ds^3=u(uu''+u'^2)$, where dashes denote differentiation with respect to x, and Eq. (6) becomes

$$(2k+1)\left\{u''+\frac{u'^2}{u(1+u^2)}\right\}-\frac{au'}{u}=\frac{3}{2}\frac{(1+u^2)^{\frac{1}{2}}}{u^2x^2}. \quad (7)$$

To discuss the behavior of the solution of Eq. (7) it is convenient to transform from u to y, where

$$y=(1+u^2)^{\frac{1}{2}}, \quad (8)$$

and y is taken to have positive values only. The cases u positive and u negative lead to different differential equations. For $u>0$, we have

$$(y^2-1)^{\frac{1}{2}}y''+\lambda y'+\mu x^{-2}=0, \quad (9)$$

where $\lambda=-a/(2k+1)>0$, $\mu=-3/2(2k+1)>0$; and for $u<0$, we have

$$(y^2-1)^{\frac{1}{2}}y''-\lambda y'-\mu x^{-2}=0, \quad (10)$$

where λ and μ are as before.

Suppose initially the electron is projected towards the proton from the point $x=x_0$. Its motion immediately afterwards is given by Eq. (10). The electron may be brought to rest at some point $x=\beta$ before it reaches the proton, or it may collide with the proton. In the former case the solution will be defined only in $\beta\leqslant x\leqslant x_0$, and in the latter case in $0\leqslant x\leqslant x_0$. Equation (10) is such that if $y'=0$ at any point where the solution is defined, then $y''>0$; and therefore y has no maxima and therefore at most one minimum. Also, if $y''=0$, then $(y^2-1)^{\frac{1}{2}}y'''$

$+2\mu x^{-3}=0$, and hence $y'''<0$. Therefore $y''=0$ at most once in $0\leqslant x<x_0$. We may therefore choose an interval $0\leqslant x\leqslant\gamma$ so that y' and y'' are of fixed signs in the portion of the interval in which the solution is defined.

Suppose $y'<0$. Then y increases as x decreases. Hence, as $x\to 0$, y tends to a finite limit $l(>1)$ or to infinity. It is easy to see that y cannot tend to a finite value l, because if it does, the equation obtained by integrating Eq. (10) with respect to x gives

$$y'-\lambda\cosh^{-1}y=\mu\int\frac{dx}{x^2(y^2-1)^{\frac{1}{2}}}+\text{constant}$$
$$=-\frac{\mu}{x}(1+\xi)(l^2-1)^{-\frac{1}{2}}+\text{constant}, \tag{11}$$

where $\xi\to 0$ as $x\to 0$. Integrating again with respect to x we see that the left-hand side remains finite as $x\to 0$, but the right-hand side becomes infinite, being approximately of the order $\mu(l^2-1)^{-\frac{1}{2}}\log(1/x)$, for small x. We thus obtain a contradiction. Hence, if $y'<0$, the solution is such that $y\to\infty$ as $x\to 0$.

We have already seen that if initially y' is negative, then as x decreases, y' will continue to be negative, since y has no maximum. The condition that y' is negative is equivalent to the condition that the acceleration is towards the proton, since $d^2x/ds^2=uu'=yy'$. We thus have the result that if the electron is projected with initial velocity and acceleration which are both towards the proton, then the electron continues to approach the proton with steadily increasing velocity and ultimately collides with the proton, with velocity tending to that of light.

It is of interest to obtain an approximate solution valid for small values of x. To do this, we transform from x to τ where $x=e^{-\tau}$. We then have

$$(y^2-1)^{\frac{1}{2}}\frac{d^2y}{d\tau^2}+\{(y^2-1)^{\frac{1}{2}}+\lambda e^{-\tau}\}\frac{dy}{d\tau}-\mu=0. \tag{12}$$

To obtain a solution which is valid for large values of τ, we note that $y\to\infty$ and $dy/d\tau>0$ as $\tau\to\infty$. It is seen that $dy/d\tau$ cannot tend to infinity, because if it did, then from Eq. (12), $d^2y/d\tau^2$ must tend to minus infinity; hence $dy/d\tau$ must be decreasing with increasing τ, which gives a contradiction. Hence $dy/d\tau\to k\geqslant 0$ as $\tau\to\infty$, and therefore $d^2y/d\tau^2\to -k$.

But according to a theorem on limits, if a function $f(x)$ tends to a finite limit as x tends to infinity, and $f'(x)$ is known to tend to a limit, then this limit must be zero. Hence $k=0$. Thus $dy/d\tau\to 0$ and $d^2y/d\tau^2\to 0$ as $\tau\to\infty$.

For large values of τ, Eq. (12) is approximately

$$\frac{d^2y}{d\tau^2}+\frac{dy}{d\tau}-\frac{\mu}{y}=0. \tag{13}$$

Writing $dy/d\tau=p$ and changing the independent variable to p and the dependent variable to y we have

$$p\frac{dp}{dy}+p-\frac{\mu}{y}=0,$$

which is easily seen to have approximate solution $p=\mu/y$, and hence $y^2=2\mu\tau+\text{constant}$. Hence

$$\frac{dx}{ds}\sim -\left(2\mu\log\frac{1}{x}\right)^{\frac{1}{2}},\ \text{for small } x, \tag{14}$$

where $\mu=-3/2(2k+1)$.

If initially $y'>0$, that is, if the initial acceleration is directed away from the proton, then in the subsequent motion, y' may change sign soon afterwards or may continue to remain positive until the electron is brought to rest at $x=\beta$. If y' changes sign and thus becomes negative, the subsequent motion will then be as discussed above, namely a motion in which the electron continues to approach the proton with steadily increasing velocity and ultimately collides with it. If, however, y' remains positive until y reaches the value 1, then the electron is brought to rest at $x=\beta$ before it reaches the proton. The approximate solution near $x=\beta$ is of the form

$$y=1+l(x-\beta)+\frac{2\sqrt{2}}{3l^{\frac{1}{2}}}\left(\lambda l+\frac{\mu}{\beta^2}\right)(x-\beta)^2, \tag{15}$$

where $y\to 1$ and $y'\to l(>0)$ as $x\to\beta$. If, however, $y'\to 0$ as $x\to\beta$ then the approximate solution near $x=\beta$ is given by

$$(y-1)^{\frac{3}{2}}=\frac{9}{4}\frac{\mu}{\beta^2}(x-\beta)^2. \tag{16}$$

The electron being thus brought to rest before it reaches the proton, it would then turn back and move away from the proton. The subsequent motion is then described by Eq. (9). We do not discuss here the motion in detail, but merely give a summary of the results. It is easy to show that there is no solution with $y\to\infty$, $x\to\infty$. The only possible types of solution are those in which either the electron is brought to rest while moving away from the proton, or in which the electron escapes to infinity while the velocity ultimately decreases and tends to a finite value. In the first case, the approximate solution near the point where the electron is brought to rest is similar in form to (15). The subsequent motion is such that the electron steadily approaches the proton with increasing velocity. In the second case, the solution for large x is given approximately by

$$y=C+\frac{\mu}{\lambda}\frac{1}{x}(1+\theta), \tag{17}$$

where $C\geqslant 1$ and $\theta\to 0$ as $x\to\infty$.

Under given initial conditions the final motion consists of either:—(i) the electron moves towards the proton with increasing velocity and collides with it, the approximate solution being (14); or (ii) the electron recedes away from the proton with steadily diminishing velocity, the approximate solution being of the form (17). An interesting feature in this result is that in both motions the final acceleration is directed towards the proton, which is in accord with the ideas of Newtonian mechanics where the acceleration is always in the direction of the force, which, in the problem under investigation, is towards the proton.

IV. TWO-DIMENSIONAL MOTION

Now we shall consider the more general motion of an electron which is spiralling round a fixed proton. The general motion will be three-dimensional, since the initial position, velocity and acceleration can all be arbitrary. Under certain initial conditions, however, the motion will be two-dimensional, and it is these motions that we shall consider here, as these are the only ones which are likely to be physically permissible. We shall chiefly be concerned here with the existence of solutions in which the electron spirals round and falls into the nucleus. Such motions were found to be not allowable by the Lorentz-Dirac equations[3] and it is of interest to determine whether the modified Eqs. (2) lead to different results. Taking axes in the plane of motion and the origin at the position of the proton, the equations of motion are

$$\left.\begin{aligned} a\frac{d^2x}{ds^2}-(2k+1)\left\{\frac{d^3x}{ds^3}+\frac{dx}{ds}\left(\frac{d^2t^2}{ds^2}-\frac{d^2x^2}{ds^2}-\frac{d^2y^2}{ds^2}\right)\right\}&=-\frac{3}{2}\frac{dt}{ds}xr^{-3},\\ a\frac{d^2y}{ds^2}-(2k+1)\left\{\frac{d^3y}{ds^3}+\frac{dy}{ds}\left(\frac{d^2t}{ds^2}-\frac{d^2x^2}{ds^2}-\frac{d^2y^2}{ds^2}\right)\right\}&=-\frac{3}{2}\frac{dt}{ds}yr^{-3}\end{aligned}\right\} \tag{18}$$

where

$$\frac{dt^2}{ds}=1+\frac{dx^2}{ds}+\frac{dy^2}{ds}.$$

We transform to V and ψ, where

$$V=\left(\frac{dx^2}{ds}+\frac{dy^2}{ds}\right)^{\frac{1}{2}}$$

is the velocity, and $\pi-\psi$ is the angle between the radius vector and the velocity, so that

$$\frac{dx}{ds}=V\cos(\pi+\theta-\psi),\quad \frac{dy}{dt}=V\sin(\pi+\theta-\psi).$$

We change the independent variable from s to τ where $r=e^{-\tau}$ and let dashes now denote differentiation with respect to τ. The equations of motion then give

$$(2k+1)\left\{V''+V'+\frac{V'^2}{V(1+V^2)}-V'\psi'\tan\psi-V(1+V^2)(\psi'-\tan\psi)^2\right\}-\frac{ae^{-\tau}V'}{V\cos\psi}=-\frac{3}{2}\frac{(1+V^2)^{\frac{1}{2}}}{V^2\cos\psi}, \tag{19}$$

$$(2k+1)\left\{\frac{d}{d\tau}(\psi'\cos\psi-\sin\psi)+\left(1+\frac{3V'}{V}\right)(\psi'\cos\psi-\sin\psi)\right\}$$
$$-\frac{ae^{-\tau}}{V\cos\psi}(\psi'\cos\psi-\sin\psi)=\frac{3}{2}\frac{(1+V^2)^{\frac{1}{2}}}{V^3}\tan\psi. \tag{20}$$

We seek those solutions according to which the electron spirals round and ultimately falls into the nucleus. Assuming that this happens, let us consider the behavior of the solution as $r\to 0$, that is, as $\tau\to\infty$.

We follow the method used in an earlier paper and easily see that except possibly for the case when $V\to\infty$ and $\psi\to 0$ as $\tau\to\infty$, there is no solution in which the electron falls into the nucleus. We shall therefore concentrate our attention on the possible solution in which $V\to\infty$ and $\psi\to 0$. For large τ, Eq. (20) is approximately

$$\psi''-\psi'+(1+3V'/V)(\psi'-\psi)=0,$$

which has solution

$$\psi'-\psi=Ce^{-\tau}/V^3, \tag{21}$$

where C is an arbitrary constant of integration. Substituting in (19) and approximating we obtain

$$(2k+1)\{V''+V'+V'^2/V^3\}=-3/2V. \tag{22}$$

This equation can be shown to be equivalent to Eq. (10) which has already been considered above. Hence we have as an approximate solution of (22) $V=(2\mu\tau)^{\frac{1}{2}}$. Substituting in (21), we obtain

$$\psi'-\psi=Ce^{-\tau}/(2\mu\tau)^{\frac{3}{2}}.$$

Hence

$$\psi e^{-\tau}=C\int_{\infty}^{\tau}\frac{e^{-2\tau}}{(\mu\tau)^{\frac{3}{2}}}d\tau \tag{23}$$

$$=\frac{Ce^{-2\tau}}{\mu^{3/2}}\left[-\frac{1}{2\tau^{3/2}}+\frac{3}{2^2}\cdot\frac{1}{\tau^{5/2}}-\frac{3\cdot 5}{2^3}\cdot\frac{1}{\tau^{7/2}}+\cdots\right]. \tag{24}$$

Hence for a first approximation we take

$$\psi=Ae^{-\tau}\tau^{-\frac{3}{2}}, \tag{25}$$

where A is a positive constant of integration. Therefore

$$rd\theta/dr=-\tan\psi=-Ae^{-\tau}\tau^{-\frac{3}{2}},\ \text{for large}\ \tau;$$

and hence

$$d\theta/d\tau=Ae^{-\tau}\tau^{-\frac{3}{2}}.$$

Therefore

$$\theta=A\int e^{-\tau}\tau^{-\frac{3}{2}}d\tau=Ae^{-\tau}\left\{-\frac{1}{\tau^{3/2}}+\frac{3}{2}\cdot\frac{1}{\tau^{5/2}}-\frac{3\cdot 5}{2^2}\cdot\frac{1}{\tau^{7/2}}+\cdots\right\}.$$

Hence the equation of the curve is given approximately, for small r, by

$$\theta=Ar\,(\log 1/r)^{-\frac{3}{2}}. \tag{26}$$

Thus, the equations of motion (2) allow the electron to spiral round and fall into the nucleus, whereas the Lorentz-Dirac equations do not permit such a collision.