separated isotopes. As soon as projected experiments to determine the spectrum and to count absolutely the density of quanta in the x-ray beam have been completed, it will be possible to measure excitation functions and absolute cross sections for these disintegration processes.

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The Paths of Ions and Electrons in Non-Uniform Crossed Electric and Magnetic Fields*

NORMAN D. COGGESHALL
Gulf Research and Development Company, Pittsburgh, Pennsylvania
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The integration of the force equations for a charged particle moving in the presence of particular types of crossed, non-uniform, electric, and magnetic fields is shown to be possible in a very simple manner. The cases which admit of the integration are those such that: the magnetic field is a function of only one coordinate (either Cartesian or the radius vector in a polar system); the electric field is a function of only one coordinate which for any particular case is the same coordinate with which the magnetic field varies; the electric field has a component only in the direction of the variable with which it varies; and the two fields are orthogonal. Since these conditions, in most cases, are only met on a median plane symmetrically situated relative to magnetic pole faces and electrostatic electrodes, the calculations refer to the motion in this plane. The equations are solved and discussions made of the orbits for several different field arrangements and for one a new type of perfect focusing. The method can be used with numerical integration when the analytical difficulties are too great or when the fields are only known empirically.

INTRODUCTION

THE integration of the Lorentz force equation for charged particles moving in the presence of certain types of non-uniform magnetic fields has been reported earlier.[1] The cases dealt with were those such that the magnetic field varied as a function of one coordinate only, either Cartesian or the radius in cylindrical or polar coordinates; the motion of the charged particles was in a plane perpendicular to the magnetic field (this in some cases restricts the motion to the median plane between the pole faces of a magnet or electromagnet); and there was no electric field present. This integration was of such a nature that it could be done analytically if the functions were simple enough or, if not, it could always be done numerically to as fine an accuracy as desired. Furthermore, for cases where the magnetic field was known only experimentally, the integration could be done numerically to an accuracy equal to that of the data. Several cases of focusing were reported, some of which do not correspond to anything obtainable with uniform fields.

In the present paper the method of integration has been extended to cover certain cases of combined magnetic and electric fields which may or may not be uniform. The restrictions are: The magnetic field is constant or is a function of only one coordinate (either Cartesian or the radius vector in spherical or cylindrical coordinates); the electric field is constant or a function of only one coordinate which for any particular crossed field arrangement is the same coordinate with which the magnetic field varies; the electric and magnetic fields are perpendicular to each

* An abstract of a preliminary report is given in Phys. Rev. **68**, 98 (1945).
[1] N. D. Coggeshall and M. Muskat, Phys. Rev. **66**, 187 (1944).

other; and the electric field has a component only in the direction of the variable with which it varies. A simple example of a crossed field arrangement which meets the above requirements is afforded by a cylindrical condenser located in the center of a long solenoid and coaxial with it. Here the magnetic field is along the length of the solenoid and varies only with the cylindrical radius measured from the center; the electric field between the condenser electrodes is perpendicular to the magnetic field and varies only with the same radius. Orbits for this and other combinations will be discussed later. Some of these have appeared in the literature earlier but using different methods of calculations, usually based on approximations.

GENERAL THEORY

When a charged particle moves in the presence of both an electric and magnetic field the force it experiences is given by:

$$\mathbf{F} = e\mathbf{E} + e\mathbf{v}\times\mathbf{H}/c, \qquad (1)$$

where e is the charge of the particle in e.s.u., c is the velocity of light in cm/sec., and v is the velocity of the particle in cm/sec.

If the electric field is a function of only one Cartesian coordinate, let us say x, and has only a component parallel to the x axis we have a kinetic energy equation:

$$\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) = \tfrac{1}{2}mv_0^2 + e\int_{x_0}^{x} E(x)dx,$$

where v_0 is the initial velocity of the particle at a value of $x=x_0$. If the magnetic field is in the z direction, the component of the velocity in the z direction is constant and we may use instead of the above equation the following one:

$$\dot{x}^2+\dot{y}^2 = \frac{2eV_0}{m} + \frac{2e}{m}\int_{x_0}^{x} E(x)dx, \qquad (2)$$

where V_0 represents an initial kinetic energy expressed as potential.

In view of the special conditions named above Eq. (1) may be written as:

$$\ddot{y} = (e/mc)H\dot{x}, \qquad (3)$$

$$\ddot{x} = eE/m - (e/mc)H\dot{y}. \qquad (4)$$

If we define a function $f(x)$ as:

$$f(x) = \int^{x} H(x)dx + \bar{c}, \qquad (5)$$

where $\bar{c}$ is a constant of integration, we may integrate Eq. (3) to obtain:

$$\dot{y} = (e/mc)f(x). \qquad (6)$$

Substituting Eq. (6) in Eq. (2) and solving for the ratio of the velocity components we obtain:

$$\frac{dy}{dx} = \pm\frac{f(x)}{[hV_0 + hV(x) - f^2(x)]^{\frac{1}{2}}}, \qquad (7)$$

where

$$V(x) = \int_{x_0}^{x} E(x)dx$$

and

$$h = 2mc^2/e.$$

If the electric field is everywhere zero, Eq. (7) becomes just the equation obtained in the earlier paper[1] in which the paths of charged particles in non-uniform magnetic fields were considered.

It is to be noticed that Eq. (7) contains three constants which are effective in characterizing the orbits, i.e., $\bar{c}$, h, and V_0. The constant $\bar{c}$ is associated with the initial slope of the orbit as it may be seen that, for a group of orbits of charged particles of the same mass and of the same initial kinetic energy, the only difference in Eq. (6) for the different ones will be a variation in the value of $\bar{c}$. From their definitions, h and V_0 obviously specify the mass and initial energy of the charged particle.

The above named constants completely define an orbit and from them certain information as to its extension may be obtained without integrating Eq. (7). Since the terms in Eq. (7) must be real, we have the extension of the orbit limited by:

$$hV_0 + hV(x) - f^2(x) \geqslant 0.$$

If for a given set of constants the equation:

$$hV_0 + hV(x) - f^2(x) = 0, \qquad (8)$$

has only one real root the orbit may extend to either positive or negative infinity in the x direction whereas if it contains two real roots the orbit may be limited in extension in the x direction. In the latter case the roots of Eq. (8)

may represent an x_{max} and an x_{min} as for these values $dy/dx=\pm\infty$.

Since the right-hand side of Eq. (7) is a function of x only, the integration of it to obtain the orbit in terms of y as a function of x is always possible, if not analytically, at least numerically. It is to be further noted that in case the magnetic field variation is known only experimentally or if the function defining $H(x)$ is too difficult, the integration involved in Eq. (5) may be done numerically with a subsequent numerical integration of Eq. (7).

If the electric field is such that it varies only with the radius vector r in cylindrical coordinates and has a component only in this direction, the kinetic energy equation may be written as:

$$\dot{r}^2+(r\dot{\theta})^2=\frac{2eV_0}{m}+\frac{2e}{m}\int_{r_0}^{r}E(r)dr. \quad (9)$$

If in addition the magnetic field has only a component in the z direction and if it varies only as a function of r, the force equations can be written as:

$$\ddot{r}-r(\dot{\theta})^2=eE-(e/mc)Hr\dot{\theta}, \quad (10)$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})=(e/mc)H\dot{r}. \quad (11)$$

If we at this time define a function $g(r)$ by:

$$g(r)=\frac{1}{r}\int_{r_0}^{r}H(r)rdr+\frac{\bar{c}}{r}, \quad (12)$$

$\bar{c}$ again being a constant of integration, we may integrate Eq. (11) to get:

$$\dot{\theta}=\frac{1}{r}\frac{e}{mc}g(r).$$

Substituting this result in Eq. (9) and solving for the ratio of the time derivatives we obtain:

$$\frac{d\theta}{dr}=\pm\frac{g(r)}{r[hV_0+hV(r)-g^2(r)]^{\frac{1}{2}}}, \quad (13)$$

where h has the same meaning as before and,

$$V(r)=\int_{r_0}^{r}E(r)dr.$$

Equation (13), for cylindrical coordinates, is analogous to Eq. (7) for Cartesian coordinates and here also information as to extension of the orbits is obtainable. The orbits may extend to $r=+\infty$ or lie between definite values of r depending upon whether the equation:

$$hV_0+hV(r)-g^2(r)=0 \quad (14)$$

has one or two real roots. Here again it is to be noticed that the integration necessary to obtain any orbit is always possible at least numerically if not analytically. This follows as the integration in Eq. (12) can be done numerically if necessary with a subsequent numerical integration of Eq. (13).

Since for free space

$$\operatorname{div} H=0 \quad \text{and} \quad \operatorname{div} E=0,$$

it is not possible, except for a few special cases, to obtain crossed fields that satisfy the conditions named above in a three-dimensional region. Usually the conditions of orthogonality and functional variation necessary for the above integrations exist only on a median plane which may be symmetrically located relative to magnetic pole faces and properly located relative to electrodes supplying an electrostatic field. In the development of Eqs. (7) and (13) it was assumed that the motion is in the median plane and that $\dot{z}=0$. If $\dot{z}\neq 0$, Eqs. (7) and (13) will give the projection of the orbit on the same plane if the required field conditions exist in a three-dimensional region. If the field conditions are satisfied only on the median plane and $\dot{z}\neq 0$, Eqs. (7) and (13) will give an approximate representation of the projection of the orbit onto this plane in a region around the point at which the orbit crosses the plane.

As was seen in the earlier paper[1] and as will be seen later in this one, there are many cases of electron and ion orbits which are spatially periodic. This periodicity, or wave-length, is of great interest in focusing applications. The wave-length will depend upon the constants $\bar{c}$, h, and V_0 and sometimes upon others which are used in specifying the functional variation of E and H. In a mass spectrometer use may be made of the periodicity by placing the ion source and ion collector in the proper geometrical arrangement

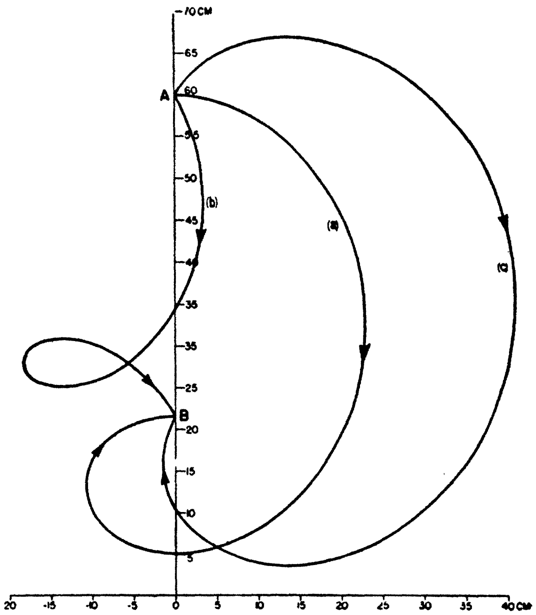

FIG. 1. Typical cycloidal orbits for case of E and H constant in Cartesian coordinates. $H_0=1500$ gauss, $E_0=30$ volts per cm, $V_0=600$ volts, $M=44$, (a) $\bar{c}=0$, (b) $\bar{c}=2.03\times10^4$, (c) $\bar{c}=-2.03\times10^4$.

and with a certain distance of separation between them. If then the fields can be varied in intensity so as to give ions of a particular mass an orbit wave-length equal to this separation, they will enter the collector. The design of an ion source for a particular field arrangement may depend upon the manner in which the orbit wave-length depends upon the parameters associated with the ion source. If the spatial period or wave-length λ is independent of the parameter $\bar{c}$ but is a function of the initial energy V_0 the ion source must be carefully designed to yield as nearly monoenergetic ions as possible. If the period is independent of the initial energy V_0 but depends upon the starting direction parameter $\bar{c}$ the ion source must have a precise slit system to produce a parallel beam of ions. Examples of the different manners in which the wave-length depends upon the parameters will be seen below. Considerations of symmetry lead to the fact that if the orbits are periodic then the solution of Eq. (7) gives x implicitly as a periodic function of y or the solution of Eq. (13) gives r implicitly as a periodic function of θ. Since the integrand is either a function of x or r, the direct deduction of the type of periodicity and the functional dependence on the parameters poses a mathematical problem of some interest. If this were possible it would allow the focusing properties of certain field arrangements to be obtained without the integration of Eq. (7) or Eq. (13). This would be valuable as a number of arrangements of physical interest do not give integrals which are in the form of a finite number of terms. Also in cases such as this, numerical integration will not lead to complete conclusions as to the wave-lengths of the orbits and their dependence on the various parameters.

Let us suppose it is desired to find fields such that a pre-selected type of orbit is obtained. Theoretically this should be possible if for the chosen orbit dy/dx is a function of x alone or if $d\theta/dr$ is a function of r alone. If either the electric or the magnetic field is chosen then substitution in Eq. (7) or (13) will yield the other. Only in special cases would it be possible to calculate fields which would produce a pre-selected family of orbits. This is true as only in special cases will the parameter which defines the family of orbits enter into the equations in the same manner as one of the above named parameters, i.e., h, V_0, E_0, H_0, $\bar{c}$.

SPECIFIC EXAMPLES

(A) Constant Electric and Magnetic Fields

The orbits and their properties for this case, deduced by other means, have been known for some time[2,3] and an instrument has been built which utilized them.[3] This treatment therefore produces no essentially new physical aspects but is included as it allows a more direct approach to calculations attendant to the orbits and it illustrates the application of the method. If we designate the constant electric and magnetic fields by E_0 and H_0 the differential equation for the orbit is:

$$\frac{dy}{dx}=\pm\frac{(H_0x+\bar{c})}{[hV_0+hE_0x-(H_0x+\bar{c})^2]^{\frac{1}{2}}}. \tag{15}$$

[2] Cf. Page and Adams, *Principles of Electricity* (D. Van Nostrand Company, Inc., New York), Chap. VIII.

[3] W. Bleakney and J. A. Hipple, Phys. Rev. **53**, 521 (1938).

This may be readily integrated to yield:

$$y=\pm\frac{1}{H_0}\left\{\left[(hV_0-\bar{c}^2)+(hE_0-2H_0\bar{c})x-H_0^2x^2\right]^{\frac{1}{2}}-\frac{hE_0}{2H_0}\sin^{-1}\left(\frac{2H_0^2x+2H_0\bar{c}-hE_0}{\left[h^2E_0^2+4hH_0^2V_0-4hE_0H_0\bar{c}\right]^{\frac{1}{2}}}\right)\right\}+F, \tag{16}$$

F = constant defining position in x, y plane. Equation (16) is that of a cycloid and the periodicity can be recognized from the arcsin function. From this the periodicity or wave-length λ of the orbits is easily found to be:

$$\lambda=2\pi mc^2E_0/eH_0^2,$$

the same result as obtained by Bleakney and Hipple. The dependence of λ upon the various parameters is ideal for focusing applications in a mass spectrometer or similar instrument. This is true since λ is independent of the initial energy and direction of the ions, which allows greater latitude in the design of the ion source and also permits ion currents of greater intensity. Furthermore the period λ is seen to be linearly dependent upon the mass. These facts were recognized by the previously named investigators and using them they built a mass spectrometer of improved focusing properties.[3] In Fig. 1 may been seen the manner of utilizing the periodicity for focusing. Here all ions of the same mass which leave point A will follow different orbits, each of which, however, has the same period, i.e., the distance between A and B. Therefore, if the ion source is located at A and a collector at B we have an example of perfect focusing. In Fig. 1 the three curves are for ions of the same mass but different angles of departure from A, equivalently, different $\bar{c}$'s.

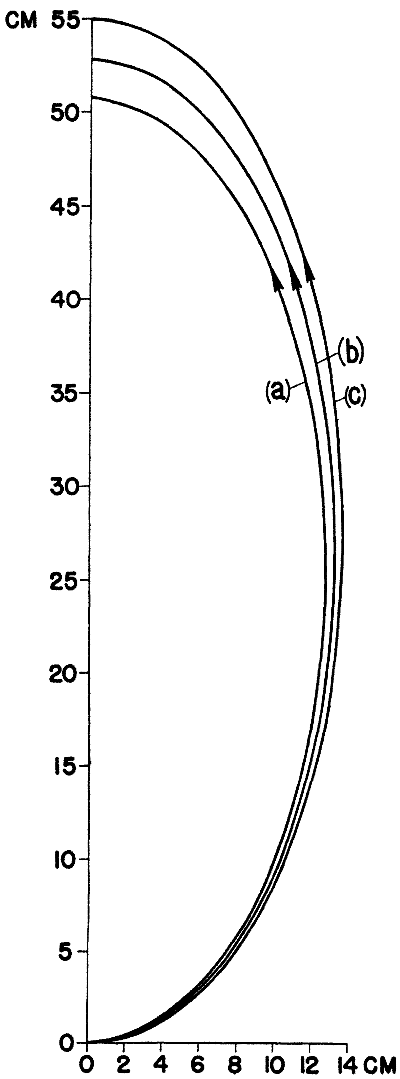

FIG. 2. Nearly elliptical orbits for case of H constant, $E=E_0x$. $H=H_0=2.5\times10^3$ gauss, $E_0=0.0333$ e.s.u., $M=44$, $V_0=300$ volts, (a) $\bar{c}=2.88\times10^2$, (b) $\bar{c}=0$, (c) $\bar{c}=-2.88\times10^2$.

(B) $E=E_0x$, $H=H_0$ = constant

The differential equation for this case is:

$$\frac{dy}{dx}=\pm\frac{(H_0x+\bar{c})}{\left[hV_0+\frac{hE_0}{2}x^2-(H_0x+\bar{c})^2\right]^{\frac{1}{2}}}. \tag{17}$$

If we define the quantities A, B, and D as follows:

$$A=(hV_0+hE_0\bar{c}^2/2H_0^2),$$

$$B=-\bar{c}hE_0/H^2,$$

$$D=(hE_0/2H_0^2-1),$$

F = constant defining position of orbit in x, y plane,

$$v=H_0x+\bar{c}.$$

Eq. (17) may be integrated to yield:

$$y=\pm\frac{1}{H_0}\left\{\frac{(A+Bv+Dv^2)^{\frac{1}{2}}}{D}-\frac{B}{2D\sqrt{D}}\log\left[(A+Bv+Dv^2)^{\frac{1}{2}}+v\sqrt{D}+\frac{B}{2\sqrt{D}}\right]\right\}+F \text{ for } D>0, \qquad (18)$$

and

$$y=\pm\frac{1}{H_0}\left\{\frac{(A+Bv+Dv^2)^{\frac{1}{2}}}{D}-\frac{B}{2D(-D)^{\frac{1}{2}}}\sin^{-1}\left[\frac{-2Dv-B}{(B^2-4AD)^{\frac{1}{2}}}\right]\right\}+F \text{ for } D<0. \qquad (19)$$

It is to be noticed that the numerical sign of D is a criterion of the type of orbit since Eq. (18) gives open or aperiodic orbits and Eq. (19) gives closed or periodic orbits. Since E is an odd function of x, then for any admissible orbit or portion thereof on one side of the line $x=0$ there may be a corresponding orbit or portion thereof on the other side and they will be symmetrical with respect to the line $x=0$.

From the nature of the terms in Eq. (19) it can be seen that orbits passing through $x=0$ and with small values of dy/dx at $x=0$ will be nearly ellipses. It is therefore of interest to investigate this case to see if a type of focusing equivalent to that used in a 180° analyzer exists. In the 180° analyzer all ions leaving the ion source travel in circles and a beam of ions of one momentum value that diverges from the exit slit will converge to form a focus approximately at a point which is 180° removed from the exit slit and on the circle traveled by the principal ray.[4] This focusing is characterized by the fact that the principal ray or orbit of the ion that leaves the exit slit in a direction perpendicular to the diameter of the circle, on which are located the ion source and the point of focus, will recross this diameter at a maximum distance from the source, i.e., ions of different starting directions will recross the diameter at points nearer the ion source. If we can demonstrate the existence of such an extremum for the principal ray in the present case, it will show if such focusing is possible.

The distance along the y axis which separates the point where an ion passing in the positive direction crosses the line $x=0$, from the point where it recrosses this line traveling in a negative direction, is given by:

$$\Delta y=2\{y(x=x_{\max})-y(x=0)\}.$$

[4] Cf. E. B. Jordan and L. B. Young, J. App. Phys. 9, 526 (1942).

A necessary condition for the extremum discussed above is that the following equation be satisfied:

$$(\partial\Delta y/\partial\bar{c})_{\bar{c}=0}=0. \qquad (20)$$

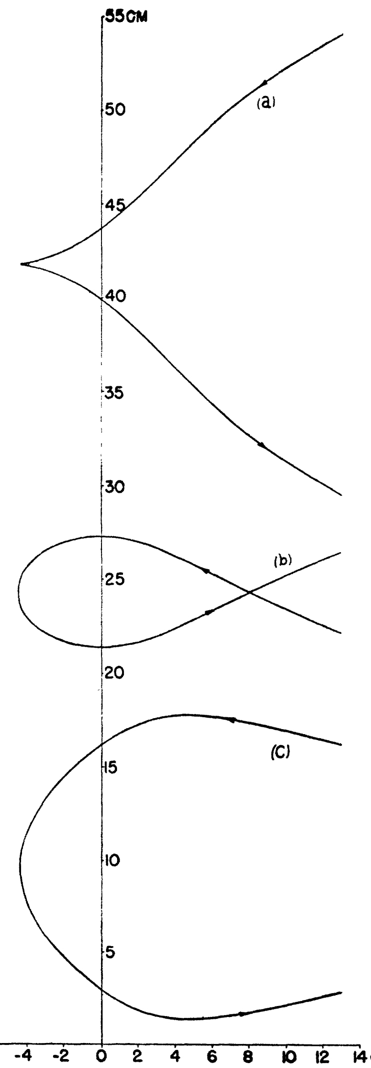

FIG. 3. Various orbits obtained for case of H constant, $E=E_0x^2$. $V_0=300$ volts, $H_0=2.5\times10^3$ gauss, $E_0=0.0333$ e.s.u., $M=44$, (a) $\bar{c}=1.175\times10^4$, (b) $\bar{c}=0$, (c) $\bar{c}=-1.175\times10^4$.

The validity of Eq. (20) can be easily tested by evaluating Δy in terms of the parameters involving $\bar{c}$. When the differentiation is carried out it is seen that Eq. (20) is not satisfied; hence we cannot expect the nearly elliptical orbits passing through $x=0$ to show focusing. This is further confirmed by an actual plot of three orbits which have only small differences in starting directions. These are seen in Fig. 2. Here (a), (b), and (c) are orbits with starting angles of $+1°$, $0°$, and $-1°$ relative to the x axis.

(C) $E=E_0x^2$, $H=H_0=$ constant

For this case the differential equation for the orbits is:

$$\frac{dy}{dx}=\pm\frac{H_0x+\bar{c}}{[hV_0+(hE_0/3)x^3-(h_0x+\bar{c})^2]^{\frac{1}{2}}}. \quad (21)$$

Equation (21) may be integrated in terms of elliptic functions but the trajectory of an ion or electron resulting from a given set of initial conditions is more conveniently obtained by numerical integration. In Fig. 3 may be seen three trajectories for an ion of mass 44 which crosses the line $x=0$ with energy equal to 300 electron volts and with a slope of $+1$, 0, -1. In curve (a) what appears to be a cusp in reality is a smooth turning point with an extremely small radius of curvature at that point. From the denominator of Eq. (21) it is clear that there can be either periodic or aperiodic orbits for a given set of parameter values depending upon where the particle starts its trajectory relative to the three roots of the cubic in the radical. When Eq. (21) is set up in terms of standard elliptic integrals it is seen that the periods of the orbits depend upon $\bar{c}$ and thus upon the starting slopes.

(D) $E=E_0e^{bx}$, $H=H_0e^{bx}$

In case the electric and magnetic fields vary in an exponential manner as above we have the following relationships:

$$f=\frac{H_0}{b}e^{bx}+\bar{c}, \quad V=V_0+\frac{E_0}{b}e^{bx},$$

$$\frac{dy}{dx}=\pm\frac{(H_0/b)e^{bx}+\bar{c}}{\{hV_0+(hE_0/b)e^{bx}-[(H_0/b)e^{bx}+\bar{c}]^2\}^{\frac{1}{2}}}. \quad (22)$$

If we make the transformation $v=e^{bx}/b$, Eq. (22) becomes:

$$\frac{dy}{dv}=\pm\frac{1}{b}\frac{(H_0v+\bar{c})}{v[hV_0+hE_0v-(H_0v+\bar{c})^2]^{\frac{1}{2}}}. \quad (23)$$

Using the following definitions:

$$A=(hV_0-\bar{c}^2),$$
$$B=(hE_0-2H_0\bar{c}),$$
$$D=-H_0^2.$$

We find Eq. (23) may be readily integrated to yield, upon substitution for v:

$$y=\pm\left[\frac{1}{b}\frac{H_0}{\sqrt{-D}}\sin^{-1}\left\{\frac{-2De^{bx}-bB}{b(B^2-4AD)^{\frac{1}{2}}}\right\}-\frac{\bar{c}}{b\sqrt{A}}\log\left\{\frac{(b^2A+bBe^{bx}+De^{2bx})^{\frac{1}{2}}+b\sqrt{A}}{e^{bx}}+\frac{B}{2\sqrt{A}}\right\}\right]+F \quad (24)$$

for $A>0$, and

$$y=\pm\left[\frac{1}{b}\frac{H_0}{\sqrt{-D}}\sin^{-1}\left\{\frac{-2De^{bx}-bB}{b(B^2-4AD)^{\frac{1}{2}}}\right\}-\frac{\bar{c}}{b\sqrt{-A}}\sin^{-1}\left\{\frac{Be^{bx}+2bA}{e^{bx}(B^2-4AD)^{\frac{1}{2}}}\right\}\right]+F \text{ for } A<0. \quad (25)$$

An examination of the roots of the quadratic under the radical sign in Eq. (22) as well as an examination of the forms of Eqs. (24) and (25) shows that Eq. (24) represents aperiodic orbits which extend to $x=-\infty$, and that Eq. (25) represents periodic orbits bounded in the x dimension. Equation (25) represents periodic orbits with the spatial period λ given by:

$$\lambda=2\pi\left|\frac{H_0}{b\sqrt{-D}}-\frac{\bar{c}}{b\sqrt{-A}}\right|. \quad (26)$$

These orbits are clearly not suited for focusing of the type illustrated in Section A (cycloidal

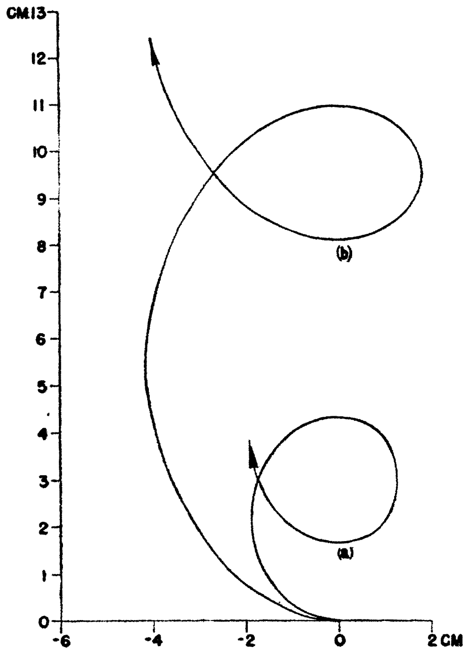

FIG. 4. Type of orbits obtainable for case of $H=H_0e^{bz}$ and $E=E_0e^{bz}$. $H_0=10^4$ gauss, $E_0=1/30$ e.s.u., $b=\frac{1}{3}$, $V_0=600$ volts, (a) $M=16$, (b) $M=44$.

orbits) as Eq. (26) shows the periodicity to depend upon two of the parameters of which it would be desirable to have the wave-length independent. These two parameters are V_0 and $\bar{c}$ which are related, respectively, to the initial energy and starting direction of each orbit. Figure 4 shows two such periodic orbits as represented by Eq. (25).

(F) $H=H_0=$ constant, $E=E_0/r$

The properties of orbits of charged particles in this arrangement of fields are particularly interesting for several reasons. One reason is that such an arrangement may be achieved in electronic vacuum tubes of ordinary size. If a cylindrical condenser is placed inside a solenoid and with the axes of the condenser and the solenoid coincident we have the necessary conditions for this arrangement satisfied. There have been a number of theoretical papers written about the orbits for this set of conditions[5–7] although in each case they have been concerned with orbits which are nearly circular. The properties of these nearly circular orbits may be obtained by approximations which do not eliminate the time t from Eqs. (10) and (11). These investigators have shown that, if these nearly circular orbits diverge from a point on the orbit of the principal ray, they will reunite to converge to a focus at a point which is on the same circle as the starting point but which is angularly displaced from it by an angle of $\pi/\sqrt{2}$. The circular orbit of the principal ray in this case is the equilibrium orbit wherein the combined effect of the electric and magnetic fields is to produce the angular acceleration necessary for uniform circular motion. A mass spectrograph utilizing this focusing principle has been built and operated.[8]

As will be seen, the present treatment does not conveniently give the same information about the nearly circular orbits but may on the other hand give information about orbits which are not nearly circular and for which the above mentioned approximations do not hold.

Making the proper substitutions in Eqs. (12) and (13) we arrive at the following differential equation for orbits with these field conditions:

$$\frac{d\theta}{dr}=\pm\frac{1}{r}\frac{(H_0r/2)+(\bar{c}/r)}{\{hV_0+hE_0\log r-[(H_0r/2)+(\bar{c}/r)]^2\}^{\frac{1}{2}}}. \tag{27}$$

Unfortunately Eq. (27) cannot be integrated to give a closed system of simple functions. The integration by series is complicated by the nature of the poles where $d\theta/dr=\infty$. Numerical integration seems the best approach with particular care being taken as r approaches a point of singularity. It is to be noticed that the squared term in the radical in the denominator will dominate the remainder of the radical for small values of r and also for large values of r. This means then that all orbits in this system are bounded in the r dimension. They are thus, then, either closed or periodic, in general, periodic

[5] W. Bartky and A. J. Dempster, Phys. Rev. **33**, 1019 (1929).
[6] R. Herzog, Zeits. f. Physik **19**, 335 (1934).
[7] R. G. E. Hutter, Phys. Rev. **67**, 248 (1945).
[8] H. Bondy, G. Johannsen, and K. Popper, Zeits. f. Physik **95**, 46 (1935).

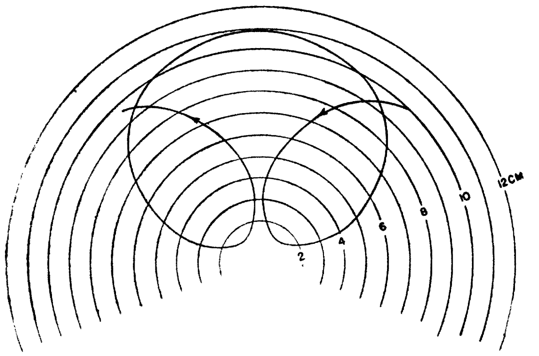

FIG. 5. Periodic type of orbit for case of H constant, $E=E_0/r$. $H_0=2.5\times10^3$ gauss, $E_0=0.175$ e.s.u., $V_0=100$ volts, $M=44$, $\bar{c}=-1.125\times10^4$.

with r as a periodic function of θ. The dependence of the periodicity upon the parameters, V_0, h, H_0, E_0, and $\bar{c}$, would be highly interesting to know and perhaps of value for practical application but the nature of Eq. (27) makes this quite difficult if not impossible. It may be possible to deduce this dependence of the periodicity from the functional form of the right hand of Eq. (27) but the author has not been able to do so. A numerical integration approach is at best unsatisfactory.

Figure 5 shows more than a full cycle of a periodic orbit. This was obtained by numerical integration. Presumably orbits could be determined in this way for ions diverging from a point with a small angular spread and reconverging to form a focus, if such existed. This, however, must be done with considerable care as the integration for the orbit through points such that $d\theta/dr=\infty$ means numerically evaluating an improper integral. For Fig. 5 it is interesting to note the upper and lower limit for r in the orbits for the parameters used. These limits may, of course, be brought nearer each other by adjusting H_0, E_0, and V_0.

For every cycle of such an orbit as seen in Fig. 5, it passes twice through a value of r such that $d\theta/dr=0$, and from Eq. (27) we can see that there will be only one such value of r. Those portions of the orbit for greater values of r as well as those for smaller values may be regarded as analogous to the semi-circular paths in a 180° analyzer. Because of the difficulties of numerical integration the focusing effect cannot easily be determined although a knowledge of the dispersion of such a system can be easily obtained. Figure 6 shows a group of four orbits for ions of atomic masses 1, 2, 12, and 44. The orbits are shown for the values of r less than such that $d\theta/dr=0$. The orbits shown correspond to those that would exist if a beam of ions were originated at a point 10 cm from the center, with an initial energy of the ions of 1000 electron volts, and with the ions all directed radially toward the center. It will be noticed that the separation on the $r=10$ cm circle between the points where the orbits for ions of one mass and those of a different mass again return to the condition $d\theta/dr=0$ is very large for small values of M but becomes smaller rapidly for increasing values of M.

In Figs. 5 and 6 the orbits are for positively charged ions with a field applied radially outward from the center. In Fig. 7 are four orbits for the case of positive ions and a field applied radially in towards the origin. These are for ions of atomic mass 12, with a starting direction radially inwards, from a point 10 cm from the center and with initial energies of 300, 1000, 1800, and 3600 electron volts.

In conclusion regarding this arrangement of fields, it is highly desirable that an analytical means of integrating Eq. (27) become available as the numerical method is difficult because of the type of infinity of the integrand encountered and also an analytical treatment might give clearer information regarding periodicity and focusing.

(F) $H=H_0=$constant, $E=E_0r$

This case, for which the electric field varies as the first power of the radial distance from the center, and is directed along this radius vector

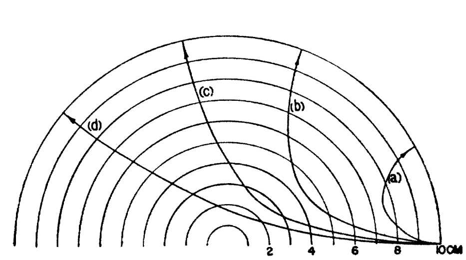

FIG. 6. Portions of orbits obtainable for case of H constant, $E=E_0/r$. $H_0=10^3$ gauss, $E_0=0.175$ e.s.u., $V_0=1000$ volts, $\bar{c}=-5\times10^4$, (a) $M=1$, (b) $M=2$, (c) $M=12$, (d) $M=44$.

is not a natural one in the sense of the field just previously treated, i.e., for a cylindrical condenser. The present field could be approximated by a double system of concentric guard rings or electrodes with the proper impressed voltages.

The differential equation of the orbits for this case is:

$$\frac{d\theta}{dr}=\pm\frac{(H_0r/2)+(\bar{c}/r)}{r\left[hV_0+\frac{hE_0}{2}r^2-\left(\frac{H_0r}{2}+\frac{\bar{c}}{r}\right)^2\right]^{\frac{1}{2}}}. \quad (28)$$

If we make the following definitions:

$$A=-\bar{c}^2,$$
$$F=(hV_0-H_0\bar{c}),$$
$$D=(\tfrac{1}{2}hE_0-\tfrac{1}{4}H_0^2).$$

Equation (28) may by simple substitutions be integrated to give:

$$\theta=\frac{H_0}{4D}\log\left\{[A+Br^2+Dr^4]^{\frac{1}{2}}+r^2\sqrt{D}+\frac{B}{2D}\right\}$$
$$+\tfrac{1}{2}\sin^{-1}\left\{\frac{Br^2+2A}{r^2(B^2-4AD)^{\frac{1}{2}}}\right\}\text{ for } D>0, \quad (29)$$

and

$$\theta=\frac{H_0}{4}\frac{1}{\sqrt{-D}}\sin^{-1}\left\{\frac{-2Dr^2-B}{(B^2-4AD)^{\frac{1}{2}}}\right\}$$
$$+\tfrac{1}{2}\sin^{-1}\left\{\frac{Br^2+2A}{r^2(B^2-4AD)^{\frac{1}{2}}}\right\}\text{ for } D<0. \quad (30)$$

As Eq. (29) is for aperiodic orbits and Eq. (30) for periodic ones we have $\frac{1}{2}hE_0>\frac{1}{4}H_0^2$ or $\frac{1}{2}hE_0<\frac{1}{4}H_0^2$ as the criterion for such. Here we are interested principally in the case of periodic orbits. We note that Eq. (30) is for orbits where in r is a periodic function of θ. When Eq. (30) is evaluated for a specific case it becomes evident that the period is given by:

$$\lambda_\theta=\pi-(\pi H_0/2(-D)^{\frac{1}{2}}). \quad (31)$$

Since the term D in Eq. (31) involves only m, E_0, and H_0 and not V_0 or $\bar{c}$ we see that here again we have the perfect focusing of the type found for the cycloidal orbits. This means that for any orbit the angular period λ_0 as given by

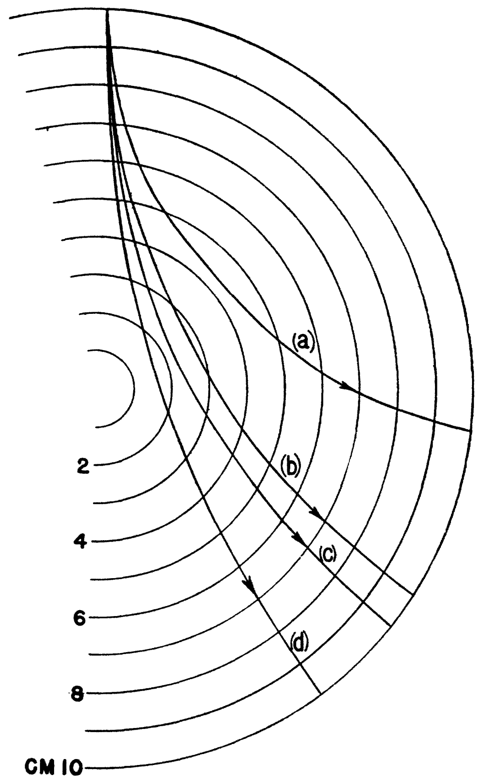

FIG. 7. Other variations in orbits obtainable for H constant, $E=E_0/r$. $H=H_0=10^3$ gauss, $E_0=-0.175$ e.s.u., $M=12$, $\bar{c}=-5\times10^4$, (a) $V_0=300$ volts, (b) $V_0=1000$ volts, (c) $V_0=1800$ volts, (d) $V_0=3600$ volts.

Eq. (31) is independent of the starting energy V_0 and of the starting velocity vector which is characterized by $\bar{c}$. Thus an instrument using such fields could afford considerable latitude in the design of the ion source or electron source. This would make it possible to achieve more intense ion currents as wide slits and a strong ion or electron collecting voltage may be used. In an ordinary mass spectrometer, for instance, the slit widths and ion collecting voltage must both be small in order to insure good focusing and resolution. However, these conditions for good focusing result in weak ion currents with the resultant need of electrometer amplifiers which by their nature are difficult to use.

In Fig. 8 can be seen a family of this type of orbits for ions of the same mass and charge. Here (a), (b), and (c) are orbits with considerably different starting directions from the point A. As is seen these orbits converge at the point B which is one angular wave-length displaced from

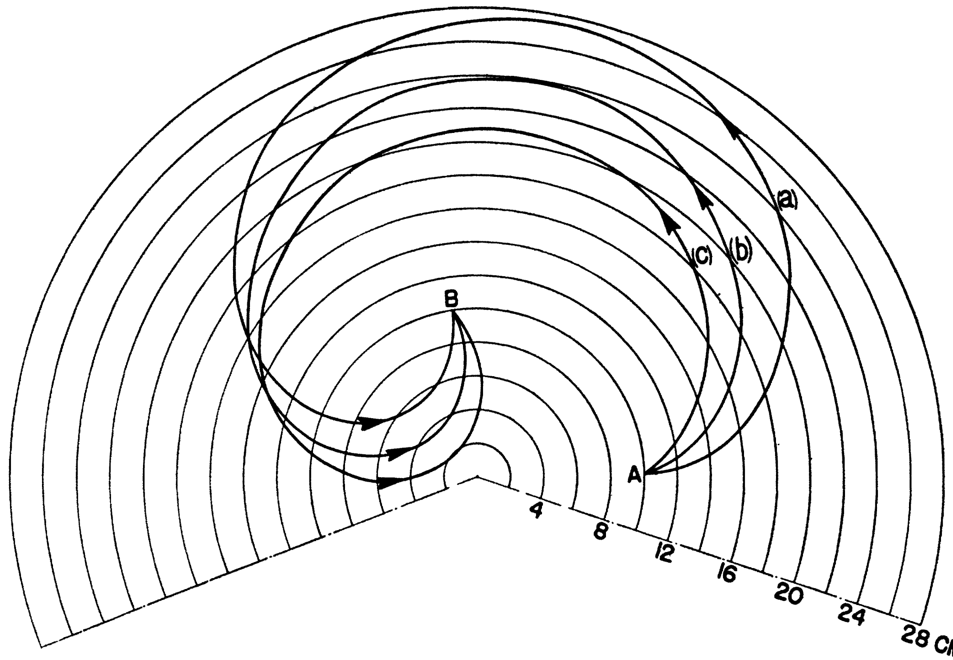

FIG. 8. Periodic orbits with preferred focusing properties for case of H constant, $E=E_0r$. $H_0=4\times10^3$ gauss, $E_0=5$ volts per cm, $M=44$, $V_0=600$ volts, (a) $c=-2\times10^5$, (b) $c=-1.13\times10^5$, (c) $c=-5\times10^4$.

A and on the same circle. In an instrument utilizing this focusing the ion source could be located at A and the ion collector at B. Since the focusing is independent of the starting voltage, relatively small voltages could be used in the ion source thus reducing the troubles due to stray fields from within the ion source interfering with the field outside the source. Also from the nature of the periodicity it is not necessary that the slits be aligned as carefully as in other types of instruments. Furthermore the angular separation need not be adjusted with particular care as the periodicity can be adjusted using either E_0 or H_0. Since the constant h is proportional to the mass of the ion and since h and E_0 enter in the definition of D as:

$$D=\tfrac{1}{2}hE_0-\tfrac{1}{2}H_0^2$$

we see that if H_0 is held constant it is necessary to vary E_0 alone to bring in different ion masses to the collector. The relation, $hE_0=\text{constant}$, is convenient for this procedure.

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