

THE DISTRIBUTION OF ENERGY IN THE NORMAL RADIATION SPECTRUM.

By LEIGH PAGE.

IN their attempt to derive a radiation formula from classical dynamics Rayleigh and Jeans have been led to an expression which is in disagreement with experiment in so far as it demands that in the state of equilibrium the radiant energy in the ether should consist mainly of waves of infinitesimally short wave-length, and that the total energy per unit volume should be infinite. Lorentz has been led to the same expression by a consideration of the absorption and emission due to the free electrons contained in a piece of metal placed in a field of stationary radiation. In fact it has been asserted that this radiation formula is based only on the most general principles of classical dynamics and electrodynamics, and that any other expression for the distribution of energy in the normal radiation spectrum must be in contradiction to these principles.

The only radiation formula agreeing with experimental observation is that due to Planck. In obtaining his formula Planck considers the absorption and emission of ideal linear oscillators capable of executing only simple harmonic vibrations. His justification in using this special mechanism lies in the fact that Kirchhoff's law shows that the distribution of energy among the waves of different frequencies in a stationary field of radiation depends only upon the temperature of the material bodies with which the radiation is in equilibrium, and not at all upon their nature. However Planck finds it necessary, in order to obtain his formula, to make a number of revolutionary assumptions, at least one of which is in contradiction with classical electrodynamics. The following are the more important assumptions on which Planck bases his theory:

(a) While an oscillator absorbs energy continuously according to classical electromagnetic theory, the radiation of energy demanded by the electrodynamic equations is replaced by an emission by quanta. To be more specific, Planck assumes that an oscillator of frequency ν can emit only when its energy is an integral multiple of $h\nu$, h being Planck's constant, and that when emission does take place the oscillator loses all its energy and starts absorbing afresh. Such an assumption

not only involves a form of emission of energy utterly outside that contained in the electromagnetic theory, but is in contradiction with classical electrodynamics in that it supposes the radiation of energy demanded by the electrodynamic equations to be non-existent.

(b) The ratio of the probability of no emission, when the energy accumulated by an oscillator is an integral multiple of $h\nu$, to the probability of an emission is proportional to the intensity of the vibrations exciting the oscillator. If we grant assumption (a) assumption (b) seems reasonable, although its only justification lies in the fact that, in addition to the other assumptions, it enables Planck to deduce a formula which is in agreement with experimental facts.

(c) The constant of proportionality contained in assumption (b) is determined in such a way that the formula derived shall reduce to the Rayleigh-Jeans expression for long wave-lengths.

These three assumptions enable Planck to find the energy density of radiation for any frequency in terms of the average intrinsic energy of the oscillators of the same frequency with which the radiation is in equilibrium. To determine the value of the average energy of the oscillators in terms of their temperature, another assumption is necessary:

(d) If w is the probability that any given oscillator lies in any particular element of extension-in-phase, then in passing from one element of extension-in-phase to the next $\Delta w/w$ is not sufficiently small to make it possible to replace the summation by an integration in the expression for the entropy of the oscillators. In fact the elements of extension-in-phase are supposed to be of a size equal to the action quantum h . Consequently Planck finds for the average energy of a linear oscillator of frequency ν the expression

$$\frac{h\nu}{2} + \frac{h\nu}{e^{kT} - 1}$$

instead of the expression

$$kT \tag{2}$$

given by classical dynamics.

From these four assumptions Planck deduces the expression

$$u_\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{kT} - 1} \tag{3}$$

for the energy density per unit frequency of homogeneous radiation of frequency ν and temperature T , c being the velocity of light in vacuum.

The object of this paper is to show that the radiation formula (3)

may be obtained from the expressions given by classical dynamics and electrodynamics for the absorption and radiation of energy, provided we make use of a single supplemental assumption in determining statistically the distribution of energy among the absorbing and emitting oscillators. We shall, like Planck, make use of Kirchhoff's law in that we shall consider a stationary field of radiation in equilibrium with ideal linear oscillators capable of executing simple harmonic vibrations. We shall imagine a large number of these oscillators of every frequency placed inside a rigid perfectly reflecting envelope, and allowed to come into equilibrium with one another and with the enclosed radiation. These oscillators will absorb and emit according to classical dynamics and electrodynamics. Hence if f is the electric moment of an oscillator in terms of Lorentz's unit of electric charge, the energy of its oscillatory motion at any instant will be given by

$$U = \frac{1}{2}Lf^2 + \frac{1}{2}Kf^2 = 2\pi^2\nu^2C^2L, \quad (4)$$

where L and K are constants, and where C is the maximum value of the electric moment, or the product of the amplitude of vibration by the charge. The rate of absorption of energy from the electromagnetic field will be given by¹

$$\frac{dA}{dt} = \frac{u_\nu}{12L} \quad (5)$$

and the rate of radiation by¹

$$\frac{dR}{dt} = \frac{2\pi\nu^2U}{3c^3L}. \quad (6)$$

So far we have made use only of classical dynamical and electrodynamical theory. To determine statistically the average value $\bar{U}$ of the energy of oscillations of frequency ν we shall now introduce a supplemental assumption, to wit:

The motion of an emitting and absorbing linear oscillator of frequency ν is stable only for those amplitudes for which the energy of its oscillations is an integral multiple of $h\nu$.

Hence the energy associated with an oscillation of frequency ν must be equal to $nh\nu$ where n is an integer. However it is not necessary for the development of our theory that the energy of the oscillation should be *exactly* $nh\nu$ but merely that it should lie between $nh\nu - \delta$ and $nh\nu + \delta$, where δ is very small compared to $h\nu$. Moreover it is not necessary to assume that the energy is, at every instant, an integral multiple of $h\nu$

¹ See "Planck's Heat Radiation" (translation by Masius) pages 160 and 165 for corresponding expressions in electrostatic units.

within the limits assigned above, but merely that the time during which the energy is *not* an integral multiple of $h\nu$ shall be very small compared to the time during which the energy is an integral multiple of $h\nu$.

In attempting to explain how this condition of the oscillator may be maintained—in view of the fact that the rate of absorption of energy is changing from instant to instant, and that even if the average absorption were just sufficient to balance the emission from an oscillator of energy $nh\nu$ it would not balance the emission from one of the energy $(n+1)h\nu$ —it is necessary to remember that there are forms of restricted motion which involve neither emission nor absorption of energy. For example, a ring of evenly spaced electrons revolving in a circle about a positive charge placed at the center would neither emit nor absorb energy, so far as the motion in the plane of the circle is concerned. If such a ring, however, vibrated as a whole in a direction perpendicular to its plane, it would constitute one of the simplest types of linear oscillators consistent with Rutherford's model of the atom. So far as the degree of freedom involved in this vibration is concerned, there would be both absorption and emission of energy. However there might be some connection between the two degrees of freedom referred to, such that the energy associated with the absorbing and emitting degree of freedom might always tend toward an integral multiple of $h\nu$, any surplus or deficit of energy being taken care of by the other degree of freedom.

Again, there is no reason to suppose that an electron itself might not possess considerable energy of rotation. If the charge on the electron is distributed symmetrically with respect to its axis, there will be no emission of energy in consequence of its rotation. In addition to its rotation about a diameter, the electron might constitute the movable element of an ideal linear oscillator. Some connection between the vibratory motion and the rotation might result in a distribution of energy such that the energy of the oscillations would always tend toward an integral multiple of $h\nu$.

This speculation as to the method by which the vibratory energy of an oscillator is maintained at some integral multiple of $h\nu$ does not provide any mechanism by which the energy of the oscillations may pass from one integral multiple of $h\nu$ to another. Such a transition, however, would probably take place only during the encounters between different oscillators. In fact, if we are going to apply the theory of probability to the derivation of an expression for the average energy of an oscillator, we must suppose that such transfers of energy may occur. This does not limit the transfers of energy to cases where the ratio of the frequencies of the colliding oscillators has certain particular values. For if the

collision results in any surplus or deficit of oscillatory energy above or below the nearest integral multiple of $h\nu$, such surplus or deficit will be rapidly extinguished by transfer of energy between the emitting and non-emitting degrees of freedom.

A slightly different interpretation of our fundamental assumption would consist in the supposition that only when the energy of an oscillator was between $nh\nu - \delta$ and $nh\nu + \delta$ could the oscillator exist in the emitting and absorbing form. For energies outside of these limits the electrons and positive nuclei constituting the oscillator might form new aggregations such that their energy would be associated with types of motion involving no absorption or emission. A change of the energy content, as a result of collision, to the proper value would cause the elements of the oscillator to so rearrange themselves as to again constitute an absorbing and emitting mechanism. From this point of view we would have particles of all energies—the distribution of energy following the Maxwell-Boltzmann law—but only those whose energies were integral multiples of $h\nu$ would count in determining the conditions of equilibrium between matter and ether.

AVERAGE ENERGY OF AN IDEAL LINEAR OSCILLATOR IN TERMS OF ITS FREQUENCY AND TEMPERATURE.

Suppose that we have inside a rigid adiabatic envelope N_1 linear oscillators of frequency ν_1 , N_2 of frequency ν_2 , etc. in thermal equilibrium with one another and with N_0 non-absorbing and non-emitting degrees of freedom. Since the energy of each oscillation is completely determined by its amplitude of vibration, the state of the system so far as the oscillations are concerned will be determined by the way in which N_1 oscillations of frequency ν_1 , N_2 oscillations of frequency ν_2 , etc., are distributed among the elements of extension-in-phase in a four-dimensional generalized space having as coördinates the three positional coördinates x , y , z , and q , where $q = C^2$. Since the energy of an oscillation of frequency ν is limited to values between $nh\nu - \delta$ and $nh\nu + \delta$, where n can assume integral values only, the elements of extension-in-phase will not fill the whole of this generalized space, and will have different positions for oscillations of different frequencies. For example, the projection on the x , y , q space of the elements of extension-in-phase for oscillations of frequency ν will be the solids bounded by the planes obtained by giving integral values to n in the equations

$$2\pi^2\nu^2qL = nh\nu - \delta,$$

$$2\pi^2\nu^2qL = nh\nu + \delta.$$

So far as the N_0 non-emitting degrees of freedom are concerned, the elements of extension-in-phase will fill the whole of the generalized space having as coördinates x, y, z and the variables determining the energy, since the energy associated with each of these degrees of freedom is in no way restricted to particular values.

Let us suppose that the state under consideration is one in which N_{00} non-emitting degrees of freedom are in element of extension-in-phase O_0 , N_{01} in element of extension-in-phase I_0 , etc., N_{10} oscillations of frequency ν_1 are in element of extension-in-phase O_1 , N_{11} in element of extension-in-phase I_1 , etc. Then the thermodynamic probability of the state, or the number of ways in which the state can be formed, is

$$W = \left\{ \frac{N_0!}{N_{00}!N_{01}!N_{02}!\cdots} \right\} \left\{ \frac{N_1!}{N_{10}!N_{11}!N_{12}!\cdots} \right\} \left\{ \frac{N_2!}{N_{20}!N_{21}!N_{22}!\cdots} \right\} \{ \} \cdots,$$

where, of course, the second figure in the subscripts does *not* refer to the same element of extension-in-phase for oscillations of one frequency as for those of another, nor for oscillations as for non-emitting degrees of freedom, at least in so far as the coördinates specifying the energy are concerned.

Now the state of equilibrium is that state the probability of which is a maximum. Since the logarithm of a positive quantity is an increasing function of the quantity itself we may find the maximum of $\log W$ instead of that of W . If by w_{ij} we denote N_{ij}/N_i , *i. e.*, the mathematical probability of an oscillation of frequency ν_i being in element of extension-in-phase j , it may easily be shown that¹

$$\log W = - \sum_i N_i \sum_j w_{ij} \log w_{ij}. \quad (7)$$

¹ In getting (7) from the expression for W use is generally made of Sterling's formula. A simpler and more direct method which obviates the use of Sterling's formula is the following:

If x is a large integer

$$\log x! = \log 2 + \log 3 + \cdots + \log x$$

$$\doteq \int_1^x \log x dx$$

$$\doteq x \log x - x$$

since the curve $y = \log x$ approaches parallelism to the x axis as x increases.

Hence

$$\begin{aligned} \log W &= \sum_i \{ \log N_i! - \sum_j \log N_{ij}! \} \\ &= \sum_i \{ N_i \log N_i - N_i - \sum_j N_{ij} \log N_{ij} + \sum_j N_{ij} \} \\ &= \sum_i N_i \{ \sum_j w_{ij} \log N_i - \sum_j w_{ij} \log N_{ij} \} \end{aligned}$$

as $\sum_j w_{ij} = 1$.

Hence

$$\log W = - \sum_i N_i \sum_j w_{ij} \log w_{ij}.$$

For equilibrium $\log W$ must be a maximum subject to the conditions that the total volume and the total energy remain constant.

For maximum of $\log W$ we have

$$\sum_i N_i \sum (\log w_{ij} + 1) \delta w_{ij} = 0.$$

From the constant volume condition we have as many equations as there are values of i , of the form

$$N_i \sum_j \delta w_{ij} = 0.$$

From the condition of constant total energy we get

$$\sum_i N_i \sum_j U_{ij} \delta w_{ij} = 0,$$

where U_{ij} is the average energy of a degree of freedom of type i in element of extension-in-phase j .

Using Lagrange's method of undetermined multipliers we get

$$w_{ij} = \alpha_i e^{-\beta U_{ij}}, \quad (8)$$

where β is independent of the type i of the particular degree of freedom under consideration while α_i is not.

So far as the non-emitting degrees of freedom are concerned U_{0j} may have any value from zero to infinity, although for oscillations of frequency ν_i , U_{ij} is limited to values between $n h \nu_i - \delta$ and $n h \nu_i + \delta$, except possibly for inappreciable periods of time immediately following collisions. Consequently expression (8) does not lead to the principle of equipartition of energy, namely that the average kinetic energy associated with each degree of freedom is $1/2\beta$ quite irrespective of its type or frequency. In fact, for the emitting and absorbing oscillations we have

$$w_{ij} = \alpha_i e^{-\beta n h \nu_i}. \quad (9)$$

Hence the average energy of an oscillation of frequency ν will be

$$\begin{aligned} \bar{U}_\nu &= \frac{\sum_0^\infty N_\nu \alpha_\nu e^{-\beta n h \nu} n h \nu}{\sum_0^\infty N_\nu \alpha_\nu e^{-\beta n h \nu}} \\ &= h \nu \frac{\sum_0^\infty n e^{-\beta n h \nu}}{\sum_0^\infty e^{-\beta n h \nu}}. \end{aligned}$$

If we put $x = \beta h\nu$, we have

$$\begin{aligned}\bar{U}_\nu &= h\nu \frac{e^{-x} + 2e^{-2x} + 3e^{-3x} + \dots}{1 + e^{-x} + e^{-2x} + \dots} \\ &= h\nu \frac{e^{-x}}{(1 - e^{-x})^2} (1 - e^{-x}) \\ &= \frac{h\nu}{e^x - 1}.\end{aligned}\tag{10}$$

In order to find x in terms of the temperature T we must first determine α_ν and β in terms of the total volume V and the total energy E_ν of all the oscillations of frequency ν . Let G_i denote the size of an element of extension-in-phase for oscillations of frequency ν_i . Then

$$\begin{aligned}G_i &= \iiint dx dy dz dq \\ &= g_i \iiint dx dy dz,\end{aligned}$$

where $g_i = \int dq$.

Hence, for oscillations of frequency ν_i

$$\begin{aligned}\sum_j w_{ij} &= \frac{\alpha_i g_i}{G_i} \sum_i e^{-\beta n h \nu_i} \iiint dx dy dz \\ &= \frac{V g_i}{G_i} \frac{\alpha_i}{1 - e^{-x}}\end{aligned}$$

since the elements of extension-in-phase fill up the whole of the x, y, z space, even though there are unoccupied gaps in the q space.

But

$$\sum_j w_{ij} = 1.$$

Therefore

$$\alpha_i = \frac{G_i}{V g_i} (1 - e^{-x}).\tag{11}$$

If E_i is the total energy of all the oscillations of frequency ν_i

$$\begin{aligned}E_i &= N_i h \nu_i \frac{\alpha_i g_i}{G_i} \sum_i n e^{-\beta n h \nu_i} \iiint dx dy dz \\ &= \frac{N_i h \nu_i}{e^x - 1}\end{aligned}\tag{12}$$

as can be seen at once from (10).

Hence

$$e^x = 1 + \frac{N_i h \nu_i}{E_i}.\tag{13}$$

So, if we put

$$y = \frac{E_i}{N_i h \nu_i} = \frac{\bar{U}_i}{h \nu_i},$$

$$\beta = \frac{1}{h \nu_i} \log \left(1 + \frac{h \nu_i}{\bar{U}_i} \right) = \frac{1}{h \nu_i} \log \left(\frac{1 + y}{y} \right) \quad (14)$$

and substituting the value of e^x from (13) in (11)

$$\alpha_i = \frac{G_i}{V g_i} \frac{1}{\left(1 + \frac{\bar{U}_i}{h \nu_i} \right)} = \frac{G_i}{V g_i} \frac{1}{(1 + y)}. \quad (15)$$

Hence the probability w_{in} of an oscillation of frequency ν_i having an intrinsic energy between $nh\nu_i - \delta$ and $nh\nu_i + \delta$ is, from (9)

$$w_{in} = \frac{y^n}{(1 + y)^{1+n}}. \quad (16)$$

Now we are ready to write down the expression for the entropy of the oscillations. If W_i is the thermodynamic probability of the state of equilibrium of the N_i oscillations of frequency ν_i , then the entropy S_i of these N_i oscillations will be, by definition,

$$S_i = k \log W_i. \quad (17)$$

As we are dealing with oscillations of a single frequency we can drop the subscript i . Hence

$$\begin{aligned} S &= k \log W \\ &= -kN \sum_j w_j \log w_j \\ &= -kN \left[\log \alpha - \frac{x}{e^x - 1} \right] \\ &= kN \log \left\{ \frac{Vg}{G} \frac{(1 + y)^{1+y}}{y^y} \right\}. \end{aligned} \quad (18)$$

Now, by definition

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E} \right)_v.$$

Hence

$$\begin{aligned} \frac{1}{T} &= \frac{k}{h\nu} \log \frac{1 + y}{y}, \\ \frac{1}{y} &= e^{\frac{h\nu}{kT}} - 1 = \frac{Nh\nu}{E} = \frac{h\nu}{\bar{U}} \end{aligned}$$

and

$$E = \frac{N h \nu}{e^{kT} - 1} \quad (19)$$

or¹

$$\bar{U} = \frac{h \nu}{e^{kT} - 1} \quad (20)$$

Also

$$\beta = \frac{1}{h \nu} \log \frac{1 + y}{y} = \frac{1}{kT}, \quad (21)$$

which is independent of the frequency or nature of the oscillation, as it should be.

Moreover, we have from general thermodynamic theory

$$\frac{p}{T} = \left(\frac{\partial S}{\partial V} \right)_E$$

giving the laws of Charles, Boyle, and Avogadro, namely:

$$pV = NkT.$$

THE RADIATION FORMULA.

The radiation formula follows from the results of the preceding analysis at once. In the state of equilibrium the rate of radiation must, on the average, be equal to the rate of absorption. Hence from (5) and (6)

$$u_\nu = \frac{8\pi\nu^2 \bar{U}}{c^3}. \quad (22)$$

Substituting the value of $\bar{U}$ from (20) we get the radiation formula

$$u_\nu = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{kT} - 1}. \quad (23)$$

THE PARTITION OF ENERGY.

From the radiation formula we can find the way in which energy is distributed among the different degrees of freedom in the ether in the case of thermodynamic equilibrium. It is, however, necessary to emphasize one essential difference between the ether and material media.

¹ This expression is the same as that obtained from the first form of Planck's theory. Indeed much of our formal analysis is similar to that by means of which Planck first obtained his radiation formula, although the interpretation given is quite different. See *Vorlesungen über die Theorie der Wärmestrahlung*, first edition, page 157.

Energy travelling in the form of waves of frequency ν through a material medium may be converted in part or in whole into energy of another frequency by the agency of the medium itself. Such is never the case in the ether. Energy travelling through the ether in the form of waves of frequency ν will remain unchanged forever unless transformed into waves of another frequency by being absorbed and reemitted by matter. Hence the distribution of energy among the different degrees of freedom in the ether is conditioned entirely by the characteristics of the matter with which it is in equilibrium. Consequently this distribution would be quite different for ether in equilibrium with matter so constituted that the energy per degree of freedom associated with each radiating and absorbing oscillation of frequency ν is an integral multiple of $h\nu$, from what it would be if the material oscillations were capable of containing any amounts of energy whatsoever.

Consider N_1 linear oscillators of frequency ν_1 , N_2 of frequency ν_2 , etc., inside a rigid adiabatic perfectly reflecting envelope in thermal equilibrium with each other and with N_1' ethereal modes of vibration of frequency ν_1 , N_2' ethereal modes of vibration of frequency ν_2 , etc. The state of equilibrium is the one having maximum probability. Following the same line of reasoning as before, the probability that a material oscillation of frequency ν has an energy between $nh\nu - \delta$ and $nh\nu + \delta$ is, as before,

$$w = \alpha e^{-\beta nh\nu} \quad (24)$$

while the probability that any one degree of freedom of vibration in the ether of frequency ν has an energy U' is

$$w' = \alpha' e^{-\beta U'},$$

where U' is not restricted in value in the same way as is the energy of the material oscillations. Now the elements of extension-in-phase must be all of equal probability. Hence U' , which denotes the *average* energy represented by an element of extension-in-phase, must change in going from one to the next by a constant amount, say ϵ . So we have

$$w' = \alpha' e^{-\beta n\epsilon}. \quad (25)$$

Consequently we can show by exactly the same method as used in the case of the linear oscillations in matter that the average energy $\bar{U}'$ associated with each degree of freedom in the ether of frequency ν is given by

$$\bar{U}' = \frac{\epsilon}{e^{\frac{\epsilon}{kT}} - 1}. \quad (26)$$

Now the number of degrees of freedom per unit volume of the ethereal modes of vibration of frequencies between ν and $\nu + d\nu$ is given by the well known expression

$$Nd\nu = \frac{8\pi\nu^2 d\nu}{c^3}. \quad (27)$$

Consequently

$$u_\nu d\nu = \frac{\frac{8\pi\epsilon\nu^2}{c^3} d\nu}{\frac{\epsilon}{e^{kT}} - 1}$$

or

$$u_\nu = \frac{\frac{8\pi\epsilon\nu^2}{c^3}}{\frac{\epsilon}{e^{kT}} - 1}. \quad (28)$$

Comparing with (23) we see that

$$\epsilon = h\nu$$

and that *the average energy associated with each degree of freedom of a simple harmonic vibration of frequency ν , whether in matter or in ether, is given by*

$$\bar{U} = \bar{U}' = \frac{h\nu}{\frac{h\nu}{e^{kT}} - 1}. \quad (29)$$

SUMMARY.

Planck's radiation formula has been derived from the expressions given by classical dynamics and electrodynamics for the absorption and radiation of energy with the aid of the supplemental assumption that the motion of an absorbing and emitting linear oscillator of frequency ν is stable only when the energy of its oscillations is an integral multiple of $h\nu$. This assumption avoids the necessity of replacing the emission of energy demanded by the electrodynamic equations by an emission that is discontinuous in time and difficult, if not impossible, to reconcile with the phenomena of diffraction and interference.

The partition of energy among the different degrees of freedom in the ether has been investigated, and found to be the same as in material oscillators.

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