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A SPECTRAL LUMINOSITY CURVE EQUATION AND ITS USE.

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SEVERAL attempts have been made to obtain an equation that would express the luminosity curve of the average eye as a function of an equal energy spectrum. Goldhammer,[1] Hertzsprung,[2] and Nutting[3] have each proposed equations that have the general form required. They have all been simple, transcendental equations and have fitted the experimental curves only approximately. The experimental curves have likewise been only approximate and there has been little value in obtaining an equation to fit any one closely.

Recently, however, considerable work[4] has been done which seems to define the luminosity curve rather better than heretofore. While much more work with a still larger number of observers remains to determine it even more accurately, it seemed worth while at this time to get an equation that would fit the latest determination within the limits of these values. Such an equation would be of value in a definition of the luminosity curve and in various computations of radiant efficiency, the mechanical equivalent of light, etc.

In an attempt to obtain an equation the application of a resonance, or absorption equation, at once suggests itself if we desire to solve it from a theoretical viewpoint. At the present time such an equation would necessarily be largely empirical in its details, and an attempt to make it fit the observed points closely would make it too complicated to be useful. The present effort was therefore to obtain by some simple means an arbitrary equation. The equations that have been suggested have

[1] Ann. Ph., XVI., p. 621 (1905).

[2] Z. Wiss. Phot., IV., p. 43 (1906).

[3] "The Luminous Equivalent of Radiation," Nutting, Bull. of Bur. of Stds., V., 1908–9, p. 273.

[4] "Physical Photometry with a Thermopile Artificial Eye," Ives & Kingsbury, Phys. Review, VI, 5 (1915), p. 319.

probably included too few constants and where agreement has been obtained between the calculated and the experimental curves, it seems to have been accidental.

For example, the form proposed by Goldhammer,

$$L_\lambda = R^n e^{n(1-R)}, \tag{1}$$

where $R = \lambda_{\text{max}}/\lambda$, has two parameters, one of which, λ_{max}, is the value of the argument at which the function is a maximum. This is fixed by experiment. Consequently, the only change that can be made is to vary the remaining parameter, n, which merely narrows or broadens the curve, fitting only part of the observed points.

Nutting[1] found that with $n = 181$, function (1) would fit the luminosity curve he had determined experimentally very well from about $\lambda = .48\mu$ to $\lambda = .65\mu$, using $\lambda_{\text{max}} = .555$. This formula below $.48\mu$ gives values too small, and beyond $.65\mu$ too large, with a total area $(\mu \times L_\lambda)$ of

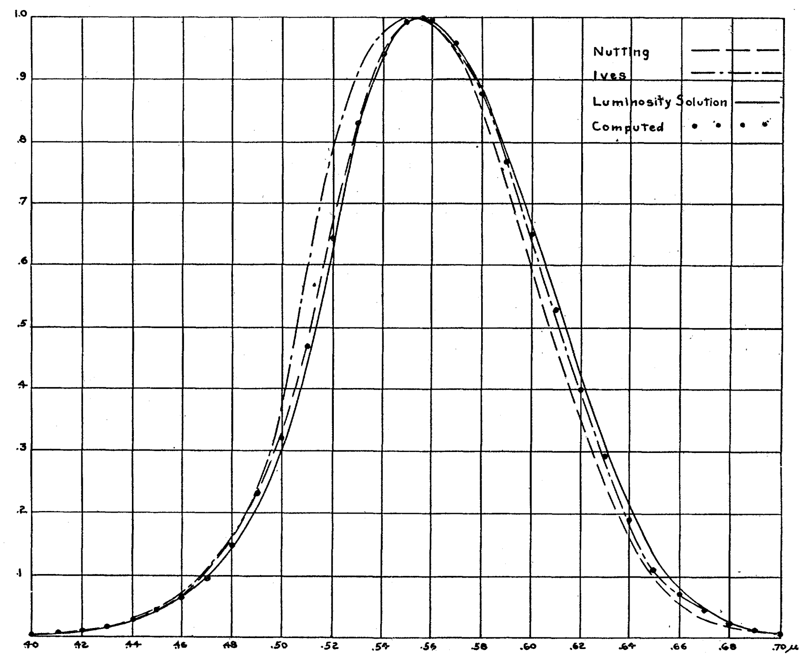

Fig. 1.

Various luminosity curves.

about .103, which is practically the same as the experimental one. The calculated curve evidently adds to the red end what it lacks at the blue end. The accuracy of the luminosity curve at the red end is especially

[1] "The Visibility of Radiation," Nutting, Phil. Mag., 29, p. 301, 1915.

important, because most of our illuminants, at the temperatures we can obtain, have most of the energy on the red side and in multiplying the luminosity curve by the energy curves any errors in the former are magnified considerably.

The method of obtaining the equation given below was as follows: The exponent, n, of the Goldhammer function (1) was given such a value that it filled the experimental curve throughout a good portion of the blue side and through the maximum, but was low at all other points, *i. e.*, the extreme blue end and all the red side. It was then assumed that the differences between the calculated and the true curves were in each case of the same general form as the one already used. If this were true, then by changing the exponent, n, and the maximum, λ_{max}, and adding a constant coefficient to the whole expression, these new expressions could be added to the first one and their sum should be very close to the true value at all points. Of course chance wholly determined how many terms would be needed. As it happened three in all were sufficient. The equation obtained by the above method was as follows:

$$L_\lambda = .999(R_1 e^{1-R_1})^{200} + .04(R_2 e^{1-R_2})^{400} + .095(R_3 e^{1-R_3})^{1000}, \qquad (2)$$

where

$$R_1 = \frac{.556}{\lambda}, \quad R_2 = \frac{.465}{\lambda}, \quad R_3 = \frac{.610}{\lambda}.$$

The experimental curve taken as the criterion was the one described elsewhere and shown in the solid line in Fig. 1. The heavy dotted line is the curve computed according to equation (2). Fig. 2 shows each of the terms plotted separately and their sum. Where the calculated differs from the other it has been left so purposely as being within the best experimental points. In the future, as the luminosity curve becomes better determined, the calculated curve can be changed easily and more terms added if necessary.

Attention should be called to the fact that function (2) has no relation to the trichromatic theory of color vision and the terms have no connection with the three elementary sensation curves as given by Konig. It is true that if we knew these elemental curves sufficiently well we could obtain three spectral functions corresponding to them whose sum would be equal to the luminosity curve of the eye. This would be very desirable as we could carry the study of the luminosity curve to greater detail.

One of the most important applications of the above function is to compute the luminous power emitted by a black body at various temperatures. If J_λ be the quantity represented by Wien's or Planck's equation, and L_λ the quantity represented by the equation (2), then the luminous power, expressed in watts per steradian per square centimeter (projected area) of radiating surface can be expressed as follows:

$$\text{Light watts} = \frac{1}{\pi}\int_0^\infty L_\lambda J_\lambda d\lambda = \text{L}. \qquad (3)$$

The factor, $1/\pi$, enters because it is the normal flux density, σ_0, of radiation per square centimeter per degree[4] we are considering and not the total hemispherical flux, σ, their relation being, $\sigma/\pi = \sigma_0$.

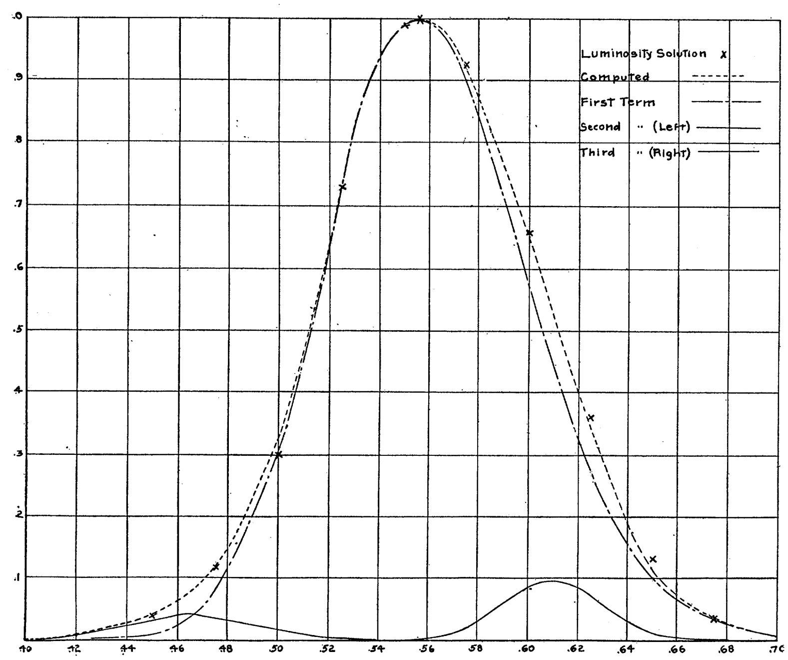

Fig. 2.

Computed luminosity curve showing effect of different terms.

Function (3) has heretofore been evaluated graphically. By means of the new luminosity curve equation it will now be integrated exactly. A formula for the integral will be obtained first by integrating the product of (1), which has the form of each term of (2), by J_λ

$$J_\lambda = c_1\lambda^{-a}(e^{c_2/\lambda k} - 1)^{-1}. \qquad (4)$$

Remembering that

$$c_1 = \frac{\sigma c_2{}^{a-1}}{1.0823(a-2)!}$$

we get

$$L = \frac{\sigma}{\pi} A \sum_{\infty}^{m=1} \left(\frac{1}{\dfrac{mB}{k}+1}\right)^{n+a-1}, \qquad (5)$$

where

$$A = \frac{e^n}{1.0823(a-2)!}\left(\frac{c_2}{\lambda_m}\right)^{a-1}\frac{\Gamma(n+a-1)}{n^{n+a-1}}$$

and

$$B = \frac{c_2}{n\lambda_m}.$$

Since $\Gamma(n+a-1) = (n+a-2)!$ and from Stirling's formula[1] for large values of n

$$(n+a-2)! = \frac{\sqrt{2\pi}}{e^{n+a-2}}(n+a-2)^{n+a-3/2}$$

we have

$$A = \frac{\sqrt{2\pi}e^{2-a}}{1.0823(a-2)!}\left(\frac{c_2}{\lambda_m}\right)^{a-1}\frac{(n+a-2)^{n+a-3/2}}{n^{n+a-1}}$$

For Wien's law $a = b$ and $m = 1$ (5) reduces to

$$L = \frac{\sigma}{\pi}\frac{A}{\left(\frac{B}{k}+1\right)^{n+4}}, \tag{6}$$

where

$$B = \frac{c_2}{n\lambda_m}$$

and

$$A = .01922\left(\frac{c_2}{\lambda_m}\right)^4\frac{1}{\sqrt{n}}\left(\frac{n+3}{n}\right)^{n+7/2}.$$

If $c_2 = 14370$ and $\sigma = 5.65 \times 10^{-12}$ watts per sq. cm. per degree[4] and using (2) for L_λ instead of (1), (6) becomes

$$L = \frac{5.65 \times 10^4}{\pi}\left[\frac{1.254}{\left(\frac{129.23}{k}+1\right)^{204}} + \frac{.0715}{\left(\frac{77.26}{k}+1\right)^{404}} + \frac{.0359}{\left(\frac{23.56}{k}+1\right)^{1004}}\right]. \tag{7}$$

In Fig. 3 the logarithm of this function is plotted against the logarithm of the absolute temperature, k, and marked A. B is the total energy radiated normally, calculated from the Stefan-Boltzmann law. C is the log of the radiant luminous efficiency, the difference between B and A. The maximum efficiency is about 13.5 per cent. at about 6500° K.

The values used in plotting curves A and D are tabulated in Table 1.

[1] Planck's Heat Radiation, Eng. ed., p. 218.

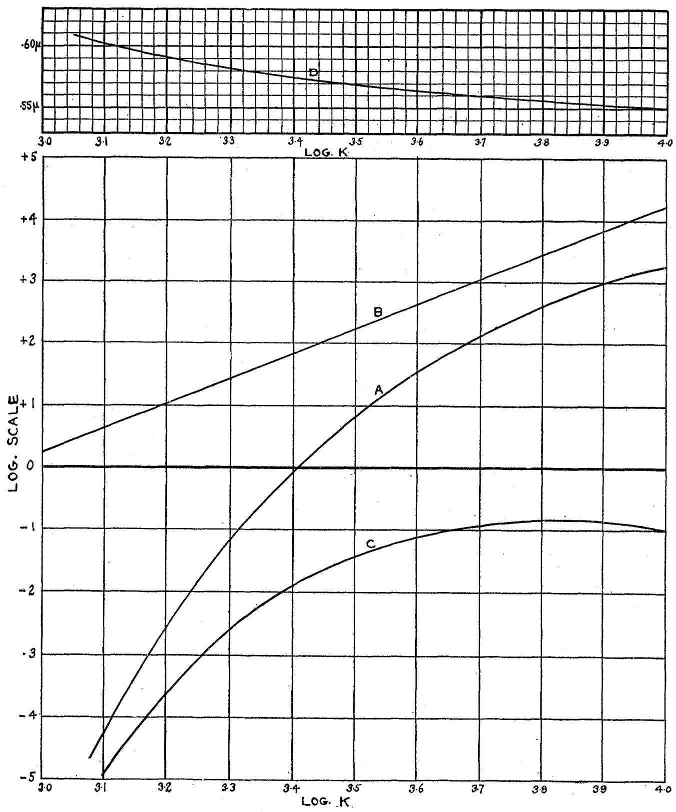

Fig. 3.

A is the log. of the light watts, L, computed from equation (7). B is the log. of the total energy calculated from the Stefan-Boltzmann law. C is the log. of the radiant luminous efficiency $= A - B$. D is the Crova wave-length calculated from equation (13). K. = degrees Kelvin.

It is possible to transform these values, which are in terms of light watts per steradian, to candle-power, which is lumens per steradian.

The relation is

$$b_0 = \frac{L}{m}, \tag{8}$$

where m is the constant known as the "mechanical equivalent of light," *i. e.*, the number of watts of luminous flux in the lumen. Conversely, from experimental values for the brightness of the black body the mechanical equivalent of light may be derived.

Experimental values of the candle power per mm.^{2} of a B. B. have been obtained by Lummer and Pringsheim[1] and by Nernst.[2] The values of

[1] Phys. Z., 3, 97 (1901–02).

[2] "The Brightness of a B. B.," Nernst, Physikal. Zeit., Vol. 7, p. 380 (1906).

TABLE I.

K.	L.	Log K.	Log L.	Crova Wave-length.	Luminous Efficiency.
1,200	2.173×10^{-5}	3.079	-4.663	.605μ	.00000582
1,400	3.75×10^{-4}	3.146	-3.426	.597	.0000555
1,600	3.25×10^{-3}	3.204	-2.488	.591	.000275
1,800	1.76×10^{-2}	3.255	-1.754	.586	.000933
2,000	6.95×10^{-2}	3.301	-1.158	.582	.00247
2,500	8.30×10^{-1}	3.397	$-$ .081	.576	.0118
3,000	4.45	3.477	$+$ .648	.570	.0305
4,000	3.66×10	3.602	$+1.563$	.564	.0794
5,000	1.32×10^{2}	3.700	$+2.121$	.560	.118
6,000	3.12×10^{2}	3.778	$+2.494$	.557	.134
7,000	5.78×10^{2}	3.845	$+2.762$	.555	.134
8,000	9.18×10^{2}	3.903	$+2.963$	.553	.125
10,000	1.76×10^{3}	4.000	$+3.246$	.551	.098

the mechanical equivalent obtained by using their figures in (8) scatter widely on both sides of the recent experimental values. They appear also to correspond to a different luminosity scale than that used by us as shown from a consideration of the equivalent or Crova wave-lengths to which these values were supposed to correspond. Fig. 3, curve D, shows the variation of the Crova wave-length with the log. of the temperature for equation (7). This was derived from the following definition.

The Crova wave-length applying to a black body over a small temperature interval is that wave-length whose radiant power varies with any change of temperature by the same fractional part that the total luminous power varies for that temperature change. Thus

$$\frac{\Delta_k J_\lambda}{J_\lambda} = \frac{\Delta_k L}{L}.$$

For the limit, as the increments approach zero, we may write[1]

$$\frac{\partial_k J_\lambda}{J_\lambda} = \frac{\partial_k L}{L}. \qquad (9)^2$$

[1] This definition and equation can be generalized to apply to radiators whose light is varying from other causes than temperature changes. For example, for different Welsbach mantles we can substitute for J_λ, a function of the wave-length and the percentage Ceria. Then we could get a Crova wave-length for mantles of varying composition, which has been done experimentally. (Washington Convention I. E. S., Sept., 1915.)

[2] By the following steps the result obtained by Foote (Journal Wash. Acad. of Sciences, V, 15 (1915), p. 526) can be derived.
Since

$$\partial_k L = \frac{c_2}{\pi k^2}\int_0^\infty \frac{L_\lambda J_\lambda}{\lambda}\, d\lambda dk$$

(10) becomes

$$\lambda_c = \frac{L}{\frac{1}{\pi}\int_0^\infty \frac{L_\lambda J_\lambda}{\lambda}\, d\lambda},$$

which is the same as his equation (5).

Since

$$\partial_k J_\lambda = \frac{c_2 J_\lambda}{\lambda k^2} dk$$

we get

$$\frac{c_2}{\lambda k^2} dk = \frac{\partial_k L}{L}. \tag{10}$$

From (10) we get the relation to the Crova wave-length of the slope of Curve A in Fig. 3, thus

$$\frac{\partial_k \log L}{d \log k} = \frac{c_2}{\lambda_c k}. \tag{11}$$

If we differentiate (6) to get $\partial_k L$, we get for the Crova value the simple form

$$\lambda_c = \frac{n\lambda m}{n+4}\left(\frac{B}{k} + 1\right). \tag{12}$$

If we take (7) instead of (6) we get

$$\lambda_c = \frac{L}{\frac{\sigma}{\pi}\left[\frac{2.3005}{\left(\frac{129.23}{k}+1\right)^{205}} + \frac{.1553}{\left(\frac{77.26}{k}+1\right)^{405}} + \frac{.0591}{\left(\frac{23.56}{k}+1\right)^{1005}}\right]}. \tag{13}$$

From this equation was calculated the Crova wave-lengths given in the table. Now, Nernst's extrapolation formula

$$k = \frac{11{,}230}{5.367 - \log b_1}$$

calls for a value of λ_c about $.56\mu$ and Lummer and Pringsheim's for λ_c about .54, while .58 is called for by the present luminosity curve for the neighborhood of 2000° K. A Crova value of $.577\mu$ has been found experimentally for the 1.25 w.p.c. tungsten lamp, which is in excellent agreement with the above luminosity curve.[1]

Detailed discussion of the application of this work to the determination of the mechanical equivalent of light is reserved for a subsequent paper in which will be given new experimental work on the brightness of the black body.

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THE UNITED GAS IMPROVEMENT COMPANY PHYSICAL LABORATORY,
PHILADELPHIA, PA.,
June, 1915.

[1] The Application of Crova's Method of Colored Light Photometry to Modern Incandescent Illuminants, Ives & Kingsbury, Washington Convention I. E. S., Sept., 1915.