The assumption of Eq. (10.3) means that the second mode is bottled in by its barrier much more than the first. The assumption of Eq. (10.3a) means that the coupling energy H_1 of the first mode is kept sufficiently small. The form of A_2 given by (10.4) is interesting in that it contains the irregular function for the final state in the denominator and the regular function for the initial state in the numerator.

For

$$V_1(r)=V_2(r)=V_0 \quad (0<r<a), \tag{10.5}$$

the equations for A_1, A_2 reduce to

$$A_1=-\frac{2H_1{}^2f_1{}^2(k_1a)}{k_1a\Delta'}, \quad A_2=-\frac{2H_1H_2f_1(k_1a)f_2(k_2a)}{k_2a\Delta'}, \tag{10.6}$$

where Δ' is given by

$$\Delta'-2b_1H_1{}^2f_1{}^2/(k_1a)-2b_2H_2{}^2f_2{}^2/(k_2a)=\left[\sum_n W_n\left(\frac{\gamma_1}{E-W_{n1}}+\frac{\gamma_2}{E-W_{n2}}+\frac{\gamma_3}{E-W_{n3}}\right)\right]^{-1}, \tag{10.6a}$$

$$=2(\gamma_1x_1\cot x_1+\gamma_2x_2\cot x_2+\gamma_3x_3\cot x_3)^{-1};$$

with

$$W_{n1}=W_n+V_0/2+[(V_0/2)^2+H_1{}^2+H_2{}^2]^{\frac{1}{2}}, \tag{10.7}$$

$$W_{n2}=W_n+V_0/2-[(V_0/2)^2+H_1{}^2+H_2{}^2]^{\frac{1}{2}}, \tag{10.7a}$$

$$W_{n3}=W_n+V_0, \tag{10.7b}$$

and

$$\gamma_1=(W_n-W_{n2})^{-1}(W_{n1}-W_{n2})^{-1}, \tag{10.8}$$

$$\gamma_2=(W_n-W_{n1})^{-1}(W_{n2}-W_{n1})^{-1}, \tag{10.8a}$$

$$\gamma_3=(W_{n2}-W_n)^{-1}(W_{n1}-W_n)^{-1}=-(H_1{}^2+H_2{}^2)^{-1}, \tag{10.8b}$$

$$x_s=[2Ma^2\hbar^{-2}(E+W_n-W_{ns})]^{\frac{1}{2}}; \quad (s=1,2,3). \tag{10.9}$$

One sees from Eqs. (10.6), (10.6a) that the numerators of these formulas for the A_q contain the factors $f_q(k_qa)/(k_qa)^{\frac{1}{2}}$ and that the denominators contain imaginary terms in the combinations $2iH_q{}^2f_q{}^2/(k_qa)$ so that the damping constants depend essentially on the regular function and the interaction energy H_q. It is of interest to note that Eq. (10.6a) gives an explicit expression in terms of cotangents with the arguments x_s defined by Eq. (10.9).

The example just considered shows that the representation of the scattering amplitudes as a sum of interfering terms can be far from being the simplest.

PHYSICAL REVIEW VOLUME 69, NUMBERS 9 AND 10 MAY 1 AND 15, 1946

A Note on the Relativistic Problem of Uniform Rotation

E. L. Hill
Department of Physics, University of Minnesota, Minneapolis, Minnesota
(Received January 12, 1946)

A study of uniform rotational motion about an axis is made on the basis of a definition of hydrokinetic character. A solution is found in which the particle speed is linear with distance from the axis of rotation to terms in $(R\omega_0/c)^2$, but approaches the speed of light at great distances. This result is unchanged by the introduction of relativistic accelerated Euclidean axes. Reasons are given for concluding that Ehrenfest's paradox in the problem of the rotating disk, and the question of the "geometry" of the motion, in the sense of general relativity theory, can be answered only on the basis of a theory of the generation of the rotation.

1. INTRODUCTION

The difficulties presented in the theory of relativity by the "rigid" motion of a body about a fixed axis have been discussed by a number of authors. Among these difficulties, the apparent paradox pointed out by Ehrenfest[1] has received particular attention. For simplicity,

[1] P. Ehrenfest, Physik. Zeits. 10, 918 (1909).

consider any section of the body by a plane perpendicular to the axis of rotation, and any circle in this plane having its center on the axis. Since the elements of the circumference of this circle everywhere move in the direction of their instantaneous velocities, one would anticipate a diminished value for the length of the radius. On the other hand, since the elements of the radii of the circle everywhere move normally to the directions of their velocities, no such contraction would be expected. The ratio of circumference to diameter of the circle would therefore be expected to be smaller than the Euclidean value, and to depend on the radius of the circle. This is the standard argument from which it is usually concluded that the "geometry" of the rotating body ceases to be Euclidean.[2]

Solutions of this paradox have been proposed by Lorentz[3] and Eddington[4] on the basis of the general theory of relativity. In a recent paper Berenda[5] has given a careful summary of this work, and has himself proposed yet another solution.

In this note a different approach will be presented, based on an altered view of the *kinematics* of the motion. Previous discussions generally have started with the classical definition of uniform rotation, according to which each particle seems to the observer to move on the circumference of a circle about the axis of rotation with a speed which is a linear function of the radius of the circle. It does not appear to have been a worrisome point to other writers that this requirement imposes a necessary condition on the size of a body performing such a rotational motion. For if this speed-distance law were to hold at large distances from the axis of rotation, points at a great distance from the axis would have speeds exceeding that of light, which is a violation of the basic principles of relativity theory. Rather than consider that there is such a limitation on the size of a rotating body, the conclusion is drawn that the speed-distance law must be non-linear.

The thesis developed here is that the definition of uniform rotational motion to be used cannot be that of the classical theory. It is anticipated that, with a suitable definition, the speed-distance relation should satisfy two general *a priori* criteria: (a) for *small* distances from the axis of rotation the particle speed should be a (nearly) linear function of the distance, as in the classical theory, and (b) for *large* distances from the axis the particle speed should approach the speed of light.

It will be shown that by a reasonable definition of uniform rotation a consistent solution can be found which satisfies these criteria and which also degenerates into the classical theory in the limit $c\rightarrow\infty$, as it must be expected to do.

2. STATEMENT OF THE PROBLEM

In order to simplify the discussion, we consider the motion as that of a system of points, or of a "fluid," in which each particle moves on a circle about the axis of rotation, the particle speed being a function only of the distance from the axis. In the observer's system of coordinates, which is supposed Euclidean, the z axis is taken along the axis of rotation. Figure 1 shows a section perpendicular to the axis of rotation, the observer's system being at rest. If Q is an arbitrary point in this plane, and $v(R)$ is the linear speed of the particles at a distance R from the

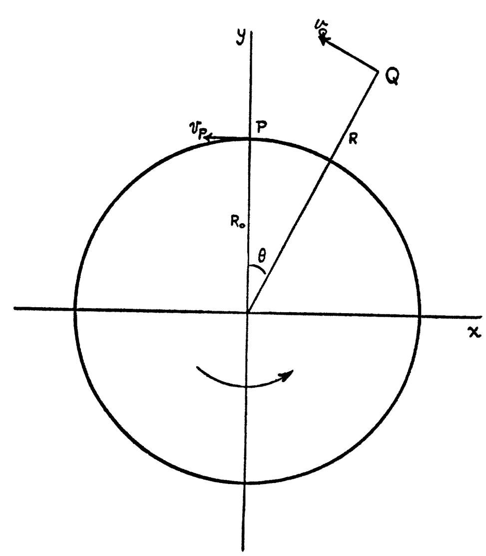

FIG. 1. Kinematical relations in the observer's coordinate system.

[2] For instance in A. Einstein and L. Infeld, *The Evolution of Physics* (Simon and Shuster, New York, 1938), p. 240.

[3] H. A. Lorentz, Nature **106**, 795 (1921).

[4] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, 1924), second edition, p. 112.

[5] C. W. Berenda, Phys. Rev. **62**, 280 (1942). Further references will be found in this paper.

origin, the velocity components of Q are

$$v_x(Q) = -v(R)\cos\theta = -y\xi(R),\quad v_y(Q) = v(R)\sin\theta = x\xi(R), \tag{1}$$

with

$$\xi(R) = v(R)/R,\quad R = (x^2+y^2)^{\frac{1}{2}}.$$

Definition. *The motion of the fluid will be considered to be a uniform rotation with constant angular velocity ω_0 if the relative angular velocity of the material about any point P, as measured in a set of axes with respect to which P is momentarily at rest, has the value ω_0 independently of the choice of the point P.*

No transformations other than those of the special theory of relativity are contemplated in this definition, so that all axis systems used will be Euclidean. It is to be expected on the basis of symmetry considerations that this requirement can be satisfied by a suitable choice of the function $v(R)$, and that at each point the relative angular velocity vector will be parallel to the z axis.

3. DETERMINATION OF THE FUNCTION $v(R)$

We start by choosing a point P which, for convenience, is taken to be on the y axis at the instant considered, at a distance R_0 from the center of rotation. From Eqs. (1) the instantaneous velocity components of P, in the observer's system, are

$$v_x(P) \equiv v_0 = -R_0\xi(R_0),\quad v_y(P) = 0.$$

A set of moving axes, with origin at P and in which P is instantaneously at rest (at time $t=t'=0$), is now introduced by the Lorentz transformation

$$x' = \frac{x-v_0t}{(1-v_0^2/c^2)^{\frac{1}{2}}},\quad y' = y-R_0,\quad z'=z,\quad t' = \frac{t-v_0x/c^2}{(1-v_0^2/c^2)^{\frac{1}{2}}}, \tag{2}$$

for which the corresponding velocity transformation equations are

$$v_x' = \frac{v_x-v_0}{1-v_xv_0/c^2},\quad v_y' = \frac{v_y(1-v_0^2/c^2)^{\frac{1}{2}}}{1-v_xv_0/c^2},\quad v_z' = \frac{v_z(1-v_0^2/c^2)^{\frac{1}{2}}}{1-v_xv_0/c^2}. \tag{3}$$

For the determination of the apparent instantaneous angular velocity of the fluid in the neighborhood of P, as measured in this rest-system of P, it is sufficient to set $t'=0$, and to expand the relative velocity components of the points about P in power series in x' and y', retaining only linear terms. The angular velocity vector is then computed from the usual hydrokinetic formula

$$\boldsymbol{\omega}' = \tfrac{1}{2}\,\mathrm{curl}'\,\mathbf{v}'. \tag{4}$$

With these conditions we find the coordinate relations

$$x = x'/(1-v_0^2/c^2)^{\frac{1}{2}},\quad y = y'+R_0,\quad z = z'. \tag{5}$$

Making use of Eqs. (1), (3), and (5) we obtain the following expressions for the instantaneous relative velocity components of the points near P, to first-order terms,

$$v_x' = -y'\left[\frac{\xi+Rd\xi/dR}{1-v^2/c^2}\right],\quad v_y' = x'[\xi/(1-v^2/c^2)],\quad v_z' = 0. \tag{6}$$

The bracketed expressions are to be given their values at the point P.

The only non-vanishing component of the relative angular velocity vector is now found to be

$$\omega_z' = \frac{1}{2}\left[\frac{2\xi+Rd\xi/dR}{1-v^2/c^2}\right]. \tag{7}$$

Following our definition of rigid rotational motion, we set

$$\omega_z' \equiv \omega_0 \equiv \text{constant},$$

from which we obtain the differential equation for the speed-distance function

$$\frac{dv}{dR} + \frac{2\omega_0}{c^2}v^2 + \frac{v}{R} - 2\omega_0 = 0. \tag{8}$$

This is a special case of the general Riccati equation for which the solution is

$$v(R) = -ic\frac{J_1(i2R\omega_0/c)}{J_0(i2R\omega_0/c)}, \tag{9}$$

J_0 and J_1 being the usual Bessel functions.

4. DISCUSSION OF THE SOLUTION

The nature of this result is examined most readily by writing out the expansions for the limiting cases of large and small distances from the axis of rotation. We find

$R\omega_0/c \ll 1$

$$v(R) = R\omega_0[1 - \tfrac{1}{2}(R\omega_0/c)^2 + \tfrac{1}{3}(R\omega_0/c)^4 \cdots], \quad (10)$$

$R\omega_0/c \gg 1$

$$v(R) = c[1 - \tfrac{1}{4}(c/R\omega_0) - \tfrac{1}{32}(c/R\omega_0)^2 \cdots]. \quad (11)$$

Equation (10) shows that, to a high degree of approximation, the velocity distribution in the neighborhood of the axis is that of the classical theory, the speed being a nearly linear function of the distance. Equation (11) shows that, if $\omega_0 \neq 0$, points at a great distance from the axis have speeds approximating that of light. Since this solution degenerates to the classical result for $c \rightarrow \infty$, it is entirely consistent with the general criteria laid down in the introductory remarks.

The apparent rate of volume dilatation at the point P is given by

$$d\Sigma'/dt' = \mathrm{div}' \, \mathbf{v}' = 0,$$

so that our solution is in agreement with the classical result in this test of the "rigidity" of the motion.

On the other hand, there is a divergence from the classical theory in that the tensor giving the apparent rate of the shearing motion about P does not vanish, having the non-zero component

$$\gamma_{xy}' = \tfrac{1}{2}(\partial v_x'/\partial y' + \partial v_y'/\partial x')$$
$$= (1 - v^2/c^2)^{-1} \cdot [-\omega_0(1 - v^2/c^2) + v/R].$$

In the limiting cases we find

$R\omega_0/c \ll 1$

$$\gamma_{xy}' = \omega_0 \cdot \tfrac{1}{2}(R\omega_0/c)^2[1 - \tfrac{1}{3}(R\omega_0/c)^2 \cdots],$$

$R\omega_0/c \gg 1$

$$\gamma_{xy}' = \omega_0 \cdot [1 - \tfrac{1}{2}(c/R\omega_0) \cdots].$$

Near the axis the rate of shear is a small quantity of second-order, but at large distances it approaches the limiting value ω_0. The rotation axis is unique in that it is only on it that the shearing motion vanishes.

5. CONCLUSION

In order to furnish a test of the uniqueness of the definition of uniform rotational motion, a calculation has been made in which the coordinate system used was such that in it the point P was instantaneously at the origin at rest and also without acceleration. The relativistic method of introducing accelerated Euclidean axis systems has been developed in a previous paper.[6] It was found that exactly the same results are obtained as above, so that our solution has a certain claim to uniqueness.

The question naturally occurs whether one could use a set of moving coordinates in which the point P would be *permanently* at rest. If such a set of proper coordinates could be defined in a relativistically significant manner it might be possible to define uniform rotational motion with respect to them in such a way as to satisfy the classical result that the rotation, as observed in the proper coordinate system of *any* point, would appear to be an uniform rotation about an axis through that point. No such system of coordinates is now known.

In conclusion, it is to be noted that this discussion does not anwer the problem posed in Ehrenfest's paradox, since it does not establish a correlation between the particles of the rotating medium with their positions in a non-rotating condition. In order to give an answer, and even a clear meaning, to this question, it seems essential that one give a theory of the generation of the motion from a state of rest, for it is only in this way that one can tell what has happened to any portion of the medium. However, there is no apparent reason why the result of such a theory should be unique. One might, for instance, consider the generation of the motion in a fluid under the action of elastic forces, or one could simulate it by an infinite set of charged particles moving in an electromagnetic field. So long as only the initial state of rest and the kinematics of the final state of motion are given, the correlation found between these states might well depend on the details of the process used. For this reason it is the writer's present belief that Ehrenfest's paradox, as well as the question of the type of "geometry" which is applicable to the motion, are ill-defined questions to which no definitive answers can be given without further specifications of the systems to be discussed.

[6] E. L. Hill, Phys. Rev. **67**, 358 (1945).