From the second equation we obtain then

$$\psi_1(k'\kappa's') = (k'\kappa's'|H'|k_0\kappa_0s_0)\left\{\frac{1}{E-E_I} - i\pi\delta(E-E_I)\right\},$$

where $E_I$ is the energy which belongs to the state $(k'\kappa's')$. The second term is so chosen that this solution when transformed to $X$-space represents only outgoing spherical waves provided that we define an integration over the pole $E=E_I$ always by its principle value.[25] The second-order solution which is of interest in our problem becomes then

$$\psi_2(k,\kappa,s) = \frac{1}{E-E_F}\sum_{k'\kappa's'}(k,\kappa,s|H'|k'\kappa's')(k'\kappa's'|H'|k_0\kappa_0s_0) \times\left\{\frac{1}{E-E_I} - i\pi\delta(E-E_I)\right\}.$$

The summation on the right-hand side is equivalent with an integration without $\delta$-term in such a way that the path of integration is displaced in the complex $\kappa$-plane around the pole in the negative imaginary half-plane. This was the procedure used in the text.

[25] Compare P. A. M. Dirac, The Principles of Quantum Mechanics, Clarendon Press, Oxford, second ed., p. 195ff.

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# On the Heavy-Electron Pair Theory in the Limit of Strong Coupling

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The pair theory of Marshak and Weisskopf was investigated assuming strong coupling. The strong coupling criterion is $A=(Nf/\mu)>5$ where $f$=coupling constant, $\mu$=heavy electron mass, $N=\int U^2(x)d^3x$, $U(x)$=source function of the nucleon. $N\cong 1/\pi a^3$, where $a$=source radius (all in units where $\hbar=c=1$). With this condition, the magnetic moment turns out to be of order $A^{-1}$, and of magnitude too small to account for the observed anomalies. The leading term in the potential of the force between two nucleons (for $r_{AB}>2a$) is independent of the spin orientations, of the coupling constants, and of the type of coupling (as long as no derivatives of the heavy-electron field quantities occur in the coupling term). This potential is identical with the one calculated by Jauch and Houriet, but was not considered by Nelson and Oppenheimer. The next term in the potential is of order $A^{-3}$. It corresponds to a superposition of a $(\Sigma_A\cdot\Sigma_B)$ term, a tensor force term, and an ordinary force. Being of order $A^{-3}$ it is too small, however, to fit the experimental results.

## 1. INTRODUCTION

THIS paper deals with the strong coupling version of the meson theory proposed by Marshak and Weisskopf.[1] In this theory the meson field quantity is assumed to be a complex spinor. The associated quanta (mesons) are then heavy electrons in all respects. They satisfy Dirac's equation in the absence of interactions with nucleons. The negative energy states are taken into account by the usual hole-theory approach. The interaction term with the nucleon is assumed to be quadratic in the meson field quantities. This identifies the theory as a "pair theory." The name refers to the fact that in the weak coupling approximation the mesons are emitted and reabsorbed by the nucleons in pairs. The particular interaction term proposed by Marshak and Weisskopf is of the form

$$\bar{H}_i = f\psi^+(z)P_s\psi(z). \qquad (1.1)$$

Here $\bar{H}_i$ is the interaction term in the Hamiltonian, $f$ is a coupling constant, $\psi$ is the meson field quantity (an operator), a cross denotes its Hermitean conjugate, $z$ is the position of the nucleon in question, and $P_s$ is one of several operators which are possible if we want the theory to be relativistically invariant. Marshak points out that it is possible, in a weak coupling theory, to eliminate all but one of them by a consideration of the deuteron problem.

It is true, however, that a weak coupling theory based on an interaction term of the form (1.1) can never explain the anomalous magnetic moments of the proton and neutron. Since $\bar{H}_i$ contains both $\psi^+$ and $\psi$, the isotopic spin of the

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[1] R. E. Marshak, Phys. Rev. **57**, 1101 (1940); R. E. Marshak and V. F. Weisskopf, Phys. Rev. **59**, 130 (1941).

nucleon is an exact constant of motion. The operator $P_s$ cannot contain the operators transforming a neutron into a proton or *vice versa* because that would destroy the conservation of charge in an elementary process. The only way in which the isotopic spin degree of freedom can enter the problem at all is through a choice of the coupling constant $f$. But it is easy to see that it is impossible to make a satisfactory choice, provided we limit ourselves to a weak coupling theory. If the magnetic moment arises in a perturbation of even order, there is no possibility at all to fit the opposite signs of the anomalies for the proton and neutron. If the magnetic moment arises in an odd-order perturbation, we might try to pick coupling constants of opposite sign for the two heavy particles. But the nuclear force between particles $A$ and $B$ will contain the factor $f_A f_B$ for every operator $P_s$ considered by Marshak. Our tentative choice of coupling constants would, therefore, lead to opposite signs for like and unlike particle forces, and this is unacceptable. It may be remarked that all meson pair theories proposed so far suffer from this same defect.

For this reason, and because a rough estimate seems to show that it is not possible to choose the coupling constants big enough to give the correct binding energy of the deuteron without violating the weak coupling criterion, it was considered advisable to investigate the strong coupling limit of this theory. The results are:

(1) The strong coupling condition is $A = Nf/\mu \gg 5$ where $N$ is the reciprocal of the volume of the source, and $\mu$ is the meson mass.

(2) The magnetic moment changes sign when $f$ changes sign, but it is much too small to give the observed anomalies as long as (1) is satisfied and $\mu a \ll 1$, where $a$ is the radius of the source.

(3) The leading term in the potential of the force between two nucleons (for $r_{AB} > 2a$) is independent of the spin orientations, of the coupling constants, and of the type of coupling, i.e., of the choice of operator $P_s$ in Eq. (1.1). This potential is identical with the one calculated by Jauch and Houriet,[2] but was not considered by Nelson and Oppenheimer.[3]

(4) The spin-dependent part of the force arises in order $A^{-3}$. It then gives a superposition of a term of type $\mathbf{S}_A \cdot \mathbf{S}_B$ and a tensor force term. Since $A \gg 5$ by assumption, this force is very small.

[2] J. M. Jauch, Helv. Phys. Acta **15**, 175 (1942); A. Houriet, Helv. Phys. Acta **16**, 529 (1943).

[3] E. Nelson and J. R. Oppenheimer, Phys. Rev. **61**, 202 (1942).

*The theory is, therefore, unsatisfactory both in the weak coupling and the strong coupling limits.* The only remaining possibility is an intermediate coupling strength ($A$ of order unity). The present treatment is not applicable to this case.

## 2. THE ONE-BODY PROBLEM

In any strong coupling theory it is necessary to use a source function convergence method. This means that we replace $\psi(z)$ in Eq. (1.1) by

$$\bar{\psi} = \int \psi(\mathbf{x})\, U(\mathbf{x}-\mathbf{z}) d^3x, \tag{2.1}$$

where the source function $U$ is chosen so that its integral over all of space is unity; usually it is chosen spherically symmetrical, and so that it is very small beyond a certain distance from the nucleon. It is common to define an effective radius of the source by

$$(a)^{-1} = \int\int \frac{U(x)\, U(x')}{|\mathbf{x}-\mathbf{x}'|} d^3x d^3x'. \tag{2.2}$$

It then turns out that for most reasonable choices of the source function, the nuclear forces do not depend very much on $U(x)$ as long as the internuclear distance is much larger than the source diameter $2a$. It must be realized that this formalism is only a preliminary convergence device necessitated by the lack of a consistent quantum theory of the interaction of fields and point sources. Therefore the restriction to distances greater than the source diameter is not really a loss in generality. We would not be willing to trust the theory at shorter distances, anyway.

In the strong coupling case, the starting state of a valid perturbation treatment will be obtained by splitting the meson field into two parts: a static field attached to the nucleon and a wave field largely independent of it. *We then consider the complex "nucleon+static meson field" as one unit, which we shall call the stat-nucleon, and we assume that it is this unit which is observed experimentally.* Because of the various possible states of the static meson field, the stat-nucleon has many more states than the original nucleon. The excited states of the stat-nucleon are referred to as isobaric states. It is clear that the existence of

isobaric states leads to two requirements for a reasonable theory. First, the ground state of the stat-nucleon must have spin one-half. Second, the excitation energy of the lowest isobar state must be high enough to account for the fact that these states have not been observed as yet.

The Hamiltonian of our problem is

$$\bar{H}=\bar{H}_f+\bar{H}_i=\int\psi^+(\mathbf{x})(\boldsymbol{\alpha}\cdot\mathbf{p}+\mu\beta)\psi(\mathbf{x})d^3x + f\bar{\psi}^+\beta\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}\bar{\psi}. \quad (2.3)$$

Here $\bar{H}_f$ is the free meson field Hamiltonian, $\bar{H}_i$ the interaction term with the nucleon; $\boldsymbol{\alpha}$ and $\beta$ are the usual Dirac matrices (it must be remembered that $\psi$ is a spinor), $\mathbf{p}$ is $(-i)$ times the gradient operator, and $\mu$ is the meson mass. The units are chosen so that $\hbar=c=1$, and lengths are measured in cm. In $\bar{H}_i$ we have chosen the operator $P_s$ proposed by Marshak and Weisskopf; $\boldsymbol{\sigma}$ is the spin operator of the mesons (i.e., $\sigma_x=-i\alpha_y\alpha_z$) and $\boldsymbol{\Sigma}$ is the spin operator of the heavy particle. $\boldsymbol{\Sigma}$ can be represented by the usual Pauli 2-by-2 matrices.

Notice that $\bar{H}_i$ contains $\psi^+$, $\psi$, and $\Sigma$. The equations of motion will contain products of two of these operators, and will therefore be *non-linear*. The non-linearity of this theory distinguishes it from the case treated by Jauch and Houriet.[2] They use $P_s=\beta$ which leads to *linear* equations of motion. This point is discussed further in Section 4.

The meson field is quantized in accordance with the exclusion principle, so $\psi$ has to satisfy the usual anti-commutation rules

$$\psi_\gamma^+(\mathbf{x})\psi_{\gamma'}(\mathbf{x}')+\psi_{\gamma'}(\mathbf{x}')\psi_\gamma^+(\mathbf{x})=\delta_{\gamma\gamma'}\delta(\mathbf{x}-\mathbf{x}'), \quad (2.4)$$

while the other anti-commutators vanish. The index $\gamma$ denotes the spinor components of $\psi$ and runs from 1 to 4.

We now split the field into a static field and a wave field. To do this we define the integral $N=\int U^2(\mathbf{x})d^3x$ and the function $u(\mathbf{x})=N^{-\frac{1}{2}}U(\mathbf{x})$. Then we put

$$\psi(\mathbf{x})=Qu(\mathbf{x})+\psi'(\mathbf{x}) \quad (2.5)$$

with an analogous formula for $\psi^+(\mathbf{x})$. We further postulate the orthogonality condition

$$\int u(\mathbf{x})\psi'(\mathbf{x})d^3x=0. \quad (2.6)$$

From (2.5) and (2.6) it follows that

$$Q=\int u(x)\psi(x)d^3x.$$

Assuming a spherically symmetric source function, it is easily seen that

$$\bar{H}=\bar{H}_0+\bar{H}'+\Omega, \quad (2.7)$$

where

$$\bar{H}_0=Q^+(\mu\beta+Nf\beta\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma})Q, \quad (2.8)$$

$$\bar{H}'=\int\psi'^+(\mathbf{x})(\boldsymbol{\alpha}\cdot\mathbf{p}+\mu\beta)\psi'(\mathbf{x})d^3x, \quad (2.9)$$

$$\Omega=Q^+\int u(\mathbf{x})(\boldsymbol{\alpha}\cdot\mathbf{p}+\mu\beta)\psi'(\mathbf{x})d^3x + \int\psi'^+(\mathbf{x})(\boldsymbol{\alpha}\cdot\mathbf{p}+\mu\beta)u(\mathbf{x})d^3x\,Q. \quad (2.10)$$

The anti-commutation relations are to be derived from (2.4), (2.5), and (2.6). They turn out to be

$$Q_\gamma^+Q_{\gamma'}+Q_{\gamma'}Q_\gamma^+=\delta_{\gamma\gamma'}, \quad (2.11)$$

$$\psi_\gamma'^+(\mathbf{x})\psi_{\gamma'}'(\mathbf{x}')+\psi_{\gamma'}'(\mathbf{x}')\psi_\gamma'^+(\mathbf{x}) =\delta_{\gamma\gamma'}[\delta(\mathbf{x}-\mathbf{x}')-u(\mathbf{x})u(\mathbf{x}')] \quad (2.12)$$

while the other anti-commutators vanish.

The decomposition of the Hamiltonian (2.7) has the significance that $\bar{H}_0$ describes the stat-nucleon, $\bar{H}'$ describes the wave-field, and $\Omega$ is the interaction between them. The strong coupling condition will be the condition of validity for a perturbation treatment on $\Omega$.

We first discuss the $\bar{H}_0$ problem. The operators $Q$ have to satisfy the conditions (2.11). They can then be represented as 16-by-16 matrices. The occurrence of $\boldsymbol{\Sigma}$ in the Hamiltonian requires a doubling of those matrices so that they become 32-by-32. It is easily seen that $B_+=Q_1^+Q_1+Q_2^+Q_2$ and $B_-=Q_3^+Q_3+Q_4^+Q_4$ as well as the stat-nucleon spin operator $\mathbf{s}=(1/2)(Q^+\boldsymbol{\sigma}Q+\boldsymbol{\Sigma})$ are constants of motion. Starting with a representation which is diagonal in $B_+$, $B_-$ and the $z$ component of $\mathbf{s}$, it is easy to find the simultaneous eigenstates of these three operators and $\bar{H}_0$. It turns out that we get two doublet and one quadruplet states with energy 0; the remaining states occur in pairs of positive and negative energy. Listing only one of each pair, we have singlets with energies $\mu+3Nf$ and $\mu-3Nf$, doublets with energies $2\mu$ and $2(3)^{\frac{1}{2}}Nf$, and

triplets with energies $\mu+Nf$ and $\mu-Nf$, giving thirty-two states in all. For large values of $f$ the ground state will be the doublet state of energy $-2(3)^{\frac{1}{2}}Nf$. The first excited state will be the singlet state of energy $-(\mu+3Nf)$. Closer investigation shows that the *strong coupling condition* is obtained by requiring the *excitation energy of this first isobar state* to be *much larger than the rest energy $\mu$ of the meson.* This is the condition quoted in the introduction. Notice that the energies above *cannot* be written in the form $\sum_{\gamma=1}^{4} N_\gamma E_\gamma$ where $N_\gamma=0$ or 1, and the $E_\gamma$ are constants. This is due to the non-linearity of the $\bar{H}_0$-problem.

We now turn our attention to the wave-field problem. The Hamiltonian $\bar{H}'$ looks just like the free-field Hamiltonian $\bar{H}_f$. There is, however, a significant difference between the two caused by the auxiliary condition (2.6) and the corresponding anti-commutation rules (2.12). Following Pauli and Hu[4] we shall treat this problem by the method of Lagrange multipliers. We are looking for a system of $c$-number spinor functions $\phi_\rho(\mathbf{x})$ which fulfill (2.6) and diagonalize the expression (2.9) with $\phi$ substituted for $\psi'$. Having found such a system, we can put

$$\psi'(\mathbf{x})=\sum_\rho a_\rho\phi_\rho(\mathbf{x}), \tag{2.13}$$

where the $a_\rho$ are scalar operators satisfying the usual anti-commutation rules. The Hamiltonian $\bar{H}'$ then assumes the simple form

$$\bar{H}'=\sum_\rho a_\rho^+ a_\rho E_\rho, \tag{2.14}$$

where the $E_\rho$ are the values of (2.9) obtained by substituting $\phi_\rho(\mathbf{x})$ for $\psi'(\mathbf{x})$.

We now observe that the problem of finding the functions $\phi(\mathbf{x})$ is equivalent to the variation problem: to extremize the integral

$$I=\int\phi^*(\mathbf{x})(\boldsymbol{\alpha}\cdot\mathbf{p}+\mu\beta)\phi(\mathbf{x})d^3x \tag{2.15}$$

subject to the conditions

$$\int\phi^*(\mathbf{x})\phi(\mathbf{x})d^3x=1, \tag{2.16}$$

$$\int\phi^*(\mathbf{x})u(\mathbf{x})d^3x=\int\phi(\mathbf{x})u(\mathbf{x})d^3x=0. \tag{2.6}'$$

[4] W. Pauli and Ning Hu, Rev. Mod. Phys. **17**, 267 (1945).

Since the wave-field problem is one of infinitely many degrees of freedom, it is not convenient to take the volume integrals over all of $x$ space. We shall need the spherical symmetry of the one-body problem later on in the calculation of the anomalous magnetic moment. Therefore we will pick a sphere of radius $R$ as our volume of integration, and we shall let $R$ become infinitely large at the end. We now use the method of Lagrange multipliers on the variation problem just outlined. We multiply (2.16) by the scalar $-E$ and pre-multiply (2.6)′ by the spinors $-\lambda^*$ and $-\lambda$, respectively; we then add the resulting equations to (2.15) and vary $\phi^*(\mathbf{x})$. Setting the variation equal to zero we obtain

$$(\boldsymbol{\alpha}\cdot\mathbf{p}+\mu\beta-E)\phi(\mathbf{x})=\lambda u(\mathbf{x}). \tag{2.17}$$

This equation clearly indicates that the eigenwaves of the wave-field problem split into two distinct groups: those with $\lambda=0$ and those with $\lambda\neq0$. If $\lambda=0$, $\phi(\mathbf{x})$ satisfies the ordinary Dirac equation. So this class is just the *class of all eigenstates of the free meson field (before introduction of the nucleon) which are already orthogonal to the source function.* It will turn out that all our results will be independent of these eigenstates of the wave-field problem.

The other class, with $\lambda\neq0$, has to be investigated more closely. We pre-multiply (2.17) by $(\boldsymbol{\alpha}\cdot\mathbf{p}+\mu\beta+E)$ to get

$$(p^2+\mu^2-E^2)\phi(\mathbf{x})=(\boldsymbol{\alpha}\cdot\mathbf{p}+\mu\beta+E)\lambda u(\mathbf{x}).$$

This implies that we can put

$$\phi(\mathbf{x})=(\boldsymbol{\alpha}\cdot\mathbf{p}+\mu\beta+E)\lambda v(\mathbf{x}), \tag{2.18}$$

where the scalar function $v(\mathbf{x})$ satisfies

$$(p^2+\mu^2-E^2)v(\mathbf{x})=u(\mathbf{x}) \tag{2.19}$$

and (2.6) with $\psi'$ replaced by $v(\mathbf{x})$. A closer investigation shows that we are free to pick convenient boundary conditions for $v$ on the surface of the large sphere, and we shall require $v$ to be zero on that surface. To simplify calculations we shall pick as source function $U(r)=(\alpha^2/4\pi r)\exp(-\alpha r)$ with the corresponding $u(r)=(\alpha/2\pi)^{\frac{1}{2}}1/r\exp(-\alpha r)$. The solutions of (2.19) which are square integrable and have a square integrable gradient are of type $v(r)=w(r)/r$

where $w$ is given by

$$w(r) = -[\alpha/2\pi]^{\frac{1}{2}}[\alpha^2+k^2]^{-1}[\exp(-\alpha r) - \cos(kr)+B\sin(kr)].$$

Here $k^2=E^2-\mu^2$ and $B$ is a constant to be determined. The orthogonality to the source function leads to the condition $B=(\alpha^2-k^2)/2\alpha k$. The boundary condition at the surface of the large sphere leads to $B^{-1}=\tan(kR)$. These two conditions combined give the equation which determines the allowed values of $k$ and therefore of $E$. It should be noticed that these eigenvalues are not identical with the ones obtained without the auxiliary condition. This is the reason why the $\bar{H}'$ problem leads to forces in the two-body problem.

The *operator* $\Omega$ defined by (2.10) will be treated as a perturbation. We can use the result (2.17) to simplify its form considerably:

$$(\boldsymbol{\alpha}\cdot\mathbf{p}+\mu\beta)\psi'(\mathbf{x}) = (\boldsymbol{\alpha}\cdot\mathbf{p}+\mu\beta)\sum_\rho a_\rho\phi_\rho(\mathbf{x}) = \sum_\rho a_\rho(E_\rho\phi_\rho(\mathbf{x})+\lambda_\rho u(x)).$$

Because of (2.6)' $\Omega$ becomes

$$\Omega = Q^+\sum_\rho a_\rho\lambda_\rho+\sum_\rho a_\rho^+\lambda_\rho^*Q. \qquad (2.20)$$

This shows immediately why the states with $\lambda=0$ are not important for our problem. They are not contained in the perturbing Hamiltonian and are therefore exact eigenwaves of the complete Hamiltonian. In particular, if the occupation numbers are 0 for those states in zero-order approximation they will be zero in all orders of a perturbation calculation.

It is easy to see that the problem admits an *angular momentum integral of motion*. This follows immediately from the invariance under rotations about the heavy particle. The integral in question is the operator

$$\mathbf{J} = \int\psi^+(\mathbf{x})[\mathbf{x}\times\mathbf{p}+\tfrac{1}{2}\boldsymbol{\sigma}]\psi(\mathbf{x})d^3x+\tfrac{1}{2}\boldsymbol{\Sigma}. \qquad (2.21)$$

The remarkable thing about this expression is that it commutes with $\bar{H}_0+\bar{H}'$ and $\Omega$ separately. This can be shown by introducing (2.5) and noticing that there are no cross terms between the static and wave fields; it also follows from the fact that both the stat-nucleon and the wave-field problem are invariant under rotations about the heavy particle. This fact is not important for the study of the one-body case, but will prove very important for the two-body problem.

### 3. THE MAGNETIC MOMENT

To find the contribution of the meson field to the magnetic moment of the stat-nucleon, we calculate the expectation value of the magnetic moment operator in the ground state. This operator is determined in the usual way. We assume a magnetic field $\mathfrak{C}$ in the $z$ direction. A vector potential corresponding to this is given by $A_x=-\frac{1}{2}\mathfrak{C}y$, $A_y=\frac{1}{2}\mathfrak{C}x$, and $A_z=0$. We then replace $\mathbf{p}$ by $\mathbf{p}+\epsilon\mathbf{A}$ in the Hamiltonian (2.3). Since we assume no static electric field, there are no changes in the definition of the canonical conjugate $\pi(x)=i\psi^+(x)$ of the operator $\psi$. The Hamiltonian will then contain an additional term proportional to the magnetic field strength, and the magnetic moment operator is defined as the derivative of this new term with respect to the magnetic field. It is easily seen that the magnetic moment operator for our problem is

$$\mathbf{M} = \frac{\epsilon}{2}\int\psi^+(\mathbf{x})[\mathbf{x}\times\boldsymbol{\alpha}]\psi(\mathbf{x})d^3x. \qquad (3.1)$$

Upon introduction of (2.5) it is seen that the term quadratic in $Q$ becomes identically zero. This means that the $\bar{H}_0$ problem by itself does not lead to a magnetic moment. There remain two terms, one quadratic in $\psi'$ and the other bi-linear in $\psi'$ and $Q$. It can be shown without difficulty that the expectation value of the term quadratic in $\psi'$ is zero in the state for which the occupation numbers $N_\rho=a_\rho^+a_\rho$ for positive values of $E_\rho$ and $N_\rho=a_\rho a_\rho^+$ for negative values of $E_\rho$ are all zero. The contributions of this term will come in even orders of a perturbation treatment.

The cross term is given by

$$\mathbf{M}'' = \tfrac{1}{2}\epsilon Q^+\int u(\mathbf{x})[\mathbf{x}\times\boldsymbol{\alpha}]\psi'(\mathbf{x})d^3x + \text{complex conjugate}. \qquad (3.2)$$

Upon introduction of (2.13) this expression simplifies to

$$M_z'' = -(\epsilon/6)\sum_\rho s_\rho K_\rho(a_\rho^+\lambda_\rho^*Q+Q^+\lambda_\rho a_\rho). \qquad (3.3)$$

Here we have picked the Lagrange multipliers $\lambda_\rho$ as spinors with all but one component equal to zero, and that component real. The operator $s_\rho$ is defined to be $+1$ if this non-zero component is the first or third, $-1$ otherwise. It is easily seen, by the way, that it is related to the angular momentum integral for the $\bar{H}'$ problem through

$$J_z' = \tfrac{1}{2}\sum_\rho s_\rho a_\rho^+ a_\rho + \text{contribution of modes with } \lambda = 0. \quad (3.4)$$

The expression $K_\rho$ in (3.3) is given by

$$K_\rho = \int u(\mathbf{x})\mathbf{x}\cdot\text{grad}\, v(\mathbf{x}) d^3x$$

and is seen to be $-[2(\alpha^2+k^2)]^{-1}$.

The operator $M_z''$ contributes in the first order of a perturbation treatment, and it is this contribution which we are going to calculate. The method is the usual one. We expand the wave-functional $\Phi$ of the ground state in powers of the parameter $A = [Nf/\mu]^{-1}$

$$\Phi = \Phi_0 + \Phi_1 + \Phi_2 + .$$

We then obtain the equations

$$(\bar{H}_0+\bar{H}')\Phi_1 + \Omega\Phi_0 = i(d\Phi_1/dx_0),$$
$$(\bar{H}_0+\bar{H}')\Phi_2 + \Omega\Phi_1 = i(d\Phi_2/dx_0),$$

and so on. We can find $\Phi_1$ by an expansion in terms of eigenfunctionals $X$ of $(\bar{H}_0+\bar{H}')$. The expression we are looking for is $2\Phi_0^* M_z''\Phi_1$. If we start from the unperturbed state in which the eigenvalue of $\bar{H}_0$ is $-2(3)^{\frac{1}{2}}Nf$ and the occupation numbers $N_\rho$ are zero, and go to the limit as the large sphere becomes very big, we obtain

$$\Phi_0^* M_z''\Phi_1 = -\frac{\epsilon\alpha}{6\pi}\sum_{r=1}^{32}\sum_{\gamma=1}^{4}\left\{ s_\gamma (r|Q_\gamma|1)^2 \int_0^\infty dk \frac{k^2}{(\alpha^2+k^2)\epsilon_k(\epsilon_k+\mu\beta_\gamma)(\epsilon_k+E^{(r)}-E^{(1)})} + s_\gamma (r|Q_\gamma^+|1)^2 \int_0^\infty dk \frac{k^2}{(\alpha^2+k^2)\epsilon_k(\epsilon_k-\mu\beta_\gamma)(\epsilon_k+E^{(r)}-E^{(1)})}\right\}. \quad (3.5)$$

The notation of (3.5) is as follows: The first sum is taken over all eigenstates of $\bar{H}_0$, the second sum over the four components of the spinor $Q$. The quantity $s_\gamma$ is defined to be $+1$ for $\gamma = 1$ and 3, $-1$ for $\gamma = 2$ and 4; the quantity $\beta_\gamma$ is $+1$ for $\gamma = 1$ and 2, $-1$ for $\gamma = 3$ and 4. The matrix element $(r|Q_\gamma|1)$ connects the arbitrary state $r$ of the stat-nucleon problem to the starting state 1 (of energy $-2(3)^{\frac{1}{2}}Nf$) and similarly for the matrix elements of $Q_\gamma^+$. $E^{(r)}$ is the energy of eigenstate $r$ of the $H_0$ problem, while $E^{(1)} = -2(3)^{\frac{1}{2}}Nf$. Finally $\epsilon_k$ is the absolute value of $E_\rho$, that is $\epsilon_k = +(k^2+\mu^2)^{\frac{1}{2}}$.

We are clearly led to the integrals

$$I_1 = \int_1^\infty \frac{(x^2-1)^{\frac{1}{2}}dx}{(x-1)(x+\eta)(x^2+\zeta^2)},$$

$$I_2 = \int_1^\infty \frac{(x^2-1)^{\frac{1}{2}}dx}{(x+1)(x+\eta)(x^2+\zeta^2)},$$

where the strong coupling condition assures us that $\eta > 1$ for all intermediate states $r$. These integrals can be evaluated by trigonometric substitution and introduction of the half angle. The result is

$$I = (\eta^2+\zeta^2)^{-1}[A(\zeta)\eta - f(\eta) + B(\zeta)],$$

where

$$A_{1,2}(\zeta) = (\zeta^2+1)^{-\frac{1}{2}}\left[\frac{\pi}{2} \pm (\zeta)^{-1}\log[(\zeta+1)^{\frac{1}{2}}+\zeta]\right],$$

$$B_{1,2}(\zeta) = (\zeta^2+1)^{-\frac{1}{2}}\left[\zeta\log[(\zeta^2+1)^{\frac{1}{2}}+\zeta] \mp \frac{\pi}{2}\right],$$

$$f_{1,2}(\eta) = (\eta\mp 1)^{\frac{1}{2}}(\eta\pm 1)^{-\frac{1}{2}}\log[(\eta^2-1)^{\frac{1}{2}}+\eta].$$

The calculations were carried out explicitly for $\alpha/\mu = 10$ and $Nf/\mu = 5$. Larger values of the latter will make the magnetic moment even smaller. We obtain $2\Phi_0^* M_z''\Phi_1 = -0.00354(\epsilon/\mu)$. This has to be compared with the magnetic moment of the proton according to the usual Dirac theory, that is $\epsilon/\mu_P$ ($\mu_P$ = proton mass). For a mass ratio of order 10, we get the result that the contribution of the meson field is less than one-tenth the intrinsic magnetic moment of the proton. This is much too small to explain the experimental value of about 1.85. It might be objected that this is only the result of a first-order treatment and that

higher terms might increase the magnetic moment considerably. Although this is not impossible for relatively small coupling constants, it is very unlikely if $Nf/\mu \gg 5$.

It is verified without difficulty that $M$ changes sign when we start from the state in which the spin of the stat-nucleon is oppositely directed. It can also be shown that this first-order contribution to the magnetic moment changes sign when $f$ changes sign. (This implies using the state for which $\bar{H}_0 = +2(3)^{\frac{1}{2}}Nf$ as the starting state.) So we could assign $f>0$ to the proton and $f<0$ to the neutron to fit the opposite anomalies for the two particles. It will turn out that the nuclear force is largely independent of $f$ so that the objection made against this procedure in the weak coupling case is not valid. However, the magnitude of the magnetic moment is too small, and the force, while largely independent of $f$, is otherwise unacceptable

It is interesting to point out the fact that this pair theory is not quite symmetrical in the positive and negative energy states. If it were, the magnetic moment would have to vanish. The sign of the magnetic moment is determined from the sign of $\epsilon$, and $\epsilon$ is the charge of the field-quantum (meson) of a positive energy state, while the "holes" correspond to mesons of charge $-\epsilon$.

## 4. THE POTENTIAL OF NUCLEAR FORCES

We shall first give a treatment of the many-body problem, using the Fourier transforms of the wave functions. Then we shall specialize to the two-body problem.

The Hamiltonian for the many-body problem is

$$\bar{H} = \int d^3x \psi^+(x)(\boldsymbol{\alpha}\cdot\mathbf{p}+\mu\beta)\psi(x) + \sum_{A=1}^{z} f_A \bar{\psi}_A^+(\beta\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma})\bar{\psi}_A. \quad (4.1)$$

Here $A$ is an index distinguishing the nucleons, at positions $\mathbf{z}_A$. We have allowed for the possibility that the coupling constants may be different for neutrons and protons. Furthermore,

$$\bar{\psi}_A = \int d^3x\, U(\mathbf{x}-\mathbf{z}_A)\psi(x). \quad (4.2)$$

The commutation rules (2.4) are unchanged.

We shall now introduce a Fourier transform through

$$\psi(\mathbf{x}) = [1/(V)^{\frac{1}{2}}] \sum_k q_k \exp[i\mathbf{k}\cdot\mathbf{x}]. \quad (4.3)$$

The sum is taken only over those vectors $\mathbf{k}$ whose components are integral multiples of $2\pi/L$, and $V=L^3$. This means that we have introduced the *periodicity condition*

$$\psi(\mathbf{x}+\mathbf{a}) = \psi(\mathbf{x}) \quad (4.3a)$$

for all vectors $\mathbf{a}$ whose components are integer multiples of $L$. Later on we shall go to the limit $L\to\infty$.

The treatment of Section 2 was particularly suited to the one-body problem, since it allowed us to use the rotational symmetry of the problem directly. In the general many-body problem there is no rotational symmetry, and it is then advantageous to transform from differential equations to algebraic equations. This is the reason for the use of the Fourier transform.

Later on we will use the invariance of the two-body problem with respect to rotations about the axis connecting the two nucleons. It will then be necessary to transform back to $x$ space to avoid the periodicity condition, which is not invariant with respect to this operation.

We now put $N=\int d^3x U^2(x)$, as before, and define $Q_A=[1/(N)^{\frac{1}{2}}]\bar{\psi}_A$ in analogy with the one-body problem. To find the commutation rules for the $Q$'s, we substitute (2.4). This gives

$$[Q_{A\gamma}, Q_{A'\gamma'}]_+ = \delta_{AA'}\delta_{\gamma\gamma'}, \quad (4.4)$$

*provided* $|\mathbf{z}_A-\mathbf{z}_{A'}|$ *is large enough that the overlap integral*

$$\int U(\mathbf{x}-\mathbf{z}_A)U(\mathbf{x}-\mathbf{z}_{A'})d^3x$$

*can be neglected.* From now on we shall *assume this condition to hold.* This implies that all our results for the nuclear forces will be valid only when the *internuclear separation exceeds the source diameter 2a.*

It is reasonable to investigate this case only, since a force resulting from an overlap of the source distributions cannot properly be called a *field* force. For $r<2a$ the type of source function used will essentially determine the result. Since the source function is only a preliminary con-

vergence device, we would not feel justified in placing any confidence in expressions which depend upon it greatly.

The Hamiltonian becomes

$$\bar{H}=\sum_k q_k^+(\boldsymbol{\alpha}\cdot\mathbf{k}+\mu\beta)q_k + N\sum_A f_A Q_A^+(\beta\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}_A)Q_A. \quad (4.5)$$

The potential energy of the system of $Z$ nucleons placed at positions $\mathbf{z}_1\cdots\mathbf{z}_Z$ is calculated as follows: We first diagonalize the operator $\bar{H}$. It will have positive and negative eigenvalues, corresponding to positive and negative energy states of the complete system. *The ground state (vacuum) is by definition the state with the most negative energy.* Because of the infinite number of degrees of freedom of the field, this eigenvalue will diverge.

We then diagonalize $\bar{H}$ with $f_A\equiv 0$, i.e., the free meson field Hamiltonian which we will call $\bar{H}_f$. Its eigenvalues will again be of both signs, and the ground state is again defined as the state of most negative energy. Since the $\bar{H}_f$-problem is linear, the diagonalized free-field Hamiltonian can be written as a sum of terms

$$\bar{H}_f=\sum_s E_s a_s^+ a_s, \quad (4.6)$$

where the operators $a_s$ satisfy the anti-commutation relations (2.11). The most negative state is then the one in which the eigenvalues of $(a_s^+\cdot a_s)$ are $=1$ if $E_s<0$ and $=0$ if $E_s>0$. It is this state of affairs which is referred to when one speaks of "filling up all the negative energy states." It may be worth while to point out here that for $f_A\neq 0$ the problem is not linear. That is, $\bar{H}$ can be diagonalized, but it cannot be represented by the simple form (4.6); the energy of a state corresponding to two "simple" states of the field in superposition is *not* necessarily the sum of the energies of the separate states. It is therefore impossible to speak of "filling up all the negative energy states." The definition of the groundstate of $\bar{H}$ as the state of most negative energy remains meaningful, however; it is the natural generalization of the hole-theory approach to non-linear problems.

The difference between the energy of the ground state of $\bar{H}$ and that of $\bar{H}_f$ is defined to be the potential energy $W_Z$ of the system of $Z$ nucleous at positions $z_1\cdots z_Z$. $W_Z$ will of course be a function of these position variables. It is usual to subtract from $W_Z$ the contribution of the self-energies. That is, we define the potential $V_Z\equiv W_Z-Z\cdot W_1$. Since $W_1$ is independent of the positions of the nucleons in question, this amounts only to a subtraction of a constant term and is therefore not an essential step. It is useful sometimes for mathematical convenience.

So far the definition of $W_Z$ is ambiguous. The two energy-values whose difference we are to take are both infinite. To make the procedure definite, we have to specify a method of doing the subtraction. The method which suggests itself is based on the principle of adiabatic invariance. This says that a continuous change in the coupling parameters $f_1\cdots f_s$ will induce a continuous change in the eigenvalues of the operator $\bar{H}$. Now, *if the $\bar{H}$-problem is linear*, then $\bar{H}$ can be written:

$$\bar{H}=\sum_s E_s' a_s'^+ a_s'. \quad (4.7)$$

The principle of adiabatic invariance now allows us to make a *unique correspondence* between the eigenvalues $E_s$ of $\bar{H}_f$ and $E_s'$ of $\bar{H}$. Having made this correspondence, we define

$$W_Z\equiv\sum_s (E_s'-E_s), \qquad E_s, E_s'<0. \quad (4.8)$$

(4.8) is the definition used by Wentzel and others.[5] It assumes implicitly that no negative eigenvalue becomes positive as the $f_A$ are increased from zero. The modifications in case this occurs are straight-forward, however.

A more difficult question presents itself when (as in our case) the $\bar{H}$ problem is *not linear*, so that $\bar{H}$ cannot be written in the form (4.7). In our case we shall avoid the difficulty by using an expansion of the eigenvalues of the $\bar{H}$ problem according to the usual perturbation procedure. It will turn out that in zero order in $Nf/\mu$ the eigenvalues of $\bar{H}$ can be represented in the form

$$\sum_{A=1}^{Z}(E_r)_A+\sum_\rho E_\rho N_\rho, \quad (4.9)$$

[5] G. Wentzel, Zeits. f. Physik **104**, 36 (1936); Helv. Phys. Acta **10**, 107 (1937); Zeits. f. Physik **118**, 277 (1941). G. Gamow and E. Teller, Phys. Rev. **51**, 289 (1937). C. Critchfield and E. Teller, Phys. Rev. **53**, 812 (1938). E. P. Wigner, C. Critchfield, and E. Teller, Phys. Rev. **56**, 531 (1939). C. Critchfield and W. Lamb, Phys. Rev. **58**, 46 (1940). C. Critchfield, Phys. Rev. **56**, 540 (1939).

where each $(E_r)_A$ is one of the thirty-two eigenvalues of the problem of a single stat-nucleon, the $E_\rho$ are the energies of the eigenwaves of the wave-field problem, and the $N_\rho$ are occupation numbers which can assume the values 0 and 1. It is then possible to correlate the $E_\rho$ with unique $E_s$ of (4.6). It is not necessary to make such correlations for the $E_r$, since only $Z$ terms are concerned, and a finite number of terms does not cause any difficulty.

We now define the 0-order potential energy

$$W_Z^{(0)} \equiv \sum_{\substack{E_s<0 \\ E_\rho<0}} (E_\rho - E_s), \tag{4.10}$$

where we have to take differences of corresponding terms, the correspondence to be determined by the principle of adiabatic invariance.

It may be remarked that some states of the $\bar{H}_f$ problem are not subtracted, since they do not correspond to any $E_\rho$, but correspond to $\sum_{A=1}^{Z} (E_r)_A$. This does not matter since it means only an addition of a finite constant to $W_Z^{(0)}$. Similarly, the omission of $\sum_{A=1}^{Z} (E_r)_A$ itself does not do any harm because this sum is finite and does not depend upon the positions of the nucleons. (This is true only if there is no source-overlap. Otherwise the corresponding term would have to be included. It would then not be just a sum of contributions of separate stat-nucleons.)

To find higher approximations to the potential energy, we simply have to add the corrections to the energy of the ground-state of the $\bar{H}$-problem to $W_Z^{(0)}$. No more subtraction need be done. The corrections in question are found by standard perturbation procedure.

It is clear from the preceding remarks that our perturbation approach has enabled us to avoid defining $W_Z$ uniquely for the general case of non-linear $\bar{H}$. To the best of our knowledge, no such general definition has yet been proposed.

We now proceed to prove (4.9). To do this, we first define

$$g_k = [1/(V)^{\frac{1}{2}}] \int d^3x\, u(x) \exp[-i\mathbf{k}\cdot\mathbf{x}],$$

with $u(x) = [1/(N)^{\frac{1}{2}}] U(x)$, as before. This definition implies

$$\sum_{\mathbf{k}} |g_k|^2 = 1,$$

and

$$Q_A = \sum_k q_k g_k \exp[i\mathbf{k}\cdot\mathbf{z}_A].$$

We then decompose the field into a static field and a wave field (similar to 2.5).

$$q_k = \sum_A Q_A g_k \exp[-i\mathbf{k}\cdot\mathbf{z}_A] + q_k', \tag{4.11}$$

$$\sum q_k' g_k \exp[i\mathbf{k}\cdot\mathbf{z}_A] = 0. \tag{4.12}$$

The Hamiltonian (4.5) becomes $\bar{H} = \bar{H}_0 + \bar{H}' + \Omega$ where

$$\bar{H}_0 = \sum_A Q_A^+ (\mu\beta + N f_A \beta\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}_A) Q_A, \tag{4.13}$$

$$\bar{H}' = \sum_{\mathbf{k}} q_k' (\boldsymbol{\alpha}\cdot\mathbf{k} + \mu\beta) q_k', \tag{4.14}$$

$$\Omega = \sum_A \sum_k Q_A^+ g_k \exp[i\mathbf{k}\cdot\mathbf{z}_A] (\boldsymbol{\alpha}\cdot\mathbf{k} + \mu\beta) q_k' + \text{complex conjugate}. \tag{4.15}$$

Provided that a perturbation treatment based on eigenstates of $\bar{H}_0 + \bar{H}'$ is valid (i.e., the strong coupling condition is satisfied), it follows immediately that (4.9) is correct. The $E_\rho$ are energies of eigenwaves of the $\bar{H}'$-problem. We can now draw some conclusions with very little effort. Together with one more result (whose proof will be left for later) they form the *main statements* of this paper.

(1) As was remarked before, the contribution of $\bar{H}_0$ to the energy of the ground state of $\bar{H}$ (in zero-order perturbation) is a sum of terms $\sum_A (E_r)_A$, each of which refers to a single stat-nucleon. Furthermore, these terms are independent of the positions of the nucleons. Both statements follow from (4.13) immediately. Therefore the $\bar{H}_0$ problem does not lead to nuclear forces. It may be well to keep in mind the fact that (4.13) is the correct expression for the term quadratic in $Q$ only if the internuclear distances exceed $2a$.

(2) The $E_\rho$ entering (4.10) are eigenvalues corresponding to eigenwaves of the $\bar{H}'$-problem. Now this problem is defined by (4.12) and (4.14). Neither one of them contains the coupling parameters $f_A$ or the nucleon spin operators $\boldsymbol{\Sigma}_A$. The only way in which the nucleons enter into the problem at all is through the *geometrical* conditions (4.12) which involve only the positions $\mathbf{z}_A$ of the nucleons and their source functions. Therefore the 0-order potential $W_Z^{(0)}$ will be independent of the coupling constant $f_A$ and the spin variables of the nucleons (or, more properly, of the stat-nucleons). Even more can be said. The only way in which the type of coupling (i.e., the

operator $P_s$ of (1.1) enters into the $\bar{H}'$-problem is the fact that it necessitates the particular splitting (4.11) of the meson field. This same splitting will be indicated whenever $P_s$ does not involve a differential operator acting on the meson field quantity.

This proves that $W_Z^{(0)}$ is even independent of the type of coupling provided no meson field derivatives occur in the coupling term. Indeed, it will be shown in the next section that $W_2^{(0)}$ is identical with the potential already calculated by Jauch and Houriet[2] who assumed a different coupling term.

(3) The third significant result cannot be seen immediately. Nevertheless, it will be stated here, with the proof reserved for a later section. It turns out that a perturbation treatment on $\Omega$ gives 0 in all odd-order perturbations. Furthermore, for the two-body problem the second-order perturbation does not lead to a spin-dependent force. Therefore the spin-dependent part of the force between two nucleons is of order $(Nf/\mu)^{-3}$ which is by assumption a very small quantity.

These results allow us to reject the heavy-electron pair theory in the limit of *strong coupling*. In the introduction we mentioned that all *weak coupling* pair theories are unacceptable because the results for the anomalous magnetic moments cannot be reconciled with the equality of like and unlike particle forces. There remains only the possibility of an *intermediate coupling*, i.e., $Nf/\mu \sim 1$. The present treatment gives no information about that case.

## 5. THE ZERO-ORDER POTENTIAL

We treat the $\bar{H}'$-problem in almost exactly the same manner as before (Section 2). The differences are that (1) the differential equations are now algebraic equations; (2) the boundary conditions at the surface of a sphere of radius $R$ are replaced by the periodicity condition (4.3a).

Consider the auxiliary problem: it is required to extremize the quadratic form

$$\sum_k x_k^*(\boldsymbol{\alpha}\cdot\mathbf{k}+\mu\beta)x_k \tag{5.1}$$

subject to the conditions

$$\sum_k x_k^* x_k = 1, \tag{5.2}$$

$$\sum_k x_k g_k \exp[i\mathbf{k}\cdot\mathbf{z}_A] = \sum_k x_k^* g_k \exp[-i\mathbf{k}\cdot\mathbf{z}_A] = 0. \tag{5.3}$$

The $x_k$ are $c$-number spinors, and (5.3) is intended to hold for all values of $A$.

Multiply (5.2) by the scalar Lagrange factor $(-E)$ and premultiply (5.3) by the spinor $(-\lambda_A)$ for every value of $A$. Then add the equations to (5.1) and differentiate with respect to each spinor component of $x_k^*$. Equating the result to zero (in order to get an extremum) yields

$$(\boldsymbol{\alpha}\cdot\mathbf{k}+\mu\beta-E)x_k = \sum_A \lambda_A g_k \exp[-i\mathbf{k}\cdot\mathbf{z}_A]. \tag{5.4}$$

This is the determining equation for the $x_k$. To treat it, premultiply by $(\boldsymbol{\alpha}\cdot\mathbf{k}+\mu\beta+E)$ on both sides. This gives

$$(k^2+\mu^2-E^2)x_k = \sum_A g_k \exp[-i\mathbf{k}\cdot\mathbf{z}_A](\boldsymbol{\alpha}\cdot\mathbf{k}+\mu\beta+E)\lambda_A. \tag{5.5}$$

There are two types of solutions to (5.5), depending upon whether $\lambda_A = 0$ (all $A$) or not. If $\lambda_A = 0$, then

$$E^2 = \mu^2 + k^2$$

for some value of $k^2$ allowed by the periodicity condition, and $x_k = 0$ unless $k^2 = E^2 - \mu^2$. We therefore have a linear combination of plane-wave solutions of the Dirac equation with the same energy, chosen so as to be orthogonal to the source function. (With our choice of the source-functions, none of the original plane-wave solutions of Dirac's equation are orthogonal to all the $u(\mathbf{x}-\mathbf{z}_A)$.) There is clearly no energy shift involved in those solutions, and, as has been pointed out already in Section 2, the perturbation Hamiltonian is also independent of them.

The solutions for which $\lambda \neq 0$ are possible only if $k^2+\mu^2-E^2 \neq 0$ for every value of $k^2$ which belongs to a lattice point. We can then divide (5.5) by $k^2+\mu^2-E^2$ to get

$$x_k = \sum_A \frac{g_k \exp[-i\mathbf{k}\cdot\mathbf{z}_A]}{k^2+\mu^2-E^2}(\boldsymbol{\alpha}\cdot\mathbf{k}+\mu\beta+E)\lambda_A. \tag{5.6}$$

We substitute this into (5.3) to get

$$\sum_k \sum_A \frac{g_k^2 \exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_{A'})]}{k^2+\mu^2-E^2} \times(\boldsymbol{\alpha}\cdot\mathbf{k}+\mu\beta+E)\lambda_A = 0. \tag{5.7}$$

This is a set of equations, four for each value of $A'$. We have $4Z$ homogeneous linear equations

in the $4Z$ independent components of the spinors $\lambda_1 \cdots \lambda_Z$. They can be satisfied only if the determinant of the system vanishes. *This determines the allowed values of* $E$. To solve the problem completely, we would have to substitute (5.6) with these values of $E$ into (5.2) in order to normalize the $\lambda_A$. It will not be necessary to do so for our purposes. We would have sets $x_{\rho,k}$ of spinors and (normalized) spinors $\lambda_{\rho,A}$ for each allowed value $E_\rho$. We could then put

$$q_k' = \sum_\rho a_\rho x_{\rho,k}, \tag{5.8}$$

and the Hamiltonian of the wave-field problem would become

$$\bar{H}' = \sum_\rho E_\rho a_\rho^+ a_\rho. \tag{5.9}$$

It is easily verified that the $a_\rho$ obey the commutation rules (2.11).

We shall now write down the determinants involved for the one- and two-body problems. For the one-body problem, we put $\mathbf{z}_A = 0$ and get

$$D_1 = det\left\{ \sum_{\mathbf{k}} \frac{g_k^2}{k^2+\mu^2-E^2}(\boldsymbol{\alpha}\cdot\mathbf{k}+\mu\beta+E) \right\},$$
$$D_1 = (E^2-\mu^2)^2[S(E^2)]^4 = 0, \tag{5.10}$$

where

$$S(E^2) = \sum_{\mathbf{k}} \frac{g_k^2}{k^2+\mu^2-E^2}. \tag{5.11}$$

The condition involved is then simply $S(E^2)=0$ since the solutions of $(E^2-\mu^2)^2=0$ do not yield any energy-shifts from the free-field case.

It is worth while here to notice that this checks against Houriet's[5] calculations. Houriet finds as the equation determining the positions of the shifted energy levels $\phi_1^2(E^2)=0$ where

$$\phi_1(E^2) = \left\{ 1+f\sum_{\mathbf{k}} g_k^2 \left( \frac{\mu+E}{k^2+\mu^2-E^2} \right) \right\} \cdot \left\{ 1+f\sum_{\mathbf{k}} g_k^2 \left( \frac{\mu-E}{k^2+\mu^2-E^2} \right) \right\}. \tag{5.12}$$

In the limit $f\to\infty$ this reduces to (5.10).

For the two-body problem, we first define the sums

$$T(E^2, r) = \sum_k g_k^2 \frac{\exp[-i\mathbf{k}\cdot\mathbf{r}]}{k^2+\mu^2-E^2}, \tag{5.13}$$

$$iP(E^2, r) = \sum_k g_k^2 \frac{k_z \exp[-i\mathbf{k}\cdot\mathbf{r}]}{k^2+\mu^2-E^2}, \tag{5.14}$$

where $\mathbf{r}=\mathbf{z}_{II}-\mathbf{z}_I$ was used to define the $z$ axis of our coordinate system.

With these definitions, (5.7) becomes

$$\left.\begin{aligned} (E+\mu\beta)S\lambda_I + \{(E+\mu\beta)T+i\alpha_z P\}\lambda_{II} = 0 \\ \{(E+\mu\beta)T-i\alpha_z P\}\lambda_I + (E+\mu\beta)S\lambda_{II} = 0 \end{aligned}\right\}. \tag{5.15}$$

The determinant of this system is

$$D_2(E^2, r) = [(E^2-\mu^2)(S^2-T^2)-P^2]^4 = 0. \tag{5.16}$$

We can again check this against Houriet. He gets for the equation determining the shifted energy values

$$[\phi_{21}(E^2)\cdot\phi_{22}(E^2)]^2 = 0,$$

where

$$\phi_{2;1,2}(E^2) = 1+2f(\mu S \pm ET) + f^2(\mu^2-E^2)(S^2-T^2)+f^2P^2.$$

Here we have used the quantities $S$, $T$, $P$ defined in (5.11), (5.13), (5.14). They are related to Houriet's $D$, $F$, $E$, as follows:

$$D = S+T, \quad F = S-T, \quad E = -P.$$

It is seen that this reduces to (5.14) if we keep only the terms in $f^2$.

This check is, of course, no accident. It was pointed out in Section 4 that the $H'$-problem is independent of the coupling type, and that it determines the potential [except for terms of higher order in $(Nf/\mu)^{-1}$] provided the strong coupling approximation is valid. Therefore, we *had* to get the same result as Houriet (in the limit $f\to\infty$), in spite of the fact that we used a different interaction term.

Since Houriet has already calculated this potential, the calculation will not be given here. The very elegant method is due to Wentzel,[6] and reference should be made to these two papers.

## 6. THE HIGHER ORDER POTENTIAL IN THE TWO-BODY PROBLEM

In the definition of the potential energy $W_2$ of a system of two nucleons

$$W_2 = (\text{energy of ground state of } \bar{H}) - (\text{energy of ground state of } \bar{H}_f), \tag{6.1}$$

we have so far evaluated the first term only to zero-order in $A = Nf/\mu$. To get the higher ap-

[6] G. Wentzel, Helv. Phys. Acta **15**, 111 (1942).

proximations to $W_2$ we write

$$W_2 = W_2^{(0)} + W_2^{(1)} + W_2^{(2)} + , \qquad (6.2)$$

where $W_2^{(0)}$ is given by (4.10) with $Z=2$.

It is clear that the correction terms are corrections to the energy of the ground state of $\bar{H}$. These corrections will be found by means of customary perturbation procedure. This will give an expansion in inverse powers of $A$. The superscripts in (6.2) will indicate the power of $A^{-1}$ involved.

The state whose energy we have to find by the perturbation method is a degenerate state. There are four states of the two-body problem, of equal (lowest) energy in zero order in $A$. They are described by

$$\begin{aligned} X(1;1;0,0,0,\cdots) &\equiv X_1, \\ X(2;1;0,0,0,\cdots) &\equiv X_2, \\ X(1;2;0,0,0,\cdots) &\equiv X_3, \\ X(2;2;0,0,0,\cdots) &\equiv X_4. \end{aligned} \qquad (6.3)$$

Here the first index refers to the eigenstate of the first stat-nucleon, the second to the eigenstate of the second stat-nucleon, and the remaining indices denote the eigenvalues of the operators $N_\rho$ defined in Section 3.

The general procedure for finding the energy perturbation of a state degenerate in zero-order is the following: We calculate matrix elements $H_{mm'}^{(n)}$:

$$H_{mm'}^{(1)} = (m|\Omega|m'), \qquad (6.4\text{-}1)$$

$$H_{mm'}^{(2)} = \sum_{m''}{}' \frac{(m|\Omega|m'')(m''|\Omega|m')}{E(m)-E(m'')}, \qquad (6.4\text{-}2)$$

$$H_{mm'}^{(3)} = \sum_{m''m'''}{}' \frac{(m|\Omega|m'')(m''|\Omega|m''')(m'''|\Omega|m')}{[E(m)-E(m'')][E(m)-E(m''')]}, \qquad (6.4\text{-}3)$$

and similarly for higher values of $n$. We use here one index $m$ to denote the ensemble of indices necessary to define the simultaneous eigenstates of $\bar{H}_0$ and $\bar{H}'$. The eigenfunctionals for $m=1,2,3,4$ are defined in (6.3). In $H_{mm'}^{(n)}$ $m$ and $m'$ assume the values 1, 2, 3, 4 only, while $m''$, $m'''$, etc., run over all possible values. However, the prime on the sums indicates that $m''m'''$, etc., do *not* assume the values 1, 2, 3, 4.

*Let us consider the first non-zero matrix* $H_{mm'}^{(n)}$. It will be non-diagonal in general. We determine its eigenvectors $Y_1 \cdots Y_4$ and the corresponding eigenvalues. The eigenvectors $Y$ are then the linear combinations of the $X$ "appropriate" to the problem, and their eigenvalues are the lowest ($n$th) order corrections to the energy of the ground state of $\bar{H}$ corresponding to the "appropriate" zero-order states $Y$. These corrections will be different for different $Y$, in general, and $W_2^{(n)}$ will, therefore, depend upon the zero-order state chosen for the system of two stat-nucleons plus wave-field. It will be shown later that the states $X$ are eigenstates of the $z$ component of the angular momentum, with eigenvalues $+1$, 0, 0, $-1$, respectively. The "appropriate" zero-order states $Y$ will just be the usual singlet and triplet eigenvectors. The treatment will thus yield a potential $W_2^{(n)}$ which depends upon the spin-directions of the stat-nucleons.

It turns out in our case that the first non-zero matrix, $H_{mm'}^{(2)}$, is already diagonal, with the diagonal elements equal to each other. The potential $W_2^{(2)}$ is therefore spin-independent, and the spin-dependent forces will arise in the next non-zero perturbation. It will be shown later that all odd-order perturbations give zero. Therefore the spin-dependent forces are contained in $W_2^{(4)}$. A simple calculation shows that the matrix to be diagonalized is

$$G_{mm'}^{(4)} = H_{mm'}^{(4)} - W_2^{(2)} \sum_{m''}{}' \frac{(m|\Omega|m'')(m''|\Omega|m')}{[E(m)-E(m'')]^2}. \qquad (6.5)$$

The matrix elements $(m|\Omega|m')$ entering in these formulas are independent of $f$ (since (4.15) does not involve $f$). The matrix (6.5) is therefore of order $A^{-3}$, and this will be the order of the spin-dependent forces.

In view of the fact that $W_2^{(2)}$ is only an ordinary force superposed on $W_2^{(0)}$, and that the spin-dependent forces are of the third order in $A^{-1}$,

neither of them was calculated explicitly. It is possible to show, however, that $W_2^{(4)}$ can be written as a superposition of a spin-independent term, a $(\mathbf{S}_{\mathrm{I}}\cdot\mathbf{S}_{\mathrm{II}})$ term, and a tensor force term, provided that $f_{\mathrm{I}}=f_{\mathrm{II}}$.

In order to complete the argument, it remains to show that (a) $H_{mm'}{}^{(n)}$ is zero for odd $n$, (b) $H_{mm'}{}^{(2)}$ is a multiple of the (4 by 4) unit matrix.

(a) is obvious from (6.4) and the fact that $\Omega$ connects only states with occupation numbers $N$, $N'$ such that

$$\sum_\rho N_\rho=(\sum_\rho N_\rho')\pm 1;$$

so, in order to get back to one of the states $X_m$, all of which have $\sum N_\rho=0$, an even number of steps is necessary.

(b) will be proved in 3 steps. *First*, we shall show that all but two of the off-diagonal elements of $H_{mm'}{}^{(n)}$ are zero in all orders $n$. This will follow from the conservation of angular momentum around the $z$ axis. *Second*, we shall show that the two remaining off-diagonal elements are zero in second order. *Third*, it will be proved that the diagonal elements $H_{mm}{}^{(2)}$ are independent of $m$.

It is easily seen that the $z$ component of the angular momentum operator

$$J_z=\int\psi^+(x)[(\mathbf{x}\times\mathbf{p})_z+\tfrac{1}{2}\sigma_z]\psi(x)d^3x+\tfrac{1}{2}(\textstyle\sum_{\mathrm{I}})_z+\tfrac{1}{2}(\sum_{\mathrm{II}})_z \quad (6.6)$$

commutes with the Hamiltonian $\bar{H}$, (4.1).

Here we assume the two nucleons to be situated somewhere along the $z$ axis of our coordinate system.

The splitting of the field (4.11) becomes in $x$ space:

$$\psi(x)=\textstyle\sum_A Q_A u(\mathbf{x}-\mathbf{z}_A)+\psi'(x), \quad (6.7)$$

$$\int\psi'(x)u(\mathbf{x}-\mathbf{z}_A)d^3x=0. \quad (6.8)$$

Using these formulas together with the fact that $u(\mathbf{x})=u(|\mathbf{x}|)$ only, it is easily seen that

$$J_z=(J_z)_0+(J_z)', \quad (6.9)$$

where

$$(J_z)_0=\tfrac{1}{2}\sum_{A=1,2}(Q_A^+\sigma_z Q_A)+\tfrac{1}{2}(\textstyle\sum_A)_z, \quad (6.10)$$

$$(J_z)'=\int\psi'^+(x)[(\mathbf{x}\times\mathbf{p})_z+\tfrac{1}{2}\sigma_z]\psi'(x). \quad (6.11)$$

The expression (6.10) is valid only as long as the overlap integral is zero.

The important point here is that $J_z$ commutes not only with $\bar{H}=\bar{H}_0+\bar{H}'+\Omega$, but also with $\bar{H}_0+\bar{H}'$ and $\Omega$, separately. It is easily seen that the states (6.3) correspond to $(J_z)'=0$ and $(J_z)_0=+1,\ 0,\ 0,\ -1$, respectively. Now, since $\Omega$ commutes with $J_z$, there cannot be any non-zero matrix elements of $\Omega$ between states with different values of $J_z$. *This proves that the only non-zero off-diagonal elements of the matrices $H_{mm'}{}^{(n)}$ can be $H_{23}{}^{(n)}$ and its complex conjugate $H_{32}{}^{(n)}$.*

The perturbing Hamiltonian $\Omega$ is, according to (4.15) and (5.4),

$$\Omega=\sum_{A=1,2}\textstyle\sum_\rho(Q_A^+\lambda_A a_\rho+a_\rho^+\lambda_{A,\rho}Q). \quad (6.12)$$

Now the matrix elements $(1|Q|2)$ and $(1|Q^+|2)$ are zero. Therefore one needs at least *four steps* to go from $X_2$ to $X_3$. *This proves that* $H_{23}{}^{(2)}=H_{32}{}^{(2)}=0$.

The proof that the diagonal elements $H_{mm}{}^{(2)}$ are equal to each other depends upon special properties of the matrix elements $(r|Q|1)$, etc., and of the system (5.15). It is simple and will not be given here. Reference may be made to the thesis submitted to Princeton University by the author.

*These arguments complete the proof that the spin-dependent forces in the two-body problem arise in order $A^{-3}$.*

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