Kuhn.[38] For $Hg+H_2$ at the same temperature $r_c<2.1\times10^{-8}$ cm.

One could suppose that for very large r_c (large l's) the perturbation theory could be successfully applied[39] in order to obtain the intensity distribution of the central part of the broadened line, but, presumably, sufficiently accurate calculations by this method would be as tedious as those carried out by means of (20) and (21) or (20a) and (21a) and certainly less accurate.

[38] K., reference 10.
[39] J.2, reference 1.

The shift of the maximum of the intensity is not considered here. It would result from (14) for high densities of perturbers if calculations of particular cases involving the effect of multiple encounters were carried out. Apart from this, a shift may also be caused by the displacement of translational motion energy levels due to the perturbations of these levels by mutual potential energy of the radiator and the perturbers. The last effect, however, is not included in our approximation.

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A Generalization of the Dielectric Ellipsoid Problem

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The classical problem of the dielectric ellipsoid involves the determination of the field within a homogeneous, isotropic, dielectric ellipsoid when it is placed in a uniform electric field. In the present generalization, both the ellipsoid and the medium in which it is placed, although still homogeneous, are anisotropic and also possess conductivities which are anisotropic. The principal axes of the ellipsoid, of the two dielectric tensors, and of the two conductivity tensors, may all be differently oriented. The external field, although uniform in space, varies sinusoidally with time. The condition specified in the last sentence is consistent with the electromagnetic field equations only in a region whose maximum dimension is small compared with $\lambda/2\pi$ where λ is the wave-length which corresponds to the frequency in question. Thus the solution given here is restricted by the condition that the maximum dimension of the ellipsoid must be small compared with $\lambda/2\pi$.

INTRODUCTION

In the application to practical problems of Wiener's general theory[1] of the electrical and optical properties of heterogeneous materials, the writer has found it desirable to have the solution of a generalization of the dielectric ellipsoid problem. The solution of this generalized problem is presented in this paper.

The classical problem of the dielectric ellipsoid[2] involves finding the field within a homogeneous, isotropic, dielectric ellipsoid when the ellipsoid is placed in a uniform electric field. In the present generalization, the ellipsoid is anisotropic and has a finite electric conductivity, and the medium in which it is imbedded is likewise anisotropic and conducting. Furthermore, the electric field, which is uniform except for the disturbance produced by the presence of the ellipsoid, varies sinusoidally with time. The conditions that the electric field be uniform in space and that it vary sinusoidally with time are evidently inconsistent with the electromagnetic field equations, but the conditions may be satisfied to any desired degree of approximation by making the dimensions of the ellipsoid sufficiently small compared with the wave-length which corresponds to the frequency in question. Accordingly, the solution obtained here holds only for the case in which the maximum dimension of the ellipsoid is small compared with c/ω where c is the velocity

[1] Otto Wiener, Abhandl. d. Sächs. Ges. d. Wiss. **32**, 509–604 (1912).
[2] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), pp. 211–213.

of light, and ω is the angular frequency of the electric field.

A solution which does not involve the limitation described in the last sentence has been obtained[3] for the scattering of plane waves by a dielectric sphere which is isotropic and which has a finite conductivity.

STATEMENT OF THE PROBLEM

The problem to be solved is as follows: In a universe filled with a homogeneous material described by a dielectric tensor[4] $\boldsymbol{\varepsilon}_2$ and a conductivity tensor $\boldsymbol{\sigma}_2$, there exists a uniform electric field $\mathbf{E}_2e^{i\omega t}$ which varies sinusoidally with time at the radian frequency ω. Now suppose that an ellipsoidal cavity be cut from the material, and that the cavity be filled with a second homogeneous material whose dielectric tensor is $\boldsymbol{\epsilon}_1$ and whose conductivity tensor is $\boldsymbol{\sigma}_1$. The principal axes of $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ and of the ellipsoid may all be differently oriented.

The problem is the determination of the electric field within the ellipsoid as a function of the frequency, of the shape and orientation of the ellipsoid, and of the constitutive tensors $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, $\boldsymbol{\sigma}_1$, and $\boldsymbol{\sigma}_2$.

METHOD OF SOLUTION

A solution will first be obtained for the simpler case in which the external medium is a vacuum. The solution for the more general case may then be obtained by an extension of the principle that any solution of an electrostatic problem depends only on the ratios of the dielectric constants and not on their absolute values.

The solution of the simpler problem involves the provisional assumption that the field within the ellipsoid is uniform, and the subsequent proof that this assumption leads to a consistent solution of the problem.

The assumed uniform field produces a polarization which is in phase with the field, as a consequence of the dielectric polarizability of the material. Another type of polarization which lags ninety degrees behind the field is also produced, as the result of the accumulation of free charge on the surface of the ellipsoid; the transport of free charge to the surface is due to the electrical conductivity of the material. The electric field caused by these polarizations may be calculated from the classical solution for the gravitational potential within a homogeneous ellipsoid. The total electric field within the ellipsoid is then the sum of the fields caused by the two separate kinds of polarization and the external field. Taken together with the constitutive relations, the relation stated in the last sentence leads directly to the solution of the problem when the external medium is a vacuum.

The following section contains a brief derivation of the electric field produced by a uniformly polarized ellipsoid. The result of this derivation is needed in the solution of the main problem, which is contained in the section which follows the next one.

ELECTRIC FIELD PRODUCED BY A UNIFORMLY POLARIZED ELLIPSOID

The gravitational or electrostatic potential Ω at an inner point of an ellipsoid of uniform mass or charge density ρ was shown by Dirichlet[5] to be given by:

$$\Omega = -\rho(A_x x^2 + A_y y^2 + A_z z^2), \tag{1}$$

where

$$A_x = \pi abc \int_0^\infty \frac{d\theta}{(a^2+\theta)\Delta}, \tag{2}$$

$$A_y = \pi abc \int_0^\infty \frac{d\theta}{(b^2+\theta)\Delta}, \tag{3}$$

$$A_z = \pi abc \int_0^\infty \frac{d\theta}{(c^2+\theta)\Delta}, \tag{4}$$

$$\Delta = \{(a^2+\theta)(b^2+\theta)(c^2+\theta)\}^{\frac{1}{2}}. \tag{5}$$

The origin of coordinates lies at the center of the ellipsoid, and the semi-axes a, b, and c lie along the x, y, and z coordinate axes, respectively. The relations (1)–(5) are derived by Kellogg[6] with the use of ellipsoidal coordinates and are derived by Meyer[7] by straightforward integration, using ordinary Cartesian coordinates.

[3] M. Born, *Die Optik* (Verlagsbuchhandlung Julius Springer, Berlin, 1933), pp. 274–285.
[4] Reference 2, p. 11.
[5] P. G. Dirichlet, J. f. Math. **32**, 80 (1846); *Werke*, Vol. 2, p. 11.
[6] O. D. Kellogg, *Potential Theory*, (Verlagsbuchhandlung Julius Springer, Berlin, 1929), pp. 192–194.
[7] G. F. Meyer, *Theorie der Bestimmten Integralle* (B. G. Teubner, Leipzig 1871), pp. 521–525, 531–547.

It follows directly from the definitions of A_x, A_y, and A_z that

$$A_x+A_y+A_z=2\pi. \tag{6}$$

When the semi-axes a, b, and c are all different, the quantities A_x, A_y, and A_z may be expressed in terms of incomplete elliptic integrals of the first and second kinds.[8] If, however, two of the axes are equal, so that the ellipsoid becomes a prolate or oblate spheroid, then these quantities may be expressed in terms of elementary functions.[9]

The following cases are of special interest:
Long thin circular rod ($a\gg b=c$):

$$A_x=0, \quad A_y=A_z=\pi.$$

Sphere ($a=b=c$):

$$A_x=A_y=A_z=2\pi/3.$$

Thin circular disk ($a\ll b=c$):

$$A_x=2\pi, \quad A_y=A_z=0.$$

Equation (1) represents the potential at an inner point of an ellipsoid which is uniformly charged throughout its volume. The potential at an inner point of a uniformly polarized ellipsoid may be obtained by taking two similar and parallel ellipsoids of exactly opposite charge densities, letting them interpenetrate so that the distance between their centers is small compared with the dimensions of the ellipsoid, and then taking the limit as the distance of separation vanishes and the charge density ρ of the positive ellipsoid becomes infinite, the product remaining finite. If $\mathbf{r}$ is the displacement of the positively charged ellipsoid from the negatively charged one, one finds, holding $\mathbf{P}=\rho\mathbf{r}$ constant and letting $\mathbf{r}$ approach zero, the following expression for the potential Ω_P at an inner point of an ellipsoid with a uniform polarization $\mathbf{P}$:

$$\begin{aligned}\Omega_P&=-\mathbf{r}\cdot\operatorname{grad}\Omega\\&=2xA_xP_x+2yA_yP_y+2zA_zP_z.\end{aligned} \tag{7}$$

The electric field at an inner point of a uniformly polarized ellipsoid may now be obtained by differentiation:

$$\begin{aligned}\mathbf{E}_P&=-\operatorname{grad}\Omega_P,\\E_{Px}&=-2A_xP_x,\\E_{Py}&=-2A_yP_y,\\E_{Pz}&=-2A_zP_z.\end{aligned} \tag{8}$$

Or, more generally in any Cartesian coordinate system,

$$\mathbf{E}_P=-2\boldsymbol{A}\mathbf{P}, \tag{9}$$

where $\boldsymbol{A}$ is the tensor which has the form

$$\boldsymbol{A}\equiv\begin{pmatrix}A_x & 0 & 0\\0 & A_y & 0\\0 & 0 & A_z\end{pmatrix}, \tag{10}$$

when the coordinate axes are parallel with the axes of the ellipsoid.

In the following development, it is more convenient to use the tensor $\boldsymbol{B}$, which is equal to $\boldsymbol{A}$ divided by 2π:

$$\boldsymbol{B}\equiv\boldsymbol{A}/2\pi. \tag{11}$$

Equation (9) thus becomes

$$\mathbf{E}_P=-4\pi\boldsymbol{B}\mathbf{P}. \tag{12}$$

With the assistance of this last equation, the solution of the main problem will now be resumed.

DETAILED SOLUTION WHEN THE EXTERNAL MEDIUM IS A VACUUM

Let us assume provisionally that the field within the ellipsoid is uniform and is given by the expression $\mathbf{E}_1e^{i\omega t}$. The exponential time factor $e^{i\omega t}$ will be omitted in the following expressions. The electric field $\mathbf{E}_1$ will produce a polarization of the ellipsoid which is given by

$$\mathbf{P}_\varepsilon=\frac{\varepsilon_1-1}{4\pi}\mathbf{E}_1. \tag{13}$$

The electric field will also produce an electric current density given by

$$\mathbf{i}=\sigma_1\mathbf{E}_1. \tag{14}$$

This current will bring about a transport of electric charge to the surface of the ellipsoid at the time rate $|\mathbf{i}|$ units of charge per second per unit area perpendicular to the vector $\mathbf{i}$. The actual amount of charge present at any time is thus given by $\int|\mathbf{i}|dt$, in units of charge per unit area perpendicular to $\mathbf{i}$. But by a well-

[8] Reference 6, p. 196; Reference 7, pp. 540–543.
[9] Reference 2, p. 214.

known theorem[10] of electrostatics, this charge distribution is equivalent to a polarization of the ellipsoid given by

$$\mathbf{P}_\sigma = \int \mathbf{i}\,dt = \boldsymbol{\sigma}_1 \mathbf{E}_1 / i\omega. \tag{15}$$

The total polarization of the ellipsoid is thus

$$\mathbf{P} = \mathbf{P}_{\boldsymbol{\epsilon}} + \mathbf{P}_\sigma = \frac{\varepsilon_1 - 1}{4\pi}\mathbf{E}_1, \tag{16}$$

where

$$\varepsilon \equiv \boldsymbol{\epsilon} + \frac{4\pi\boldsymbol{\sigma}}{i\omega}. \tag{17}$$

The tensor ε is the complex dielectric tensor as usually defined.

By Eq. (12) of the last section, the electric field which is produced by this polarization is

$$\mathbf{E}_P = -4\pi \boldsymbol{B}\mathbf{P} = -\boldsymbol{B}(\varepsilon_1 - 1)\mathbf{E}_1. \tag{18}$$

The total field $\mathbf{E}_1$ within the ellipsoid is equal to the sum of the field $\mathbf{E}_P$ which is due to the polarization of the ellipsoid and the uniform external field $\mathbf{E}_2$. Thus,

$$\begin{aligned}\mathbf{E}_1 &= \mathbf{E}_2 + \mathbf{E}_P \\ &= \mathbf{E}_2 - \boldsymbol{B}(\varepsilon_1 - 1)\mathbf{E}_1,\end{aligned} \tag{19}$$

or

$$\mathbf{E}_2 = [1 + \boldsymbol{B}(\varepsilon_1 - 1)]\mathbf{E}_1. \tag{20}$$

This is the desired relation between the field $\mathbf{E}_1$ within the ellipsoid and the uniform external field $\mathbf{E}_2$.

It will be noted that the provisional assumption of a uniform electric field within the ellipsoid led to a total polarization, the electric field of which was uniform. The combination of the polarization field and the external uniform field is, therefore, also uniform, and the provisional assumption is thereby confirmed.

A homogeneous ellipsoid is the most general homogeneous solid which becomes uniformly polarized when it is placed in a uniform field.

DETAILED SOLUTION FOR AN ARBITRARY EXTERNAL MEDIUM

The solution just obtained satisfies the conditions that

$$\nabla \cdot \varepsilon_1 \mathbf{E} = 0 \tag{21a}$$

hold within the ellipsoid, that

$$\nabla \cdot \mathbf{E} = 0 \tag{22a}$$

hold in the region outside the ellipsoid, and that the normal component of $\varepsilon\mathbf{E}$ be continuous through the surface of the ellipsoid:

$$(\varepsilon_1 \mathbf{E})_{n1} = (\mathbf{E})_{n2}. \tag{23a}$$

In the more general problem, these relations are replaced by

$$\nabla \cdot \varepsilon_1' \mathbf{E} = 0, \tag{21b}$$

$$\nabla \cdot \varepsilon_2' \mathbf{E} = 0, \tag{22b}$$

and

$$(\varepsilon_1' \mathbf{E})_{n1} = (\varepsilon_2' \mathbf{E})_{n2}. \tag{23b}$$

Note now that if in the second set of relations, the tensor $\varepsilon_1' \varepsilon_2'^{-1}$ be written as $\boldsymbol{\alpha}$:

$$\boldsymbol{\alpha} \equiv \varepsilon_1' \varepsilon_2'^{-1}, \tag{24}$$

then the second set of relations may be written

$$\nabla \cdot \boldsymbol{\alpha}(\varepsilon_2' \mathbf{E}) = 0, \tag{21c}$$

$$\nabla \cdot \varepsilon_2' \mathbf{E} = 0; \tag{22c}$$

and

$$[\boldsymbol{\alpha}(\varepsilon_2' \mathbf{E})]_{n1} = [\varepsilon_2' \mathbf{E}]_{n2}. \tag{23c}$$

Comparison of this set of relations with the first set shows that the general problem is equivalent to the simpler problem in which the external medium is a vacuum, provided that in the more general problem, the expressions $\varepsilon_1' \varepsilon_2'^{-1}$ and $\varepsilon_2' \mathbf{E}$ are identified with the expressions ε_1 and $\mathbf{E}$ of the simpler problem. Accordingly, the solution (20) for the simpler case becomes:

$$\varepsilon_2 \mathbf{E}_2 = [1 + \boldsymbol{B}(\varepsilon_1 \varepsilon_2^{-1} - 1)]\varepsilon_2 \mathbf{E}_1, \tag{25}$$

or

$$\mathbf{E}_2 = [1 + \varepsilon_2^{-1} \boldsymbol{B}(\varepsilon_2 - \varepsilon_1)]\mathbf{E}_1, \tag{26}$$

where the primes have been omitted because they are no longer needed.

In order to obtain $\mathbf{E}_1$ as an explicit function of $\mathbf{E}_2$, it is necessary to determine the tensor which is reciprocal to the tensor $\boldsymbol{T}$:

$$\boldsymbol{T} \equiv 1 + \varepsilon_2^{-1} \boldsymbol{B}(\varepsilon_2 - \varepsilon_1), \tag{27}$$

whence the expression for $\mathbf{E}_1$ is

$$\mathbf{E}_1 = \boldsymbol{T}^{-1} \mathbf{E}_2. \tag{28}$$

The solution of the generalized problem stated at the beginning of this paper is given by the last three equations with Eqs. (2)–(5), (10), (11), and (17) serving to define the quantities which appear therein.

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[10] M. Abraham and R. Becker, *Electricity and Magnetism* (Blackie and Son Limited, London, 1932), pp. 72, 73.