Under the assumption that ka is small, we get

$$\cot \delta_0 = A/k, \tag{17}$$

and by (11),

$$\sigma = 4\pi/(k^2+A^2). \tag{18}$$

If η is small, $A \approx \alpha$, we have

$$\sigma_0 = 4\pi/(k^2+\alpha^2). \tag{19}$$

The numerical results are given in Table V, in which we take $V = -Ae^{k/r}$, and the values of α are obtained from the Table I at the point of joint. Corresponding to (18) and (19), Bethe and Bacher have derived the formulae: $\sigma_{0B} = 4\pi h^2/(\epsilon + E)$, $\sigma_B \doteq \frac{3}{2}\sigma_0$,[3] and the values calculated from these are given in the 4th and 5th columns of Table V for comparison.

It is seen that the present results are in better agreement with the experimental results than those given by Bethe and Bacher.

If we take $V = -(B/r)e^{k/r}$, the results do not differ appreciably from those given above.

In conclusion, the author wishes to express his thanks to Dr. K. C. Wang for suggesting this calculation and for helpful discussions.

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Fourier Transforms of Retarded and Advanced Potentials

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The Fourier transforms of the retarded and advanced potentials of the electromagnetic field and of the wave fields of elementary particles are obtained with the help of the invariant functions of Jordan, Pauli, and Dirac, together with their generalizations. It is shown that the Fourier transforms of these potentials are closely related to those of the outward and inward moving waves given by Dirac for the scattering problems in quantum mechanics, and their connection is discussed. It is also shown that there exists a type of potentials which represents waves with frequencies of opposite signs propagating in opposite directions.

1. INTRODUCTION

JORDAN and Pauli[1] introduced into the quantum theory of electromagnetic field a function

$$\frac{\delta(x_0+|\mathbf{x}|)-\delta(x_0-|\mathbf{x}|)}{|\mathbf{x}|},$$

which plays a very important part in the commutation relations of the field variables. Following Dirac,[2] we shall denote it by $-\Delta$ and write

$$\Delta(\mathbf{X}) = \frac{\delta(x_0-|\mathbf{x}|)-\delta(x_0+|\mathbf{x}|)}{|\mathbf{x}|}, \tag{1}$$

where $\mathbf{X}$ denotes the four-vector $(\mathbf{x}, x_0)$. This Δ function is relativistically invariant and satisfies the wave equation of electromagnetic field in vacuum

$$\Box\Delta(\mathbf{X}) = 0, \tag{2}$$

with

$$\Box = \frac{\partial^2}{\partial x_0{}^2} - \nabla^2. \tag{3}$$

It can be expressed as a triple Fourier integral of the form

$$\Delta(\mathbf{X}) = \frac{1}{2\pi^2}\int \frac{\sin kx_0}{k}\exp(\pm i\mathbf{k}\cdot\mathbf{x})d^3k, \tag{4}$$

where $k = |\mathbf{k}|$, or as a quadruple Fourier integral of the form[3]

$$\Delta(\mathbf{X}) = \frac{i}{4\pi^2}\int \Delta(\mathbf{K})\exp(-ik_0x_0) \times \exp(\pm i\mathbf{k}\cdot\mathbf{x})d^4k, \tag{5}$$

where $\mathbf{K}$ denotes the four-vector $(\mathbf{k}, k_0)$.

Some functions of the same type but more general than that of Jordan and Pauli appeared first in Dirac's[4] positron theory, which are in the

[1] Jordan and Pauli, Zeits. f. Physik **47**, 151 (1928).
[2] Dirac, Proc. Roy. Soc. **A180**, 1 (1942).
[3] Dirac, Ann. l'Inst. H. Poincaré **9**, 13 (1939).
[4] Dirac, Proc. Camb. Phil. Soc. **30**, 150 (1934).

notation of recent papers by Pauli,[5]

$$D(\mathbf{X})=\frac{1}{(2\pi)^3}\int\frac{\sin\,(k^2+\kappa^2x_0)^{\frac{1}{2}}}{(k^2+\kappa^2)^{\frac{1}{2}}}\times\exp\,(\pm i\mathbf{k}\cdot\mathbf{x})d^3k,\quad(6)$$

$$D_1(\mathbf{X})=\frac{1}{(2\pi)^3}\int\frac{\cos\,(k^2+\kappa^2x_0)^{\frac{1}{2}}}{(k^2+\kappa^2)^{\frac{1}{2}}}\times\exp\,(\pm i\mathbf{k}\cdot\mathbf{x})d^3k.\quad(7)$$

These functions are also relativistically invariant and satisfy the wave equations

$$(\Box+\kappa^2)D(\mathbf{X})=0,\quad(8)$$

$$(\Box+\kappa^2)D_1(\mathbf{X})=0.\quad(9)$$

As shown by Dirac, if we denote $|\mathbf{x}|$ by x,

$$D(\mathbf{X})=-\frac{1}{4\pi}\frac{1}{\kappa}\frac{\partial}{\partial x}F(x,\,x_0),\quad(10)$$

where

$$F(x,\,x_0)=\begin{cases}J_0[\kappa(x_0{}^2-x^2)^{\frac{1}{2}}] & (x_0>x)\\ 0 & (x>x_0>-x_0)\\ -J_0[\kappa(x_0{}^2-x^2)^{\frac{1}{2}}] & (x_0<x),\end{cases}\quad(11)$$

and

$$D_1(\mathbf{X})=\frac{1}{4\pi}\frac{1}{\kappa}\frac{\partial}{\partial x}F_1(x,\,x_0),\quad(12)$$

where

$$F_1(x,\,x_0)=\begin{cases}N_0[\kappa(x_0{}^2-x^2)^{\frac{1}{2}}] & (x_0>x,\,x_0<-x)\\ -iH_0{}^{(1)}[i\kappa(x^2-x_0{}^2)^{\frac{1}{2}}] & (x>x_0>-x).\end{cases}\quad(13)$$

The function $D(\mathbf{X})$ reduces to $\Delta(\mathbf{X})/4\pi$ when $\kappa=0$ and may be regarded as the analog of $\Delta/4\pi$ in field theories of elementary particles whose rest mass does not vanish. The function $D_1(\mathbf{X})$ reduces when $\kappa=0$ to

$$D_1(\mathbf{X})=\frac{1}{2\pi^2}\frac{1}{x^2-x_0{}^2}\text{ for }x\neq x_0.\quad(14)$$

The possibility of employing the D_1 function in quantized field theories has recently been discussed by Pauli.

It is of interest to point out that besides the positron theory and the general theory of second quantization there is a third type of mathematical problem in which these D functions can be used with advantage; namely, the Fourier resolution of retarded and advanced potentials of field theories which will be carried out in the present paper for both the electromagnetic field and the wave fields of elementary particles with the help of these D functions and their generalizations. The mathematical treatment will be developed on the lines of Dirac's[6] treatment of outward and inward moving waves in scattering problems of quantum mechanics. The connection between the Fourier transforms of the retarded and advanced potentials and those of the outward and inward moving waves will be discussed.

2. FOURIER TRANSFORMS OF RETARDED AND ADVANCED POTENTIALS OF ELECTROMAGNETIC FIELD

Let $\mathbf{A}(\mathbf{X})$ be a vector function with four components representing the scalar and vector potentials of the electromagnetic field when the charge and current densities are represented by a vector function $\mathbf{j}(\mathbf{X})$. It is known in classical electromagnetic theory that the function $\mathbf{A}$ satisfies the inhomogeneous wave equation

$$\Box\mathbf{A}(\mathbf{X})=4\pi\mathbf{j}(\mathbf{X})\quad(15)$$

and that there are two independent solutions of (15) which have simple physical meanings, namely, the retarded potentials

$$\mathbf{A}(\mathbf{X})=\int\frac{\mathbf{j}(\mathbf{x}',\,x_0-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}^2|}d^3x'\quad(16)$$

and the advanced potentials

$$\mathbf{A}(\mathbf{X})=\int\frac{\mathbf{j}(\mathbf{x}',\,x_0+|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}d^3x'.\quad(17)$$

These retarded and advanced potentials may be expressed as four-dimensional integrals with the help of Dirac's δ functions, thus

$$\mathbf{A}(\mathbf{X})=\int\mathbf{j}(\mathbf{X}')\frac{\delta(x_0-x_0'\mp|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}d^4x',\quad(18)$$

or

$$\mathbf{A}(\mathbf{X})=\int\mathbf{j}(\mathbf{X}')\frac{\delta(r_0\mp r)}{r}d^4x',\quad(19)$$

if we write $r_0=x_0-x_0'$, $\mathbf{r}=\mathbf{x}-\mathbf{x}'$, $r=|\mathbf{r}|$. Now the δ functions satisfy the relation

$$\frac{\delta(r_0-r)+\delta(r_0+r)}{2r}=\delta(r_0{}^2-r^2),\quad(20)$$

[5] W. Pauli, Phys. Rev. **58**, 716 (1940); Rev. Mod. Phys. **13**, 203 (1941).

[6] Dirac, Zeits. f. Physik **44**, 585 (1927); *Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1935), second edition.

and (1) and (20) give us

$$\frac{\delta(r_0 \mp r)}{r} = \delta(r_0^2 - r^2) \pm \tfrac{1}{2}\Delta(\mathbf{R}), \tag{21}$$

where $\mathbf{R}$ denotes the four-vector $(\mathbf{r}, r_0)$. Formulae (19) may therefore be written

$$\mathbf{A}(\mathbf{X}) = \int \mathbf{j}(\mathbf{X}') \{ \delta(r_0^2 - r^2) \pm \tfrac{1}{2}\Delta(\mathbf{R}) \} d^4x'. \tag{22}$$

It has been shown by Dirac that, if we write

$$\begin{gathered} \Box_R = \frac{\partial^2}{\partial r_0} - \frac{\partial^2}{\partial r_1^2} - \frac{\partial^2}{\partial r_2^2} - \frac{\partial^2}{\partial r_3^2}, \\ \Box_R \Delta(\mathbf{R}) = 0. \end{gathered} \tag{23}$$

Using Dirac's method we can also show that

$$\Box_R \delta(r_0^2 - r^2) = 4\pi\delta_4(\mathbf{R}), \tag{24}$$

where

$$\delta_4(\mathbf{R}) = \delta(r_0)\delta(r_1)\delta(r_2)\delta(r_3).$$

From (21), (23), (24) it follows that

$$\Box_R \frac{\delta(r_0 \mp r)}{r} = 4\pi\delta_4(\mathbf{R}). \tag{25}$$

The relations (23)–(25) correspond to the fact that the retarded potentials, advanced potentials, and their mean values satisfy the same inhomogeneous wave equation, while the differences of the retarded and advanced potentials satisfy the homogeneous wave equation for the vacuum and are free of singularity everywhere.

Let the vector functions $\mathbf{A}(\mathbf{X})$ and $\mathbf{j}(\mathbf{X})$ be expressed as Fourier integrals in the form

$$\tilde{\mathbf{A}}(\mathbf{X}) = \frac{1}{(2\pi)^2} \int \tilde{\mathbf{A}}(\mathbf{K}) \exp(i(\mathbf{K}, \mathbf{X})) d^4k, \tag{26}$$

$$\tilde{\mathbf{j}}(\mathbf{X}) = \frac{1}{(2\pi)^2} \int \tilde{\mathbf{j}}(\mathbf{K}) \exp(i(\mathbf{K}, \mathbf{X})) d^4k, \tag{27}$$

where

$$(\mathbf{K}, \mathbf{X}) = k_0 x_0 - \mathbf{k} \cdot \mathbf{x}.$$

From (22), (26), and (27) we find

$$\tilde{\mathbf{A}}(\mathbf{K}) = \tilde{\mathbf{j}}(\mathbf{K}) \int \{ \delta(r_0^2 - r^2) \pm \tfrac{1}{2}\Delta(\mathbf{R}) \} \times \exp(-i(\mathbf{K}, \mathbf{R})) d^4r. \tag{28}$$

Thus the Fourier transforms of the potentials can be expressed in terms of those of the source densities if we know the Fourier transforms of the functions $\delta(r_0^2 - r^2)$ and $\Delta(\mathbf{R})$.

Now

$$\int \delta(r_0^2 - r^2) \exp(-i(\mathbf{K}, \mathbf{R})) d^4r = \int\int \frac{\delta(r_0 - r) + \delta(r_0 + r)}{2r} \exp(-ik_0 r_0 + i\mathbf{k} \cdot \mathbf{r}) d^3r dr_0$$
$$= \int \frac{\cos k_0 r \exp(i\mathbf{k} \cdot \mathbf{r})}{r} d^3r,$$

so that, on writing

$$\Delta_1(\mathbf{K}) = \frac{1}{2\pi^2} \int \frac{\cos k_0 r \exp(i\mathbf{k} \cdot \mathbf{r})}{r} d^3r, \tag{29}$$

we have

$$\int \delta(r_0^2 - r^2) \exp(-i(\mathbf{K}, \mathbf{R})) d^4r = 2\pi^2 \Delta_1(\mathbf{K}). \tag{30}$$

From (5) applied with $\mathbf{K}$ and $\mathbf{X}$ replaced by $\mathbf{R}$ and $\mathbf{K}$, we obtain

$$\tfrac{1}{2} \int \Delta(\mathbf{R}) \exp(-i(\mathbf{K}, \mathbf{R})^2) d^4r = -2\pi i^2 \Delta(\mathbf{K}). \tag{31}$$

Substituting from (30), (31) in (28), we obtain

$$\tilde{\mathbf{A}}(\mathbf{K}) = 2\pi^2 \tilde{\mathbf{j}}(\mathbf{K}) \{ \Delta_1(\mathbf{K}) \mp i\Delta(\mathbf{K}) \}, \tag{32}$$

a relation which expresses the Fourier transforms of the potentials in terms of those of the source densities with the help of the D functions. We have written here Δ, Δ_1 for the values of $4\pi D$, $4\pi D_1$, respectively, in the case of $\kappa = 0$.

3. A MODIFIED FORM OF RETARDED AND ADVANCED POTENTIALS

Substituting from (1) and (14) in (32) we obtain for the Fourier transforms of retarded and advanced potentials, the expressions

$$\tilde{\mathbf{A}}(\mathbf{K}) = 4\pi\tilde{\mathbf{j}}(\mathbf{K})\left\{\frac{1}{k^2-k_0{}^2} \mp \frac{\pi}{2} i \frac{\delta(k_0-k)-\delta(k_0+k)}{k}\right\}, \quad (k^2 \neq k_0{}^2). \tag{33}$$

The expressions on the right-hand side of (33) are not defined for $k_0 = \pm k$, where the term $1/(k^2-k_0{}^2)$ tends to infinity. In order to give this term a definite meaning for $k_0 = \pm k$, we can replace the function $1/(k^2-k_0{}^2)$ by the function $\pi\Delta_1/2$, defined by (29), at the points of singularity. When thus interpreted, (33) substituted in (26) gives our original formulae (19).

There is, however, another way of dealing with the singularities at $k_0 = \pm k$. Following Dirac's works on scattering, we can take an integral whose integrand contains $1/(k^2-k_0{}^2)$ as a factor to be equal to its Cauchy principal value, e.g.,

$$\int \frac{\tilde{\mathbf{j}}(\mathbf{K}) \exp (i(\mathbf{K}, \mathbf{X}))}{k^2-k_0{}^2} d^4k = \lim_{\epsilon \to 0} \int d^3k \left\{\int_{-\infty}^{-k-\epsilon} + \int_{-k+\epsilon}^{k-\epsilon} + \int_{k+\epsilon}^{\infty}\right\} dk_0 \frac{\tilde{\mathbf{j}}(\mathbf{K}) \exp (i(\mathbf{K}, \mathbf{X}))}{k^2-k_0{}^2}. \tag{34}$$

It will be interesting to see what result (33) gives when it is interpreted in this way.

The potentials given by (33) are

$$\mathbf{A}(\mathbf{X}) = \frac{1}{\pi} \int \tilde{\mathbf{j}}(\mathbf{K}) \left\{\frac{1}{k^2-k_0{}^2} \mp \frac{\pi}{2} i\Delta(\mathbf{K})\right\} \exp (i(\mathbf{K}, \mathbf{X})) d^4k,$$

$$= \frac{1}{(2\pi)^3} \int\int \mathbf{j}(\mathbf{X}') \exp (-i(\mathbf{K}, \mathbf{X}')) \left\{\frac{2}{k^2-k_0{}^2} \mp \pi i \Delta(\mathbf{K})\right\} \exp (i(\mathbf{K}, \mathbf{X})) d^4k.$$

Hence, if we write

$$g(\mathbf{X}, \mathbf{X}') = \frac{1}{(2\pi)^3} \int \left\{\frac{2}{k^2-k_0{}^2} \mp i\pi\Delta(\mathbf{K})\right\} \exp (i(\mathbf{K}, \mathbf{R})) d^4k, \tag{35}$$

the retarded and advanced potentials may be written in the form

$$\mathbf{A}(\mathbf{X}) = \int g(\mathbf{X}, \mathbf{X}') \mathbf{j}(\mathbf{X}') d^4x'. \tag{36}$$

Equating the right-hand sides of (19) and (36), we find

$$g(\mathbf{X}, \mathbf{X}') = \frac{\delta(r_0 \mp r)}{r}. \tag{37}$$

Let us now find what expressions for $g(\mathbf{X}, \mathbf{X}')$ are given by (35) when Cauchy's principal values are taken.

Consider first the integral

$$\int_{-\infty}^{\infty} \frac{1}{k_0{}^2-k^2} \exp (ik_0x_0) dk_0 = \frac{1}{2k} \int_{-\infty}^{\infty} \left(\frac{1}{k_0-k} - \frac{1}{k_0+k}\right) \exp (ik_0x_0dk_0).$$

Since

$$\int_{-\infty}^{\infty} \frac{1}{k_0-k} \exp (ik_0x_0) dk_0 = \pm \pi i \exp (ikx_0) \quad (x_0 \gtrless 0),$$

$$\int_{-\infty}^{\infty} \frac{1}{k_0+k} \exp (ik_0x_0) dk_0 = \pm \pi i \exp (-ikx_0) \quad (x_0 \gtrless 0),$$

we have

$$\int_{-\infty}^{\infty} \frac{1}{k_0{}^2-k^2} \exp (ik_0x_0) dk_0 = \mp \pi \frac{\sin kx_0}{k} \quad (x_0 \gtrless 0), \tag{38}$$

so that

$$\int \frac{1}{k_0^2-k^2} \exp(i(\mathbf{K}, \mathbf{X}))d^4k = \mp 2\pi^3 \Delta(\mathbf{X}) \quad (x_0 \gtrless 0), \tag{39}$$

by (4). From (5), (35), and (39) we obtain

$$\begin{aligned} g(\mathbf{X}, \mathbf{X}') &= \Delta(\mathbf{R}) \quad (r_0>0), \\ &= 0 \qquad (r_0<0), \end{aligned} \tag{40}$$

for the retarded potentials and

$$\begin{aligned} g(\mathbf{X}, \mathbf{X}') &= 0 \qquad (r_0>0), \\ &= -\Delta(\mathbf{R}) \quad (r_0<0), \end{aligned} \tag{41}$$

for the advanced potentials.

On account of (1), formulae (37) and formulae (40), (41) agree except when $r_0=0$. On the plane $r_0=0$, (37) gives us $\delta(r)/r$ but (40) and (41) give us 0. Thus the two expressions for the function $g(\mathbf{X}, \mathbf{X}')$ are the same everywhere except at the point $\mathbf{R}=\mathbf{X}-\mathbf{X}'=0$.

4. FOURIER TRANSFORMS OF RETARDED AND ADVANCED POTENTIALS IN THE CASE OF $\kappa \neq 0$

Let us now consider a U field satisfying the equation

$$(\square+\kappa^2)U(\mathbf{X}) = 4\pi\rho(\mathbf{X}). \tag{42}$$

We assume, for simplicity, U and ρ to be scalar functions.

On expressing the functions $U(\mathbf{X})$ and $\rho(\mathbf{X})$ as Fourier integrals in the form

$$U(\mathbf{X}) = \frac{1}{(2\pi)^2} \int \tilde{U}(\mathbf{K}) \exp(i(\mathbf{K}, \mathbf{X}))d^4k, \tag{43}$$

$$\rho(\mathbf{X}) = \frac{1}{(2\pi)^2} \int \tilde{\rho}(\mathbf{K}) \exp(i(\mathbf{K}, \mathbf{X}))d^4k, \tag{44}$$

and substituting in (42), we find

$$(\kappa^2+k^2-k_0^2)\tilde{U}(\mathbf{K}) = 4\pi\tilde{\rho}(\mathbf{K}). \tag{45}$$

The result of dividing out (45) by the factor $(\kappa^2+k^2-k_0^2)$ is not unique, since when an arbitrary linear combination of the functions $\delta[k_0-(k^2+\kappa^2)^{\frac{1}{2}}]$ and $\delta[k_0+(k^2+\kappa^2)^{\frac{1}{2}}]$ is multiplied by $(\kappa^2+k^2-k_0^2)$ the product will vanish. Formulae (33) of Section 3 suggest that the Fourier transforms of the retarded and advanced U fields should be of the form

$$\tilde{U}(\mathbf{K}) = 4\pi\tilde{\rho}(\mathbf{K}) \left\{ \frac{1}{\kappa^2+k^2-k_0^2} \mp \frac{\pi}{2} i \frac{\delta[k_0^2(k^2+\kappa^2)^{\frac{1}{2}}]-\delta[k_0+(k^2+\kappa^2)^{\frac{1}{2}}]}{(k^2+\kappa^2)^{\frac{1}{2}}} \right\}. \tag{46}$$

By a piece of work analogous to that of Section 3 we find that (46) substituted in (43) gives, by (44),

$$U(\mathbf{X}) = \int g(\mathbf{X}, \mathbf{X}')\rho(\mathbf{X}')d^4x', \tag{47}$$

with

$$g(\mathbf{X}, \mathbf{X}') = \frac{1}{(2\pi)^3} \int \left\{ \frac{2}{\kappa^2+k^2-k_0^2} \mp i\pi \frac{\delta[k_0-(k^2+\kappa^2)^{\frac{1}{2}}]-\delta[k_0+(k^2+\kappa^2)^{\frac{1}{2}}]}{(k^2+\kappa^2)^{\frac{1}{2}}} \right\} \exp(i(\mathbf{K}, \mathbf{R}))d^4k, \tag{48}$$

which reduces to

$$\begin{aligned} g(\mathbf{X}, \mathbf{X}') &= 4\pi D \quad (r_0>0), \\ &= 0 \qquad (r_0<), \end{aligned} \tag{49}$$

and

$$\begin{aligned} g(\mathbf{X}, \mathbf{X}') &= 0 \qquad (r_0>0), \\ &= -4\pi D \quad (r_0<0), \end{aligned} \tag{50}$$

for the retarded and advanced potentials, respectively. It can be seen with the help of (10) and (11) that the formulae (49) and (50) agree with the results obtained by Bhabha[7] for the meson field.

5. OUTWARD AND INWARD MOVING WAVES

The work of the last two sections was carried out on the same lines as Dirac's treatment of scattering. Dirac obtained a general result which may be stated as follows: When $|\mathbf{x}|$ is large,

$$\int \exp(i\mathbf{k}\cdot\mathbf{x})f(\mathbf{k})\left\{\frac{1}{(k'^2+\kappa^2)^{\frac{1}{2}}-(k^2+\kappa^2)^{\frac{1}{2}}}\mp i\pi\delta[(k'^2+\kappa^2)^{\frac{1}{2}}-(k^2+\kappa^2)^{\frac{1}{2}}]\right\}d^3k = -4\pi^2(k'^2+\kappa^2)^{\frac{1}{2}}f\left(\pm k'\frac{\mathbf{x}}{x}\right)\frac{\exp(\pm ik'x)}{x}. \quad (51)$$

Dirac showed with the help of these formulae that by taking the combinations

$$\frac{1}{(k'^2+\kappa^2)^{\frac{1}{2}}-(k^2+\kappa^2)^{\frac{1}{2}}}\mp i\pi\delta[(k'^2+\kappa^2)^{\frac{1}{2}}-(k^2+\kappa^2)^{\frac{1}{2}}],$$

one can obtain wave functions which represent outward and inward moving particles, respectively. It was later shown by Møller[8] in the nonrelativistic approximation that Dirac's result holds not only for large distances from the source but throughout the whole space.

It will be interesting to find out the connection between our formulae for retarded and advanced potentials and Dirac's formulae for outward and inward moving waves. From (43) and (46) we have

$$U(\mathbf{X}) = \frac{1}{\pi}\int \tilde{\rho}(\mathbf{K})\left\{\frac{1}{\kappa^2+k^2-k_0{}^2}\mp\frac{\pi}{2}i\frac{\delta[k_0-(k^2+\kappa^2)^{\frac{1}{2}}]-\delta[k_0+(k^2+\kappa^2)^{\frac{1}{2}}]}{(k^2+\kappa^2)^{\frac{1}{2}}}\right\}\exp(i(\mathbf{K},\mathbf{X}))d^4k$$
$$= \frac{-1}{2\pi}\int \frac{\tilde{\rho}(\mathbf{K})}{(k^2+\kappa^2)^{\frac{1}{2}}}\left\{\left[\frac{1}{k_0-(k^2+\kappa^2)^{\frac{1}{2}}}\pm\pi i\delta(k_0-(k^2+\kappa^2)^{\frac{1}{2}}\right] + \left[\frac{1}{-k_0-(k^2+\kappa^2)^{\frac{1}{2}}}\mp\pi i\delta[-k_0-(k^2+\kappa^2)^{\frac{1}{2}}]\right]\right\}\exp(ik_0x-i\mathbf{k}\cdot\mathbf{x})d^4k. \quad (52)$$

The asymptotic values of the right-hand side of (52) for large distances are, according to (51) applied with $-\mathbf{x}$ in place of $\mathbf{x}$,

$$U(\mathbf{X}) = 2\pi\int_{-\infty}^{-\kappa}\tilde{\rho}\left[\mp(k_0{}^2-\kappa^2)^{\frac{1}{2}}\frac{\mathbf{x}}{x},\, k_0\right]\frac{\exp(i[k_0x_0\pm(k_0{}^2-\kappa^2)^{\frac{1}{2}}x])}{x}dk_0 + 2\pi\int_{\kappa}^{\infty}\tilde{\rho}\left[\pm(k_0{}^2-\kappa^2)^{\frac{1}{2}}\frac{\mathbf{x}}{x},\, k_0\right]\frac{\exp(i[k_0x_0\mp(k_0{}^2-\kappa^2x)^{\frac{1}{2}}])}{x}d^3k_0, \quad (53)$$

showing that the retarded and advanced potentials correspond to the outward and inward moving waves, respectively, both for frequencies $k_0>\kappa$ and for frequencies $k_0<-\kappa$.

Besides the retarded and advanced potentials, there is another type of solutions of the inhomogeneous wave equations which have a simple mathematical form, namely, the potentials whose Fourier transforms are given, instead of by (46), by

$$\tilde{U}(\mathbf{K}) = 4\pi\tilde{\rho}(\mathbf{K})\left\{\frac{1}{\kappa^2+k^2-k_0{}^2}\mp\frac{\pi}{2}i\frac{\delta[k_0-(k^2+\kappa^2)^{\frac{1}{2}}]+\delta[k_0+(k^2+\kappa^2)^{\frac{1}{2}}]}{(k^2+\kappa^2)^{\frac{1}{2}}}\right\}$$
$$= 4\pi\tilde{\rho}(\mathbf{K})\left\{\frac{1}{\kappa^2+k^2-k_0{}^2}\mp\pi i\delta(k_0{}^2-k^2-\kappa^2)\right\}. \quad (54)$$

Proceeding as in deriving (53) we find as the asymptotic values of the U fields whose Fourier trans-

[7] Bhabha, Proc. Roy. Soc. **A172**, 384 (1939).
[8] Møller, Zeits. f. Physik **66**, 513 (1930).

forms are given by (54):

$$U(\mathbf{X})=2\pi\left\{\int_{-\infty}^{-\kappa}+\int_{\kappa}^{\infty}\right\}dk_0\tilde{\rho}\left[\pm(k_0^2-\kappa^2)^{\frac{1}{2}}\frac{\mathbf{x}}{x},\,k_0\right]\frac{\exp\,(i[k_0x_0\mp(k_0^2-\kappa^2)^{\frac{1}{2}}x])}{x}. \tag{55}$$

Unlike the retarded and advanced potentials, this type of solutions either represents outward moving waves of positive frequencies together with inward moving waves of negative frequencies or represents inward moving waves of positive frequencies together with outward moving waves of negative frequencies, so that waves with frequencies of opposite signs travel in opposite directions. Such U fields may be written in the form of (47) with

$$g(\mathbf{X},\mathbf{X}')=\frac{1}{4\pi^3}\int\left\{\frac{1}{\kappa^2+k^2-k_0^2}\mp i\pi\delta(k_0^2-k^2-\kappa^2)\right\}\exp\,(i(\mathbf{K},\mathbf{R}))d^4k. \tag{56}$$

These expressions reduce to

$$g(X,X')=-2\pi i\{D_1(\mathbf{R})\pm iD(\mathbf{R})\}\qquad(r_0\gtreqless 0), \tag{57}$$

$$g(X,X')=2\pi i\{D_1(\mathbf{R})\mp iD(\mathbf{R})\}\qquad(r_0\gtreqless 0), \tag{58}$$

respectively, which are Hankel functions.

APPENDIX. PRINCIPAL VALUES OF IMPROPER INTEGRALS

Concerning Cauchy's principal value of an improper integral, the following theorem is frequently useful. Let $f(x)$ be a function such that the integral,

$$\int_{a-b}^{a+b}\frac{f_0(x)}{x-a}dx,$$

exists, where

$$f_0(x)=\frac{f(x)-f(2a-x)}{2},$$

and $b>0$. Then

$$P\int_{a-b}^{a+b}\frac{f(x)}{x-a}dx=\int_{a-b}^{a+b}\frac{f_0(x)}{x-a}dx. \tag{59}$$

For, if we write

$$f_e=\frac{f(x)+f(2a-x)}{2},$$

then

$$P\int_{a-b}^{a+b}\frac{f(x)}{x-a}dx=P\int_{a-b}^{a+b}\frac{f_0(x)}{x-a}dx+P\int_{a-b}^{a+b}\frac{f_e(x)}{x-a}dx=\int_{a-b}^{a+b}\frac{f_0(x)}{x-a}dx$$

$$+\lim_{\epsilon\to 0}\left\{\int_{a-b}^{a-\epsilon}\frac{f_e(x)}{x-a}dx+\int_{a+\epsilon}^{a+b}\frac{f_e(x)}{x-a}dx\right\}=\int_{a-b}^{a+b}\frac{f_0(x)}{x-a}dx.$$

As a corollary to this theorem we note that if $f(x)$ be expressible as Taylor's series in the form

$$f(x)=f(a)+(x-a)f'(a)+\tfrac{1}{2}(x-a)^2f''(a)+\cdots$$

and b is a small number, then

$$P\int_{a-b}^{a+b}\frac{f(x)}{x-a}dx=2bf'(a)+\cdots. \tag{60}$$

This corollary has been used in a recent paper by Hsüeh and the present writer.[9]

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[9] Ma and Hsüeh, Proc. Camb. Phil. Soc. 40 (1944).