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On the Meson Charge Cloud Around a Proton*

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The calculations of Fröhlich, Heitler, and Kahn for the deviation from the Coulomb law for a proton owing to mesons are re-examined and extended to the scalar meson theory. A perturbation calculation is used up to terms proportional to the square of the coupling constant, including the recoil of the nucleon to first order in μ/M. The recent Dirac theory involving negative energy states of the mesons, in conjunction with the λ-limiting process due to Wentzel, makes the theory convergent. The dissociation probability P of a proton in this theory is proportional to the square of the coupling constant and to the mass ratio μ/M. P is of the order of 2 percent. The meson charge cloud produces only a slight decrease of the Coulomb force acting on a charged test-particle, not a reversal of this force. No experimentally observable effects can be expected from the processes considered. The results of Fröhlich, Heitler, and Kahn are not reproduced by the convergent theory.

INTRODUCTION

IN this paper we are going to investigate the meson charge cloud which forms around a proton caused by the protons' interaction with a meson field. It is generally admitted that mesons must be assumed to have charge. This means physically that charge is transported in a meson beam, while no charge is transported in a (neutral) light beam. The charge of the mesons gives rise to interactions between them and electromagnetic fields. This leads, for instance, to the anomalous magnetic moments of the elementary heavy particles (neutron and proton). Not only do the heavy particles themselves interact with an external magnetic field (through their spin magnetic moment), but also the mesons which they create around themselves. The experiment gathers together the contributions from the heavy particles and from the meson field, while the usual theoretical value neglects the contribution of the mesons. This effect has been discussed by Fröhlich, Heitler, and Kemmer[1] and more recently by Jauch.[2]

In this paper we calculate another effect of the charge of the mesons. The meson field which forms around the proton contains charge, and this means an effective "smearing out" of part of the charge of the proton over a region within the range of nuclear forces. This *charge cloud around the proton* is likely to change the Coulomb force on a charged test particle brought near to the proton. It is also going to influence the scattering of charged particles on the proton. These effects might be experimentally detectable. The

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[1] Fröhlich, Heitler, and Kemmer, Proc. Roy. Soc. **A166**, 154 (1938).
[2] J. M. Jauch, Phys. Rev. **63**, 334 (1943).

change in the Coulomb law at small distances from the proton would lead to a (slight) change in the energy levels of the s states of the hydrogen atom. The anomalous scattering would be most noticeable for very high energy incident particles which can approach the nucleus closely.

Fröhlich, Heitler, and Kahn[3] have tried to calculate this change in the energy levels of hydrogen, assuming a vector meson field. They arrive at a result for the correction, but the validity of their calculation has been questioned by Lamb[4] and a controversy ensued over it.[5] The difficulty in those calculations is caused by our insufficient knowledge of field quantization. All the standard theories lead to divergencies of terms of the self-energy type. The process by which the meson charge cloud around a proton is found leads to terms of this type. Only one nuclear particle is involved, and its interaction with the meson field created by itself is considered. This is in contradistinction to the nuclear force calculations in which the interaction of a nuclear particle with the meson field created by a different nuclear particle is found.

In our case the self-energy effect gives rise to an infinite total charge in the meson charge cloud. One of the requirements of a "reasonable" theory is that, on the average, most of the time the meson field contains no mesons at all. For a relatively short fraction of each second, the meson field should contain one or more mesons in eigenstates of the field. If the meson field contains just one (positive) meson, the proton is changed into a neutron during that time. This is necessary in order to insure conservation of charge. Let ϵ be the charge of the proton, and e the total charge in the meson field. Then we define a "dissociation probability" P by $e=P\epsilon$. The failure of the usual theory appears through the fact that P not only exceeds unity, but actually diverges.

It must be emphasized that the total charge present does not diverge. Formally, at least, the usual theories conserve the charge. The total charge present consists of the charge on the heavy particle plus the charge in the meson field. Thus it is $(1-P)\epsilon+P\epsilon=\epsilon$. The equality holds formally even when P diverges.

Let U be the potential energy of a test particle of charge ϵ in the meson field. It can be shown that $\nabla^2 U=-4\pi\epsilon\rho$ where ρ is the charge density in the meson field. This is an operator equation and thus must hold for the diagonal elements (the average values). It is seen that an infinite constant in U need not bring about a divergence of ρ; but a divergence of the total charge in the meson field will make the energy U infinite. This is physically plausible since a test particle needs infinite energy to approach an infinite charge, even when that infinite charge is spread out over a small region.

Fröhlich, Heitler, and Kahn get out of this difficulty by subtracting from their calculated term the part of the term which is indistinguishable from the effect of a charge concentrated at the center (as far as the interaction energy with a test particle is concerned). The difference turns out to be convergent. Their calculations are open to grave objections, however. Both the minuend and the subtrahend are infinite, and almost anything may be considered as the difference between two infinite terms. Also, they get a strong attraction of a positive test particle at very close distances from the nucleon. This is rather hard to accept, since we would be inclined to expect only a slight decrease of the Coulomb force because of the spreading out of the charge on the nucleon, not a complete reversal of that force.

Recently Dirac[6] has proposed a theory which involves negative energy eigenstates of the meson field. Pauli[7] has discussed this theory in great detail. It turns out that if the theory is coupled with a limiting process taken over from classical physics, all the divergencies can be eliminated. This limiting process, called the λ process, was first proposed by Wentzel.[8]

It was considered worth while to test this convergent theory by application to our particular problem. If the theory is correct, then we ought to obtain a convergent total charge in the meson field, and we should be able to calculate

[3] Fröhlich, Heitler, and Kahn, Proc. Roy. Soc. **A171**, 269 (1939).
[4] W. Lamb, Phys. Rev. **56**, 384 (1939).
[5] Fröhlich, Heitler, and Kahn, Phys. Rev. **56**, 961 (1939), W. Lamb, Phys. Rev. **57**, 458 (1940).
[6] Dirac, Proc. Roy. Soc. **A180**, 1 (1942).
[7] W. Pauli, Rev. Mod. Phys. **15**, 175 (1943).
[8] G. Wentzel, Zeits. f. Physik **86**, 479 and 635 (1933); **87**, 726 (1934).

the correction to the energy levels of the hydrogen atom without recourse to any arbitrary subtraction scheme. Furthermore, this procedure might give us some additional insight into the actual workings of the Dirac theory and of the λ process. That is, we should be able to tell just how particular terms in the charge density are affected by the introduction of those modifications, and just how the new theory manages to yield a convergent result. One difficulty must be pointed out, however. It has been shown by Pauli and Jauch[9] that the Dirac theory makes the short wave-length contributions convergent, but at the expense of changing the contributions of the long wave-lengths. Practically this means a conflict with the correspondence principle and is an extremely serious objection to the whole theory.

Hence our results cannot be expected to agree with experiment, except in a qualitative way. The main object of this paper is thus a further test of the convergent meson theory of Dirac and Pauli, rather than the calculation of experimentally verifiable results.

CHARGE DENSITY CALCULATION WITH THE USUAL SCALAR THEORY

We use the Pauli-Weisskopf[10] theory in which the meson field quantity ψ is a scalar. This theory has been discussed carefully in a review paper by Pauli.[11] The main change in notation is the difference in the Fourier decomposition of the field. We set

$$\psi(\mathfrak{x})=\frac{1}{\sqrt{2}(2\pi)^{\frac{3}{2}}}\int d^3k\left(\frac{1}{k_0}\right)\times(a_k\exp(i\mathfrak{k}\cdot\mathfrak{x})-b_k^*\exp(-i\mathfrak{k}\cdot\mathfrak{x})). \quad (2.1)$$

The notation is,

$\mathfrak{x}\equiv(\mathbf{x}, ix_0)=(\mathbf{x}, ict)$, 4-position of field point,

$d^3k=dk_1dk_2dk_3$, volume element in k-space,

$k_0=+(k^2+\mu^2)^{\frac{1}{2}}$,

$\mu=\mu c/\hbar$, Compton wave number for a particle (meson) of mass μ (about 200 electron masses),

$\mathfrak{k}\cdot\mathfrak{x}=\mathbf{k}\cdot\mathbf{x}-k_0x_0$, 4-vector dot-product.

The operators a_k and b_k are, in Pauli's notation,

$$a_k=(k_0)^{\frac{1}{2}}U_+(k), \quad b_k=(k_0)^{\frac{1}{2}}U_-(k). \quad (2.2)$$

This change is helpful later on in the Dirac theory, since it throws all imaginary factors into the operators; with Pauli's original notation we would encounter factors of the type $1/(k_0)^{\frac{1}{2}}$ but also of the type $1/(-k_0)^{\frac{1}{2}}$. We wrote the Fourier decomposition as an integral rather than a sum since this is the form in which it will be used. The units are the same as in Pauli's paper (i.e., c and $\hbar$ are set equal to unity).

The wave function Ψ of the nucleon is assumed to be made up of plane-wave eigenfunctions of the Dirac Hamiltonian

$$\bar{H}_n=\boldsymbol{\alpha}\cdot\mathbf{p}+M\beta=-i\boldsymbol{\alpha}\cdot\nabla_y+M\beta. \quad (2.3)$$

Here $\boldsymbol{\alpha}$, β are the well-known Dirac matrices, and M is the mass of the nucleon. $\mathfrak{y}=(\mathbf{y}, iy_0)$ is the space-time position of the nucleon.

The Schrödinger equation is

$$(\bar{H}_0+\bar{H}_n+\bar{H}_i)\Omega=id\Omega/dx_0. \quad (2.4)$$

Here

$$\bar{H}_0=\int d^3k(a_k^*a_k+b_k^*b_k)=$$

Hamiltonian of free meson field, (2.5)

$\bar{H}_n=$ Hamiltonian of the free nucleon,

$$\bar{H}_i=g(4\pi)^{\frac{1}{2}}(\tau_-\psi^*(\mathfrak{y})+\tau_+\psi(\mathfrak{y}))\cdot\beta$$

$=$ Interaction Hamiltonian,[12] (2.6)

$\Omega=$ Schrödinger functional giving the state of the meson field and of the nucleon.

In the interaction Hamiltonian, g is a dimensionless coupling constant (of order of magnitude $\frac{1}{2}$). τ_+ and τ_- are charge creation and destruction operators, respectively. Let u_P denote the eigenfunction of a proton, u_N that of a neutron, then

$$\tau_+u_P=0, \quad \tau_-u_P=u_N, \quad \tau_+u_N=u_P, \quad \tau_-u_N=0. \quad (2.7)$$

Notice that the meson field quantity ψ in $\bar{H}_i$ has to be taken at the position $\mathfrak{y}$ of the heavy particle, giving a point interaction. The time coordinate is common for the nucleon and the meson field, i.e., $x_0=y_0$.

[9] W. Pauli and J. M. Jauch, Phys. Rev. **65**, 255 (1944).
[10] W. Pauli and V. Weisskopf, Helv. Phys. Acta **7**, 709 (1934).
[11] W. Pauli, Rev. Mod. Phys. **13**, 203 (1941).
[12] N. Kemmer, Proc. Roy. Soc. **A166**, 143 (1938).

In order to solve (2.3) by a perturbation method we expand the Schrödinger functional Ω in powers of the coupling constant g:

$$\Omega=\Omega_0+\Omega_1+\Omega_2+\cdots. \tag{2.8}$$

Equating terms of the same power in g, we get

$$(\bar{H}_0+\bar{H}_n)\Omega_0 = id\Omega_0/dx_0, \tag{2.9}$$

$$(\bar{H}_0+\bar{H}_n)\Omega_1+\bar{H}_i\Omega_0=id\Omega_1/dx_0, \tag{2.10}$$

$$(\bar{H}_0+\bar{H}_n)\Omega_2+\bar{H}_i\Omega_1=id\Omega_2/dx_0, \tag{2.11}$$

etc.

For the zero-order functional in this perturbation problem we take

$$\Omega_0=\Psi_0\omega(0)u_P. \tag{2.12}$$

Here Ψ_0 is a function of $\mathfrak{y}$ and the spin-coordinate of the nucleon. We shall specialize to a proton initially at rest, i.e.,

$$\Psi_0=\begin{pmatrix} u\pm \\ 0 \end{pmatrix} \exp(-iMy_0) \tag{2.13}$$

where $u_+=\begin{pmatrix}1\\0\end{pmatrix}$ and $u_-=\begin{pmatrix}0\\1\end{pmatrix}$. Thus (2.13) is

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\exp(-iMy_0) \quad \text{or} \quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\exp(-iMy_0)$$

according as to whether the spin of the nucleon is assumed to be in the positive or negative z direction.

$\omega(0)$ is the meson field functional corresponding to no mesons present. Generally we shall denote by $\omega(N_k^+, N_k^-)$ a simultaneous eigenfunction of the operators $a_k^*a_k=k_0N_k^+$ and $b_k^*b_k=k_0N_k^-$. From (2.2) and Pauli's paper[13] we see that

$$a_k^*\omega(N_k^+)=(k_0)^{\frac{1}{2}}(N_k^++1)^{\frac{1}{2}}\omega(N_k^++1),$$

$$a_k\omega(N_k^+)=(k_0)^{\frac{1}{2}}(N_k^+)^{\frac{1}{2}}\omega(N_k^+-1), \tag{2.14}$$

b_k same as a_k but for negative mesons.

We now proceed to find Ω_1 in (2.8). From (2.10) we see that we need only consider functionals for which the matrix elements of $\bar{H}_i$: $\Omega_1^*\bar{H}_i\Omega_0$ do not vanish. The only functionals satisfying this condition are of the type $\Psi_1(\mathfrak{y})\omega(1_k^+)u_N$. Physically, they denote the existence of one positive meson in a momentum eigenstate of the meson field, plus a neutron in a momentum eigenstate of the Dirac Hamiltonian. Thus

$$\Omega_1=\int d^3k\Psi_1(\mathfrak{y}, k)\omega(1_k^+)u_N. \tag{2.15}$$

Substitute in (2.10) and pre-multiply by $\omega^*(1_k^+)u_N^*$ to get

$$\bar{H}_n\Psi_1+\omega^*(1_k^+)u_N^*\bar{H}_i\omega(0)u_P=id\Psi_1/dx_0. \tag{2.16}$$

Here we have used the fact that

$$\bar{H}_0\omega(1_k^+)=id\omega(1_k^+)/dx_0;$$

i.e., $\omega(1_k^+)$, being an eigenstate of $a_k^*a_k$ and of $b_k^*b_k$, is also an eigenstate of the free meson field Hamiltonian $\bar{H}_0$; it is then taken to be a solution of the unperturbed Schrödinger equation for the meson field. This just fixes the time dependence of the meson field functional (we are using the Schrödinger representation here in which the operators are constants or else explicit functions of the time, while the wave functions have a time dependence given by Schrödinger's equation).

We put (2.16) in the form

$$(id/dx_0-\bar{H}_n)\Psi_1=(1_k^+, N|\bar{H}_i|0, P),$$

(the notation for the matrix element of $\bar{H}_i$ is obvious). Pre-multiplying by $(id/dx_0+\bar{H}_n)$ and using the commutation rules for the Dirac matrices, this becomes

$$-(d^2/dx_0^2+p^2+M^2)\Psi_1$$
$$=(id/dx_0+\bar{H}_n)\cdot(1_k^+, N|\bar{H}_i|0, P). \tag{2.17}$$

Using (2.17) and (2.13) together with (2.3), (2.6), and (2.1), we obtain

$$\Psi_1=f\frac{\exp(-i\mathfrak{k}\cdot\mathfrak{y}-iMy_0)}{(k_0)^{\frac{1}{2}}(\mu^2-2Mk_0)}\begin{pmatrix}(2M-k_0)u\pm\\ -\mathbf{k}\cdot\boldsymbol{\sigma}u_\pm\end{pmatrix}. \tag{2.18}$$

Here $f=g/(2\pi)$ and $\boldsymbol{\sigma}$ is the well-known Pauli spin vector:

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

[13] Reference 11, page 210, Eq. (19).

The procedure just outlined involves the following approximation: (2.14) is correct only if we have an unperturbed meson field. In general we have,

$$\frac{da_k^*}{dx_0}=i[\bar{H}_i,\,a_k^*].$$

The approximation consists in neglecting this explicit time variation of a_k^* compared to the time variation of the term $\exp(-i\mathfrak{k}\cdot\mathfrak{y})$ in the matrix element $(1_k^+, N|\bar{H}_i|0, P)$. This means neglecting a second-order term in Ω_0 and a third-order term in Ω_1.

We finally obtain,

$$\Omega_0=\begin{pmatrix}u\pm\\0\end{pmatrix}\exp(-iMy_0)\omega(0)u_P, \tag{2.19}$$

$$\Omega_1=f\int\frac{d^3k\exp(-i\mathfrak{k}\cdot\mathfrak{y}-iMy_0)}{(k_0)^{\frac{1}{2}}(\mu^2-2Mk_0)}\begin{pmatrix}(2M-k_0)u\pm\\-\mathbf{k}\cdot\boldsymbol{\sigma}u\pm\end{pmatrix}\omega(1_k^+)u_N, \tag{2.20}$$

$$\Omega_2=f^2\int\frac{d^3k\,d^3l\exp[-i(\mathfrak{k}+\mathfrak{l})\cdot\mathfrak{y}-iMy_0]\omega(1_l^+,1_k^-)u_P}{2(k_0l_0)^{\frac{1}{2}}(\mu^2-2Ml_0)[\mu^2-M(k_0+l_0)+k_0l_0-\mathbf{k}\cdot\mathbf{l}]}\cdot\begin{pmatrix}[(2M-k_0-l_0)(2M-l_0)+(\mathbf{k}+\mathbf{l})\cdot\mathbf{l}+i\mathbf{l}\times\mathbf{k}\cdot\boldsymbol{\sigma}]u\pm\\ [(2M-l_0)(\mathbf{k}+\mathbf{l})\cdot\boldsymbol{\sigma}-(k_0+l_0)\mathbf{l}\cdot\boldsymbol{\sigma}]u\pm\end{pmatrix}. \tag{2.21}$$

An infinite resonance term was omitted from Ω_2. It turns out that this term may be made convergent by the λ process and can then be combined with the Hamiltonian of the heavy particle $\bar{H}_n$, giving only a slight change in its effective mass.

It may be worth while to put down the expressions for Ω_0, Ω_1, Ω_2 in the limit as $M\rightarrow\infty$. Let $u_P'=u_p\exp(-iMy_0)u\pm$. Then these expressions are

$$\Omega'_0=\omega(0)u'_P, \tag{2.19'}$$

$$\Omega'_1=-f\int d^3k(k_0)^{-\frac{3}{2}}\exp(-i\mathfrak{k}\cdot\mathfrak{y})\omega(1_k^+)u'_N, \tag{2.20'}$$

$$\Omega'_2=f^2\int\frac{d^3k\,d^3l}{k_0^{\frac{1}{2}}l_0^{\frac{3}{2}}(k_0+l_0)}\exp(-i(\mathfrak{k}+\mathfrak{l})\cdot\mathfrak{y})\omega(1_l^+,1_k^-)u'_P. \tag{2.21'}$$

We now proceed to calculate the diagonal element of the charge density operator ρ. To the second order in f, we have

$$\bar{\rho}=\Omega^*\rho\Omega=\Omega_0^*\rho\Omega_0+2\Omega_1^*\rho\Omega_0+2\Omega_2^*\rho\Omega_0+\Omega_1^*\rho\Omega_1, \tag{2.22}$$

where we have used the fact that the charge density operator is Hermitean. In the scalar theory it is given by

$$\rho=i\epsilon(\pi^*\psi^*-\pi\psi)=\frac{\epsilon}{2(2\pi)^3}\int\int d^3k\,d^3l\left\{\left(\frac{1}{k_0}+\frac{1}{l_0}\right)(a_k^*a_l-b_k^*b_l)\exp(-i(\mathfrak{k}-\mathfrak{l})\cdot\mathfrak{x})+\left(\frac{1}{l_0}-\frac{1}{k_0}\right)(a_kb_l\exp(i(\mathfrak{k}+\mathfrak{l})\cdot\mathfrak{x})-a_l^*b_k^*\exp(-i(\mathfrak{k}+\mathfrak{l})\cdot\mathfrak{x}))\right\}. \tag{2.23}$$

It is seen that the first two terms of (2.22) vanish, and we obtain, after averaging over the two possible spin directions of the nucleon

$$\Omega_1^*\rho\Omega_1=\frac{\epsilon f^2}{2(2\pi)^3}\int\int d^3k\,d^3l\left(\frac{1}{k_0}+\frac{1}{l_0}\right)\exp[i(\mathbf{l}-\mathbf{k})\cdot(\mathbf{x}-\mathbf{y})]\frac{(2M-k_0)(2M-l_0)+\mathbf{k}\cdot\mathbf{l}}{(2Mk_0-\mu^2)(2Ml_0-\mu^2)}, \tag{2.24}$$

$$\Omega_2{}^*\rho\Omega_0=\frac{\epsilon f^2}{4(2\pi)^3}\int\int d^3k\, d^3l\left(\frac{1}{k_0}-\frac{1}{l_0}\right)\exp\left[-i(\mathbf{k}+\mathbf{l})\cdot(\mathbf{x}-\mathbf{y})\right]$$
$$\times\frac{(2M-k_0-l_0)(2M-l_0)+(\mathbf{k}+\mathbf{l})\cdot\mathbf{l}}{(2Ml_0-\mu^2)\left[M(k_0+l_0)+\mathbf{k}\cdot\mathbf{l}-k_0l_0-\mu^2\right]}. \qquad (2.25)$$

We transform (2.25) by interchanging k and l and taking half the sum of the two integrals thus obtained. After this symmetrizing operation we drop all terms for which the final integral will contain μ/M to a power higher than the first. This is easily accomplished since M is not contained in k_0 and l_0; so it goes outside the various integrals immediately. We get, after some reduction,

$$\Omega_2{}^*\rho\Omega_0\doteq\frac{\epsilon f^2}{4(2\pi)^3}\int\int d^3k\, d^3l\left(\frac{1}{k_0}-\frac{1}{l_0}\right)(k_0-l_0)\exp\left[-i(\mathbf{k}+\mathbf{l})\cdot(\mathbf{x}-\mathbf{y})\right]$$
$$\cdot\frac{1-\dfrac{1}{2M}(k_0+l_0)-\dfrac{\mathbf{k}\cdot\mathbf{l}-k_0l_0-\mu^2}{M(k_0+l_0)}}{(k_0-\mu^2/2M)(l_0-\mu^2/2M)(k_0+l_0)}.$$

Using the fact that

$$\left(\frac{1}{k_0}-\frac{1}{l_0}\right)(k_0-l_0)=-\left(\frac{1}{k_0}+\frac{1}{l_0}\right)(k_0+l_0)+4$$

we can put the integral above into a form in which it can be compared with (2.24). It must be remembered that we have to take twice (2.25) and add it to (2.24). In going through the steps it will be seen that part of the integral just written down cancels (2.24) to terms of first order in μ/M. Expanding

$$(k_0-\mu^2/2M)^{-1}=\frac{1}{k_0}\left(1-\frac{\mu^2}{2Mk_0}+\cdots\right),$$

we finally obtain

$$\bar{\rho}=\frac{2\epsilon f^2}{(2\pi)^3}\int\int\frac{d^3k\, d^3l}{k_0l_0(k_0+l_0)}\cdot\exp\left[-i(\mathbf{k}+\mathbf{l})\cdot(\mathbf{x}-\mathbf{y})\right]$$
$$-\frac{\epsilon f^2}{(2\pi)^3}\left(\frac{1}{M}\right)\int\int\frac{d^3k\, d^3l}{k_0l_0}\left(\frac{\mu^2}{k_0l_0}+1\right)\exp\left[-i(\mathbf{k}+\mathbf{l})\cdot(\mathbf{x}-\mathbf{y})\right]$$
$$+\frac{\epsilon f^2}{2(2\pi)^3}\left(\frac{1}{M}\right)\int\int\frac{d^3k\, d^3l}{(k_0l_0)^2}\left(\frac{k_0-l_0}{k_0+l_0}\right)^2(\mathbf{k}\cdot\mathbf{l}-k_0l_0-\mu^2)\exp\left[-i(\mathbf{k}+\mathbf{l})\cdot(\mathbf{x}-\mathbf{y})\right]. \qquad (2.26)$$

In the limit $M\rightarrow\infty$ this becomes

$$\bar{\rho}'=\frac{2\epsilon f^2}{(2\pi)^3}\int\int\frac{d^3k\, d^3l}{k_0l_0(k_0+l_0)}\exp\left[-i(\mathbf{k}+\mathbf{l})\cdot(\mathbf{x}-\mathbf{y})\right]. \qquad (2.26')$$

To simplify this, we put

$$\frac{1}{k_0+l_0}=\int_0^\infty\exp\left[-(k_0+l_0)q\right]dq;\quad \mathbf{z}=\mathbf{x}-\mathbf{y};\quad r=|\mathbf{z}|,$$

and interchange orders of integration. We then obtain (cf. table of integrals at the end of the paper),

$$\bar{\rho}'=\frac{\epsilon g^2\mu^2}{4\pi}\int_0^\infty\frac{\left[H_1{}^{(1)}\left[i\mu(r^2+q^2)^{\frac{1}{2}}\right]\right]^2}{(r^2+q^2)}dq=\frac{\epsilon g^2\mu^2}{4\pi r}\int_0^{\pi/2}\left[H_1{}^{(1)}(i\mu r\sec\theta)\right]^2d\theta. \qquad (2.27)$$

The limiting values of this expression for large and small distances from the nucleon are

$$\mu r \gg 1, \quad \bar{\rho}' \doteq \frac{\epsilon g^2 \mu^3}{4\pi^{3/2}} \frac{e^{-2\mu r}}{(\mu r)^{5/2}}, \tag{2.28}$$

$$\mu r \ll 1, \quad \bar{\rho}' \doteq \frac{\epsilon g^2 \mu^3}{4\pi^2} \frac{1}{(\mu r)^3}. \tag{2.29}$$

The total charge in the meson field $\bar{e} = \int \bar{\rho} d^3x$ thus diverges logarithmically on the standard scalar theory, in zero order in μ/M. The recoil correction terms [last two integrals in (2.26)] have not been evaluated explicitly. The second term of (2.26) gives a negatively infinite total charge, but it is easy to see that this doesn't help if one only goes back to (2.24) and (2.25). We integrate over x space and interchange orders of integration. Remembering that

$$\frac{1}{(2\pi)^3} \int \exp(i\mathbf{k}\cdot\mathbf{x}) d^3x = \delta(\mathbf{k})$$

we see that (2.25) gives zero total charge and (2.24) gives

$$\bar{e} = \epsilon f^2 \int \frac{d^3k}{k_0} \frac{(2M-k_0)^2 + k^2}{(2Mk_0 - \mu^2)^2}. \tag{2.30}$$

This is a positive definite form and diverges for large values of k like

$$\frac{\epsilon f^2}{2M^2} \int \frac{d^3k}{k_0}. \tag{2.31}$$

This might seem to indicate zero total charge as $M \to \infty$. In that case, however, we have to use (2.26′) and obtain a divergence of the charge in the meson field of the form

$$\epsilon f^2 \int d^3k/(k_0)^3. \tag{2.31′}$$

CHARGE DENSITY CALCULATION WITH DIRAC'S NEW METHOD OF FIELD QUANTIZATION (SCALAR THEORY)

This method, proposed by Dirac[6] in 1942, has been extensively treated by Pauli.[7] We shall use the same notation as Pauli, except for the change in Fourier decomposition corresponding to (2.1). Dirac introduces two fields ψ and ϕ. Only ψ interacts with the heavy particles, ϕ is the redundant field. The fields are decomposed into contributions of positive and negative (charge) mesons through

$$\psi = (2)^{-\frac{1}{2}}(U_p + U_n^*), \quad \phi = (2)^{-\frac{1}{2}}(U_p - U_n^*) \tag{3.1}$$

and the Fourier decompositions of U_p and U_n are

$$\begin{aligned} U_p &= (2)^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}} \int d^3k (k_0)^{-1} (a_k \exp(i\mathfrak{k}\cdot\mathfrak{x}) + c_k \exp(-i\mathfrak{k}\cdot\mathfrak{x})), \\ U_n &= (2)^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}} \int d^3k (k_0)^{-1} (b_k \exp(i\mathfrak{k}\cdot\mathfrak{x}) + d_k \exp(-i\mathfrak{k}\cdot\mathfrak{x})). \end{aligned} \tag{3.2}$$

To connect with Pauli's notation, we remark that

$$a_k = (k_0)^{\frac{1}{2}} U_{p+}(k), \quad b_k = (k_0)^{\frac{1}{2}} U_{n+}(k), \quad c_k = (k_0)^{\frac{1}{2}} U_{p-}(k), \quad d_k = (k_0)^{\frac{1}{2}} U_{n-}(k). \tag{3.3}$$

Notice that U_p and U_n contain contributions from mesons of both positive and negative energies. Dirac's proposal is to quantize the fields containing exp $(i\mathfrak{k}\cdot\mathfrak{x})$ in the usual way, those containing exp $(-i\mathfrak{k}\cdot\mathfrak{x})$ in the improved manner. The improved manner consists in an indefinite metric in Hilbert space, which gives rise to negative energies for those eigenstates of the meson field. The commutation relations are

$$[a_k, a_l^*]=[b_k, b_l^*]=k_0\delta(\mathbf{k}-\mathbf{l}), \quad [c_k, c_l^*]=[d_k, d_l^*]=-k_0\delta(\mathbf{k}-\mathbf{l}). \tag{3.4}$$

These rules allow us to interpret the operators

$$N_k^+=(k_0)^{-1}a_k^*a_k; \quad N_k^-=(k_0)^{-1}b_k^*b_k; \quad \tilde{N}_k^+=-(k_0)^{-1}c_k^*c_k; \quad \tilde{N}_k^-=-(k_0)^{-1}d_k^*d_k \tag{3.5}$$

as the densities in k space of numbers of mesons of (+charge, +energy), (−charge, +energy), (+charge, −energy), (−charge, −energy), respectively. The tilde over the N denotes negative energy states.

We denote by $\omega(N_k^+N_k^-\tilde{N}_k^+\tilde{N}_k^-)$ a simultaneous eigenfunctional of the operators (3.5) which is also a solution of the unperturbed Schrödinger equation

$$\bar{H}_0\omega=id\omega/dx_0 \tag{3.6}$$

where $\bar{H}_0$, the free meson field Hamiltonian, is given by

$$\bar{H}_0=\int d^3k(a_k^*a_k+b_k^*b_k+c_k^*c_k+d_k^*d_k)=\int d^3k\ k_0(N_k^++N_k^--\tilde{N}_k^+-\tilde{N}_k^-). \tag{3.7}$$

We then obtain the rules (2.14) for a_k and b_k, while

$$c_k^*\omega(\tilde{N}_k^+)=i(k_0)^{\frac{1}{2}}(\tilde{N}_k^++1)^{\frac{1}{2}}\omega(\tilde{N}_k^++1),$$

$$c_k\omega(\tilde{N}_k^+)=i(k_0)^{\frac{1}{2}}(\tilde{N}_k^+)^{\frac{1}{2}}\omega(\tilde{N}_k^+-1), \tag{3.8}$$

d_k same as c_k except for negatively charged mesons.

The indefinite metric in Hilbert space shows up through the fact that c_k^* does not have a $-i$ where c_k has an i, but has a $+i$ also. Another effect of the indefinite metric manifests itself in the Hermitean conjugate of the functional $\omega(\tilde{N}_k^+)$: $\omega^*(\tilde{N}_k^+)=-[\omega(\tilde{N}_k^+)]^*$. This gives the property that an operator with only positive eigenvalues has negative expectation values in the states of negative energy.

We now proceed to calculate the charge density just as in the previous section. The Schrödinger equation which we intend to solve by the perturbation method is (2.4). The only change there is the substitution of (3.7) for (2.5). Notice that the interaction Hamiltonian (2.6) is *not* changed. Only the ψ field interacts with the nucleon, the ϕ field does not. The expansion in powers of the coupling constant g, (2.8), is kept; and we again obtain Eqs. (2.9), (2.10), (2.11). It will not be necessary this time to calculate Ω_2. The zero-order functional is given by (2.12) and (2.13), just as before. But in expanding the interaction Hamiltonian into Fourier components according to (3.1) and (3.2) it will be noticed that there are non-vanishing matrix elements not only for transitions to states with one positive charge, positive energy meson plus a neutron, but also for transitions to states with one positive charge, negative energy meson plus a neutron. We thus have to write, instead of (2.15)

$$\Omega_1=\int d^3k[\Psi_1(\mathfrak{y}, k)\omega(1_k^+)+\tilde{\Psi}_1(\mathfrak{y}, k)\omega(\tilde{1}_k^+)]u_N. \tag{3.9}$$

An entirely similar argument to the one used in the previous section leads to the result

$$\Omega_1=(2)^{-\frac{1}{2}}f\int\frac{d^3k}{(k_0)^{\frac{1}{2}}}\exp(-iMy_0)u_N\left[\frac{\exp(-i\mathfrak{k}\cdot\mathfrak{y})\omega(1_k{}^+)}{(\mu^2-2Mk_0)}\cdot\begin{pmatrix}(2M-k_0)u_\pm\\ -\mathbf{k}\cdot\boldsymbol{\sigma}u_\pm\end{pmatrix}\right.$$

$$\left.+i\frac{\exp(i\mathfrak{k}\cdot\mathfrak{y})\omega(\tilde{1}_k{}^+)}{(\mu^2+2Mk_0)}\cdot\begin{pmatrix}(2M+k_0)u_\pm\\ \mathbf{k}\cdot\boldsymbol{\sigma}u_\pm\end{pmatrix}\right]. \tag{3.10}$$

The charge density operator in the Dirac theory is given by

$$\rho=\rho^++\rho^-,$$

$$\rho^+=\frac{\epsilon}{2(2\pi)^3}\int\int d^3k\,d^3l\left[\left(\frac{1}{k_0}+\frac{1}{l_0}\right)(a_k{}^*a_l\exp[-i(\mathfrak{k}-\mathfrak{l})\cdot\mathfrak{x}]-c_k{}^*c_l\exp[i(\mathfrak{k}-\mathfrak{l})\cdot\mathfrak{x}])\right.$$

$$\left.+\left(\frac{1}{l_0}-\frac{1}{k_0}\right)(a_k{}^*c_l\exp[-i(\mathfrak{k}+\mathfrak{l})\cdot\mathfrak{x}]-c_k{}^*a_l\exp[i(\mathfrak{k}+\mathfrak{l})\cdot\mathfrak{x}])\right] \tag{3.11},$$

ρ^- same as ρ^+ but with a and c replaced by b and d.

Notice that there are no pair creation or destruction terms in this operator. The "Zitterbewegung" terms correspond to transitions from positive to negative energies and vice versa. This simplifies the calculations considerably, since $\rho\omega(0)=0$ and therefore all terms of type $\Omega_n{}^*\rho\Omega_0$ vanish identically. This is the reason why we did not have to calculate Ω_2. We get

$$\bar{\rho}=\Omega_1{}^*\rho^+\Omega_1=\frac{\epsilon f^2}{4(2\pi)^3}\int\int d^3k\,d^3l\left(\frac{1}{k_0}+\frac{1}{l_0}\right)\frac{[(2M-k_0)(2M-l_0)+\mathbf{k}\cdot\mathbf{l}]\exp[-i(\mathbf{k}-\mathbf{l})\cdot(\mathbf{x}-\mathbf{y})]}{(\mu^2-2Mk_0)(\mu^2-2Ml_0)}$$

$$+(\mathfrak{k},\mathfrak{l}\rightarrow-\mathfrak{k},\mathfrak{l})+(\mathfrak{k},\mathfrak{l}\rightarrow\mathfrak{k},-\mathfrak{l})+(\mathfrak{k},\mathfrak{l}\rightarrow-\mathfrak{k},-\mathfrak{l}). \tag{3.12}$$

The integral splits into simple products of single integrals, which can be calculated to the first order in μ/M without any difficulty. We then get (see table of integrals)

$$\bar{\rho}=\frac{\epsilon g^2}{8\pi}\left(\frac{\mu}{M}\right)\frac{\mu}{r^2}e^{-2\mu r}-\frac{\epsilon g^2}{2M}\frac{e^{-\mu r}}{r}\delta(r).$$

The δ-function gives a singularity at the origin. Expanding the exponential in powers of r and noting that $r^n\delta(r)=0$ for $n\geqslant 1$, we get

$$\bar{\rho}=\frac{\epsilon g^2}{8\pi}\left(\frac{\mu}{M}\right)\frac{\mu}{r^2}e^{-2\mu r}+\frac{\epsilon g^2}{2}\frac{\mu}{M}\delta(r)-\frac{\epsilon g^2}{2M}\frac{\delta(r)}{r}. \tag{3.13}$$

The last part of this expression is not acceptable since it makes the total charge negatively infinite. It must be remembered that as yet we have not used the λ-limiting process. This will be done in the next section. So far, we can say, however, that the negative energy mesons alone do not suffice to make the theory convergent. This point is well known, of course, and (3.13) just provides an additional confirmation. Another thing worth noting is that the charge density (and therefore also the total charge) in the meson field is exactly zero in the limit as the mass M of the nucleon approaches infinity. That is, we get a change from the Coulomb law only if we consider the recoil of the nucleon upon emitting a meson.

CALCULATION OF THE TOTAL CHARGE IN THE MESON FIELD USING THE CONVERGENT (SCALAR) THEORY

In order to make the theory convergent, we still need the λ-limiting process. This is of an essentially classical nature and designed to overcome the difficulties which arise from the classical treatment of a point charge. It was first introduced by Wentzel and its application to quantum electrodynamics was simplified by Dirac.[14] Its physical basis is the following. We introduce a term into the interaction which is approximately unity at large space-time distances from the nucleon. This term becomes rapidly oscillating upon close approach to the heavy particle. In momentum space, the term becomes rapidly oscillating for large values of the wave number k. It is actually of the type $\cos(k_0\lambda_0)$ with λ_0 a real constant of the dimension of a length. To make the limiting process work, the position in space-time of the nucleon has to be approached from points within the light cone; this makes λ_0 essentially a time interval, but in our choice of units this is still a length dimensionally. λ_0 indicates in a rough manner the extent of the region around the nucleon about which we have little information.

We expect the oscillating term to cancel effectively the contribution of mesons of very high energies (small wave-lengths) to the interaction. It is from those mesons that the divergencies arise. We then go to the limit as λ_0 approaches 0. It would thus seem reasonable to expect to cancel the contribution of terms at the origin (i.e., δ-function terms) to the charge density.

The limiting process used here is regular. This means that any integral which already converges without its help is evaluated to its proper value. There is thus no point in recalculating the charge density. We do not expect a change in the first term of (3.13). Integrating over all x space gives $\frac{1}{4}\epsilon g^2\mu/M$ for the contribution of this term to the total charge in the meson field. If the λ process really cuts out the contributions of the terms at the origin to the charge density, then this is the value we would expect for the total charge in the meson field on the convergent theory.

[14] P. A. M. Dirac, Ann. de l'Inst. Poincaré 9, 13 (1939).

We now proceed to calculate this total charge directly. Since the λ process has been treated in detail by Pauli,[7] we shall only quote the necessary formulae here. The calculation of the wave functional by the perturbation method is the same as before. The only change is in the interaction Hamiltonian $\bar{H}_i$. After expanding it into Fourier components we replace a_k, a_k^*, etc. by $a_kL_k^-$, $a_k^*L_k^+$, etc., where

$$\begin{aligned} L_k^- &= \cos\left(\tfrac{1}{2}k_0\lambda_0\right)+\sin\left(\tfrac{1}{2}k_0\lambda_0\right), \\ L_k^+ &= \cos\left(\tfrac{1}{2}k_0\lambda_0\right)-\sin\left(\tfrac{1}{2}k_0\lambda_0\right). \end{aligned} \tag{4.1}$$

It must be emphasized that this change occurs only in the interaction term. The quantities which refer to the meson field alone remain unchanged. The same argument as in the previous section leads to an expression for Ω_1 which is identical with (3.10) except for a factor L_k^+ under the integral sign.

In calculating the total charge it is important to realize that Ω_1^* is not the ordinary Hermitean conjugate of Ω_1. First: $[\omega(\tilde{1}_k^+)]^* = -\omega^*(\tilde{1}_k^+)$. This comes from the indefinite metric in Hilbert space. Secondly: $(L_k^+)^* = L_k^-$. This is the generalization of the notion of Hermitean conjugate introduced by the λ process.

The total charge operator is

$$e=\epsilon\int d^3k(k_0)^{-1}(a_k^*a_k-b_k^*b_k-c_k^*c_k+d_k^*d_k). \tag{4.2}$$

Using this operator and simplifying we get for the expectation value of e

$$\bar{e}=-2\pi(M)^{-3}(2M^2-\mu^2)\epsilon f^2 \times\int_0^\infty \frac{k^4\cos(k_0\lambda_0)dk}{(k_0{}^2-\mu^4/(4M^2))^2}. \tag{4.3}$$

This is a well-known integral. We can split it into two parts as follows:

$$\int_0^\infty \cos(k_0\lambda_0)dk - \int_0^\infty \frac{(k_0{}^2-\mu^4/(4M^2))^2-k^4}{(k_0{}^2-\mu^4/(4M^2))^2}\cos(k_0\lambda_0)dk.$$

The first integral has been evaluated by Jauch[2] and gives zero in the limit as λ_0 approaches zero. The second integral converges even without the limiting process, and since the process is regular we can evaluate it directly, leaving out the factor cos $(k_0\lambda_0)$. The value of this integral is $\frac{3}{4}\pi\mu(1-(\mu/2M)^2)^{\frac{1}{2}}$. Calculating the total charge in the meson field to the first order in μ/M we get

$$\bar{e}=\tfrac{3}{4}\epsilon g^2\mu/M. \tag{4.4}$$

This is more than we expected from the first term of (3.13) only. It is easily seen that the other part comes from the second term of (3.13). Thus the limiting process did *not* cut out all the contributions from the origin, but only the divergent contribution [the third term of (3.13)].

It may be worth while to remark that it would not have been correct to keep terms of higher order in μ/M in (4.4). We used the ordinary Dirac Hamiltonian for the nucleon together with standard first quantization procedure. Thus we neglected pair creation terms in the nucleons (in the sense of Dirac's theory of holes). These terms first appear in the third order in μ/M, and that would be the order of the next term in (4.4).

DISCUSSION OF THE POSSIBILITY OF EXPERIMENTAL VERIFICATION

In the introduction we have mentioned two effects of the meson charge cloud around the proton which might provide experimental checks for the theory. One is the change in the energy levels of the s states of the hydrogen atom, with a resultant change in the fine structure of the hydrogen lines. There have been reports[15] of slight deviations from the theoretical fine structure calculated from Sommerfeld's formula; and it was mainly to see whether those deviations could be explained by the meson charge cloud around the proton that Fröhlich, Heitler, and and Kahn[3] undertook their investigation. They point out that a complete reversal of the Coulomb force at close distances from the nucleon is needed to account for the experimental data. If we take (3.13) without the last term, which is taken out by the λ process, we see that there is only a spreading out of the charge of the proton within the range of the nuclear forces $(1/\mu)$, but not a change of sign of the charge anywhere in space.

Furthermore, the effect is much too small to be observed experimentally. The influence of a potential step upon the energy levels of the s states of the H atom has been calculated by Jauch.[16] If we use the equation $\nabla^2 U=-4\pi\epsilon\rho$, where U is the potential energy of a test particle of charge ϵ, we can obtain U by simple integrations from (3.13). We then get

$$U(r)=\frac{\epsilon^2}{r}-\frac{\epsilon^2 g^2\mu}{2}\frac{\mu}{M}[Ei(-2\mu r)+(2\mu r)^{-1}e^{-2\mu r}] \tag{5.1}$$

where the exponential integral is defined by

$$Ei(-x)=-\int_x^\infty t^{-1}e^{-t}dt.$$

The potential energy used in Jauch's paper is of the Coulomb type in the outer region, and a constant in the inner region. Defining an equivalent inside energy U_0 by

$$\int_0^{1/\mu} U(r)4\pi r^2 dr=4\pi U_0/3\mu^3$$

and comparing its value with the values necessary to give an appreciable shift in the energy levels (using the curves plotted in Jauch's paper) we see that the effect is several orders of magnitude to small for experimental determination.

In the introduction we also mentioned the possibility of anomalous scattering of charged particles of high energies on protons, caused by that meson charge cloud. But the charge cloud will not become noticeable until the incident particles have energies of the order of several hundred times the rest energy of an electron. In other words, the de Broglie wave-length of the incident particles must be comparable to the dimensions of the meson charge cloud before we can expect any anomalous scattering. But for incident particles of such tremendous energies our treatment is most certainly inapplicable. We may expect cumulative processes (showers) which we have not taken into account at all.

We thus conclude that the effect of the meson charge cloud is entirely too small to be observed experimentally.

SUMMARY

The results of Fröhlich, Heitler, and Kahn[3] are not reproduced by the convergent theory.

[15] R. C. Williams, Phys. Rev. **54**, 558 (1938); Pasternack, Phys. Rev. **54**, 1113 (1938).

[16] J. M. Jauch, Helv. Phys. Acta **13**, 451 (1940).

This fact strikingly illustrates the danger of working with divergent theories near the limits of their validity. The convergent theory leads to an effect which differs completely from the earlier result, both qualitatively and quantitatively.

The author wishes to express his thanks to Dr. J. M. Jauch for the suggestion of the problem and for many helpful discussions.

APPENDIX: TABLE OF INTEGRALS

$$\int_{-\infty}^{\infty}\frac{e^{ikr}dk}{(k_0)^{2n}}=\frac{\pi^{\frac{1}{2}}i^{n+\frac{1}{2}}r^{n-\frac{1}{2}}}{(n-1)!(2\mu)^{n-\frac{1}{2}}}H^{(1)}_{n-\frac{1}{2}}(i\mu r),$$

$k_0=(k^2+\mu^2)^{\frac{1}{2}}$, n half integer, $H_n^{(1)}(z)$ . . . Hankel function of order n and first kind.

$$\int(k_0)^{-1}\exp(i\mathbf{k}\cdot\mathbf{x})d^3k=-\frac{2\pi^2\mu}{r}H_1^{(1)}(i\mu r),$$

$$\int(k_0)^{-1}\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{x})d^3k=\frac{2\pi^2\mu^2}{r}H_2^{(1)}(i\mu r)\mathbf{e}_x,$$

$$\int(k_0)^{-2}\exp(i\mathbf{k}\cdot\mathbf{x})d^3k=\frac{2\pi^2}{r}e^{-\mu r},$$

$$\int(k_0)^{-2}\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{x})d^3k=\frac{2\pi^2}{i}\frac{d}{dr}\left(\frac{e^{-\mu r}}{r}\right)\mathbf{e}_x,$$

$$\int(k_0)^{-3}\exp(i\mathbf{k}\cdot\mathbf{x})d^3k=2\pi^2 iH_0^{(1)}(i\mu r),$$

$\mathbf{e}_x$ . . . unit vector in $\mathbf{x}$ direction, $r=|\mathbf{x}|$,

$$\int\exp(-k_0q)\exp(i\mathbf{k}\cdot\mathbf{x})d^3k=2\pi^2 i\mu^2 q(r^2+q^2)^{-1}H_2^{(1)}[i\mu(r^2+q^2)^{\frac{1}{2}}],$$

$$\int(k_0)^{-1}\exp(-k_0q)\exp(i\mathbf{k}\cdot\mathbf{x})d^3k=2\pi^2\mu(r^2+q^2)^{-\frac{1}{2}}H_1^{(1)}[i\mu(r^2+q^2)^{\frac{1}{2}}].$$

The following integrals are correct to the first order in μ/M only:

$$\pm\int\frac{(2M\mp k_0)\exp(\pm i\mathbf{k}\cdot\mathbf{x})}{k_0(\mu^2\mp 2Mk_0)}d^3k=-\frac{2\pi^2}{r}e^{-\mu r}\mp\frac{\pi^2\mu}{M}\left[i\mu H_0^{(1)}(i\mu r)+\frac{1}{r}H_1^{(1)}(i\mu r)\right],$$

$$\int\frac{(2M\mp k_0)\exp(\pm i\mathbf{k}\cdot\mathbf{x})}{(\mu^2\mp 2Mk_0)}d^3k=\pm\frac{2\pi^2\mu}{r}H_1^{(1)}(i\mu r)+\frac{4\pi^3}{M}\delta(r)-\frac{\pi^2\mu^2}{Mr}e^{-\mu r},$$

$$\int\frac{\mathbf{k}\exp(\pm i\mathbf{k}\cdot\mathbf{x})}{k_0(\mu^2\mp 2Mk_0)}d^3k=\mp i\pi^2\frac{1+(\mu r)}{Mr^2}e^{-\mu r}\mathbf{e}_x,$$

$$\pm\int\frac{\mathbf{k}\exp(\pm i\mathbf{k}\cdot\mathbf{x})}{(\mu^2\mp 2Mk_0)}d^3k=-\frac{\pi^2\mu^2}{Mr}H_2^{(1)}(i\mu r)\mathbf{e}_x.$$

Limiting behavior of Hankel functions:

	$x\ll 1$	$x\gg 1$
$H_0^{(1)}(ix)$	$-\frac{2i}{\pi}\log\left(\frac{2}{\gamma x}\right)$	$-i\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{e^{-x}}{x^{\frac{1}{2}}}$
$H_1^{(1)}(ix)$	$-\frac{2}{\pi x}$	$-\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{e^{-x}}{x^{\frac{1}{2}}}$
$H_2^{(1)}(ix)$	$\frac{4i}{\pi x^2}$	$i\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{e^{-x}}{x^{\frac{1}{2}}}$.