On the Interaction of Radiation and Two Electrons* †

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The quantum theory of radiation describes the effect of retardation in the interaction of two electrons by the mutual emission and absorption of light quanta. A study has been made of a process involving the interaction of two electrons and any number of quanta. It was noted that the number of terms involving virtual quanta needed to describe the retardation in the usual first-order approximation of perturbation theory depends on the number of real quanta involved in the process under consideration. It was shown that, nevertheless, the resulting expression obtained by summing over all the intermediate states involving virtual quanta is of the same form for any process. Møller's formulas for the retarded interaction of two electrons with no light quanta and with one light quantum emitted are special cases of our equation.

I. INTRODUCTION

It is well known that the quantum mechanical Hamiltonian of a system of charged particles obeying Dirac's equation and the radiation field can be written[1]

$$H = H_p + H_r + H_s + H', \tag{1a}$$

where H_p represents the Hamiltonian of the particles, H_r that of the light quanta, H_s the electrostatic energy of the particles, and H' the interaction of the particles and the field. We have

$$H_s = (1/2) \sum e_i e_k / r_{ik} \tag{1b}$$

and

$$H' = -\sum e_k (\boldsymbol{\alpha}_k \cdot \mathbf{A}(k)), \tag{1c}$$

where e_k is the charge and $\boldsymbol{\alpha}_k$ the Dirac matrix for the kth particle, $\mathbf{A}(k)$ the vector potential evaluated at the position of e_k and r_{ik} the distance between the ith and kth particle.

Equation (1a) contains an explicit expression for the instantaneous Coulomb interaction only, as given by (1b). The effect of retardation is included in H' and is described by a mutual emission and absorption of light quanta. In the simplest possible case of retarded interaction of two particles in the absence of radiation, it consists of the emission of just one quantum by one particle and reabsorption by the other and vice versa.[2] The purpose of this paper is to show that while in the more general case of the interaction of any number of light quanta with two particles the retardation corresponds to emissions and absorptions described by a large number of terms involving virtual quanta, we can sum all these terms and obtain an expression for the retarded interaction which is of the same form for any process.

II. THE MATRIX ELEMENT

As it is not possible with the present methods to solve the Schrödinger equation $H\psi = -i\hbar(\partial\psi/\partial t)$ with the Hamiltonian (1a) directly, a perturbation theory is required. The method often used treats the terms (1b) and (1c) for the interaction as small perturbations causing transitions from a state E_0 with a certain energy and momentum of the particles and a certain number of quanta to another state E_F. The transition probability is zero unless the total energy is conserved, and then it is given by the well-known equation[3]

$$w_{E_0E_F} = (2\pi/\hbar)\rho_{E_0} |H'_{E_0E_F}|^2, \tag{2a}$$

where ρ_{E_0} is the density of the final or initial states with energy E_0, and

$$H'_{E_0E_F} = \int \psi_0^* (H_s + H')\psi_F d\tau \tag{2b}$$

is the matrix element for the transition. The

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[1] For the general theory and the notation used see W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, 1936), Chapter III.

[2] This case was first calculated with a correspondence method by C. Møller, Zeits. f. Physik **70**, 686 (1931) and Ann. d. Physik **14**, 531 (1932). The present method was first used for this problem by H. Bethe and E. Fermi, Zeits. f. Physik **77**, 296 (1932).

[3] W. Heitler, reference 1, p. 90.

matrix element has a value different from zero for the Coulomb interaction of two particles, in which case it is equal to

$$(4\pi\hbar^2c^2e_1e_2/|\mathbf{\Delta p}|^2)(\overset{*}{u}_{01}u_1)(\overset{*}{u}_{02}u_2), \tag{3a}$$

for two *free* particles, and for transitions involving emission or absorption of just one light quantum, where it is given by

$$-e(2\pi\hbar^2c^2/k_\lambda)^{\frac{1}{2}}(u_a{}^*\alpha_e u_b). \tag{3b}$$

Here the u's are the Dirac amplitudes of the electrons and α_e is the component of $\boldsymbol{\alpha}$ in the direction of polarization of the light quantum k_λ. The momentum is conserved for these transitions.

If the matrix element for the direct transition is zero, we have to use a succession of simple transitions leading from the initial to the final state through a number of intermediate states, for which we still have conservation of momentum, but no longer conservation of energy. We have to replace (2b) by

$$H'=\sum\frac{H'_{0,1}H'_{1,2}\cdots H'_{F-2,F-1}H'_{F-1,F}}{(E_0-E_1)(E_0-E_2)\cdots\times(E_0-E_{F-2})(E_0-E_{F-1})}, \tag{4}$$

where the denominator contains the product of the differences of the energies of the initial and intermediate states, and the numerator the products of the matrix elements for the successive simple transitions. These transitions have to be chosen in such a way that the total expression (4) is of the lowest possible order in the electronic charge e for which it does not vanish,[4] but otherwise we have to sum over all possible successions of intermediate states. We can see from (3a) and (3b) that the matrix element for interaction with a light quantum is proportional to e and the Coulomb interaction is proportional to e^2 and therefore equivalent to two interactions with light.

After these general considerations let us now consider a process involving $z+Z$ light quanta and two particles. Then our energy balance for the initial and final states is

$$E_0=E_1{}^0+E_2{}^0+a+b+\cdots+z+A+B+\cdots+Z$$
$$=E_1{}^F+E_2{}^F=E_F, \tag{5}$$

[4] It is a well-known difficulty of the quantum theory of radiation that any calculation of a process in a higher than the lowest possible order leads to divergencies.

where $E_1{}^0$, $E_1{}^F$ and $E_2{}^0$, $E_2{}^F$ are the energies of the first and second particle in the initial and final state, respectively, and $a\cdots z$ and $A\cdots Z$ are the energies of the light quanta. We have written the equation in a form corresponding to absorptions only, but our calculations include emissions as well, as the only difference would be a change in sign throughout for the energy and momentum of the particular light quantum. Furthermore, we can include positrons as well as electrons by changing the sign of the energy and momentum of the particle.

The conservation of momentum requires

$$\mathbf{p}_1{}^0+\mathbf{p}_2{}^0+\mathbf{a}+\mathbf{b}+\cdots$$
$$+\mathbf{z}+\mathbf{A}+\mathbf{B}+\cdots+\mathbf{Z}=\mathbf{p}_1{}^F+\mathbf{p}_2{}^F. \tag{6}$$

The first non-vanishing matrix element for such a process is of the order $z+Z+2$ in e. It involves $z+Z$ transitions due to the interaction of the particles with all the light quanta and one transition due to the instantaneous Coulomb interaction of the two particles. We obtain a matrix element of the same order if we have instead of the instantaneous Coulomb interaction a retardation described by one transition corresponding to the emission of a light quantum K' by one particle and a second transition due to the absorption of this quantum by the second particle. Any succession of transitions involving the same light quantum more than once or more than one Coulomb interaction would either be of a higher order than those just considered or would not fulfill the conditions (5) and (6).

We note that we cannot expect (4) to give finite results for any values of the energies and momenta of the particles and quanta involved subject to the conditions (5) and (6) only. We assume for the moment that we have to deal with non-vanishing denominators of (4) only and shall discuss the limitations of the first-order approximation later.

Let us now consider the possible successions of interactions in detail. We can have one particle interacting with any n quanta, and the other one with the remaining $z+Z-n$ quanta. Furthermore, the interactions of the n quanta with the first particle can take place in any order (and likewise for the $z+Z-n$ quanta and the second particle). In addition the Coulomb interaction can take place after the interaction with any $n+M$ quanta. Then one has to sum over both signs of spin and energy of the electron in the intermediate states. Finally the exclusion principle has to be taken into account by making the total matrix element antisymmetric in the coordinates of the two particles.[5]

We do not attempt to sum over all these possibilities since we want only to study the form of the retardation for a given Coulomb interaction. From now on we limit our calculations to that part of the sum, which has the same numerator in (4), namely,

$$(-e)^{z+Z+2}[(2\pi\hbar^2c^2)^{z+Z+2}/(a\cdots zA\cdots Z)]^{\frac{1}{2}} \times\prod_{1,1}^{n,M}\prod_{n+2,\,M+2}^{z,Z}(u_1{}^{*n-1}\alpha_n u_1{}^n)(u_2{}^{*M-1}\alpha_M u_2{}^M) \times\begin{cases} 2(u_1{}^{*n}u_1{}^{n+1})(u_2{}^{*M}u_2{}^{M+1})/|\Delta\mathbf{p}|^2 \text{ for the instantaneous Coulomb interaction} \\ \sum\limits_{\substack{\text{directions of}\\ \text{polarization}}}[(u_1{}^{*n}\alpha_{K'}u_1{}^{n+1})\times(u_2{}^{*M}\alpha_{K'}u_2{}^{M+1})/K'] \text{ for the retardation,}\end{cases} \tag{7}$$

with $K'=|\Delta\mathbf{p}|$, the absolute value of the change in momentum of either particle due to the Coulomb interaction.

For the moment we consider the two parts for instantaneous and retarded interaction separately. The part containing the instantaneous Coulomb interaction corresponds to the following process:

$$\left.\begin{array}{l}\text{Electron } e_1 \text{ emits (or absorbs)}\\ \quad a;\,b;\,\cdots\,n\\ \text{Electron } e_2 \text{ emits (or absorbs)}\\ \quad A;\,B;\,\cdots\,M\end{array}\right\}\text{Instantaneous}$$

$$\text{Coulomb interaction}\begin{cases} e_1 \text{ emits (or absorbs)}\\ \quad n+1;\,\cdots\,z\\ e_2 \text{ emits (or absorbs)}\\ \quad M+1;\,\cdots\,Z.\end{cases}$$

We denote the light quanta (and their energies) by small letters if they interact with the first electron and by capital letters for the second electron.

We have already arbitrarily ordered the quanta which interact with particle 1, and separately the quanta which interact with particle 2. We still have to sum over all possible orders of interaction subject to these limitations. For example, we can have one process where e_1 interacts with $a, b, \cdots n$ and then e_2 with $A, B, \cdots M$ but also another one where e_1 interacts with a, then e_2 with A, then again e_1 with $b, c, \cdots$ and similar possibilities. The only restriction (beyond the arbitrary partial ordering of quanta mentioned above) is that the order of interaction for each particle has to be the same and that the Coulomb interaction can take place only after e_1 has interacted with the light quantum n and e_2 with M, but before their interactions with $n+1$ and $M+1$, respectively.

We note that the part of our sum which contains the instantaneous Coulomb interaction can be factored into a product of the Coulomb term and of two series, the one containing all the successions of transitions taking place before the Coulomb interaction and the other containing the successions of transitions taking place thereafter.

Now we have to consider the case where the Coulomb interaction is replaced by the emission and subsequent absorption of a virtual quantum K', which transfers the same amount of momentum. It is essential to keep in mind that *this need not take place in two consecutive steps.*[6] *The only limitation is that e_1 can only emit K' after its interaction with n and before its interaction with $n+1$, and that e_2 can only absorb K' after its interaction with M and before its interaction with $M+1$ and vice versa.* The extreme case would be the succession of interactions where one particle has emitted K' and interacted with all z quanta and is therefore in its final state, while the second particle is still in its initial state.

We note that we can no longer factor the sum describing the retardation as in the case of the instantaneous Coulomb interaction. Furthermore, we see that because of the fact that the emission and absorption of K' need not take

[5] For this last point compare, e.g., N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, 1933), Chapter XV.

[6] This point has not been emphasized before. The expression given by W. Heitler, reference 1, p. 101, Eq. (31), takes account of consecutive steps only. It is therefore *not* the general matrix element for the retarded interaction of two particles. The same criticism applies to the paper by W. Heitler and L. Nordheim, J. de phys. et rad. **5**, 449 (1934), which gives the same expression. The actual calculations of this paper neglect the retardation, however, so that no error is introduced.

place in two consecutive steps, the sum describing the retardation will contain more terms than the sum containing the instantaneous interaction. The number of terms needed to describe the retardation depends on the number of quanta involved in the entire process. This appears strange, at least formally, as there seems to be no reason why the expression for the interaction of one particle with radiation and for the instantaneous interaction of two particles should be of the same form independent of the process, but not the one for the retarded interaction. But we have to keep in mind that we are still describing the retarded interaction by virtual quanta, and we shall see that the expression obtained after eliminating them is again independent of the process under consideration as it should be.

We first simplify the notation for the energy differences appearing in the denominator of (4). If the first transition were due to the absorption of light quantum a by e_1, then this particle would have the energy $E_1{}^a=(|\mathbf{p}_1{}^0+\mathbf{a}|^2+\mu^2)^{\frac{1}{2}}$ and the difference between the energy of the initial and the first intermediate state would be

$$a+E_1{}^0-E_1{}^a$$

as the second particle is unchanged. We denote this difference by $\langle E_1{}^a\rangle$. Similarly we put

$$a+b+\cdots+m+E_1{}^0-E_1{}^m=\langle E_1{}^m\rangle \tag{8a}$$

with

$$E_1{}^m=(|\mathbf{p}_1{}^0+\mathbf{a}+\mathbf{b}+\cdots+\mathbf{m}|^2+\mu^2)^{\frac{1}{2}},$$

$$A+B+\cdots+L+E_2{}^0-E_2{}^L=\langle E_2{}^L\rangle \tag{8b}$$

with

$$E_2{}^L=(|\mathbf{p}_2{}^0+\mathbf{A}+\mathbf{B}+\cdots+\mathbf{L}|^2+\mu^2)^{\frac{1}{2}}.$$

If now we have an intermediate state reached by the interaction of e_1 with m quanta and of e_2 with L quanta, then the energy difference is

$$a+b+\cdots+m+A+B+\cdots +L+E_1{}^0+E_2{}^0-E_1{}^m-E_2{}^L=\langle E_1{}^m\rangle+\langle E_2{}^L\rangle. \tag{8c}$$

III. A THEOREM

Let

$$A_0=0,\ A_1,\ A_2,\ \cdots A_n$$
$$B_0=0,\ B_1,\ B_2,\ \cdots B_m$$

be two sets of numbers, subject only to the restriction $A_i\neq -B_k$ for any $i,\ k\neq 0$.

Let (A,B) stand for any product of $n+m$ factors of the form A_i+B_k, where we obtain each factor from the preceding one by replacing *either* A or B by the following number of its set.

Theorem: The sum of the reciprocals of all possible (A,B) obtained by starting with either A_0+B_1 or B_0+A_1 is

$$S_{n,m}=\sum 1/(A,B)=1/A_1A_2\cdots A_nB_1B_2\cdots B_m.$$

We shall prove the theorem by induction. It is certainly true for $n=1$ and $m=1$ since

$$\frac{1}{A_1(A_1+B_1)}+\frac{1}{B_1(A_1+B_1)}=\frac{1}{A_1B_1}.$$

As the sum does not depend on the initial order of the numbers in the two sets, it is sufficient to show that if we add another number A_{n+1} at any place to the set $A_1\cdots A_n$ and construct $S_{n+1,m}$, it will be equal to $S_{n,m}/A_{n+1}$. The proof can be demonstrated most easily if we insert A_{n+1} between A_0 and A_1. We shall use the symbol $(A_i+B_k)\rightarrow(A_{i+1}+B_k)$ to indicate a step which changes an A to the value indicated, and likewise for a change of the B.

We now proceed to construct $S_{n+1,m}$. We start with the factor (B_0+A_{n+1}). Summing all the $1/(A,B)$ which can be obtained by $(B_0+A_{n+1})\rightarrow(B_0+A_1)$ and then proceeding according to our rule, we simply obtain $\overset{A_1}{S}_{n,m}/A_{n+1}$ where $\overset{A_1}{S}_{n,m}$ indicates those terms of $S_{n,m}$ which start with (B_0+A_1). For $(B_1+A_0)\rightarrow(B_1+A_{n+1})\rightarrow(B_1+A_1)$ we obtain $\overset{B_1}{S}_{n,m}/(B_1+A_{n+1})$ and for $(B_0+A_{n+1})\rightarrow(B_1+A_{n+1})\rightarrow(B_1+A_1)$ we obtain $B_1\overset{B_1}{S}_{n,m}/[A_{n+1}(B_1+A_{n+1})]$ where $\overset{B_x}{S}_{n,m}$ indicates the sum of those terms of $S_{n,m}$ which start at (B_1+A_0) and continue with A unchanged up to (A_0+B_x) and then change to (A_1+B_x).

Proceeding in this fashion and adding the terms with the same $\overset{B_x}{S}_{n,m}$ as factor, we obtain

$$\overset{A_1}{S}_{n,m}/A_{n+1}$$

$$\left(\frac{1}{B_1+A_{n+1}}+\frac{B_1}{A_{n+1}(B_1+A_{n+1})}\right)\overset{B_1}{S}_{n,m}=\overset{B_1}{S}_{n,m}/A_{n+1}$$

$$\left(\frac{1}{B_2+A_{n+1}}+\frac{B_2}{(B_1+A_{n+1})(B_2+A_{n+1})}+\frac{B_1B_2}{A_{n+1}(B_1+A_{n+1})(B_2+A_{n+1})}\right)\overset{B_2}{S}_{n,m}=\overset{B_2}{S}_{n,m}/A_{n+1},$$

and the general term is from the method of construction

$$B_1B_2B_3B_4\cdots B_xS_{1,x}\overset{B_x}{S_{n,m}}=\frac{B_1B_2\cdots B_x}{A_{n+1}B_1B_2\cdots B_x}\overset{B_x}{S_{n,m}}=\overset{B_x}{S_{n,m}}/A_{n+1},$$

where $S_{1,x}$ indicates the sum of all the (A, B) formed from $A_0=0$, A_{n+1} and $B_0=0$, B_1, $B_2\cdots B_x$ and for which the theorem holds true by assumption since $x\leq m$ and $1\leq n$.

Adding the right-hand side we obtain

$$S_{n+1,m}=(\overset{A_1}{S_{n,m}}+\overset{B_1}{S_{n,m}}+\overset{B_2}{S_{n,m}}+\cdots+\overset{B_x}{S_{n,m}}+\cdots+\overset{B_m}{S_{n,m}})/A_{n+1}=S_{n,m}/A_{n+1}$$

which is what was to be proved.

IV. SUMMATION OF THE SERIES

We first sum the series defined in Part II containing the instantaneous Coulomb interaction. We remember that we could factor this series and we have already defined our $\langle E_1{}^m\rangle$ and $\langle E_2{}^L\rangle$ by the relations (8) in such a manner that we can immediately apply our theorem to the first factor containing the successions of interactions taking place before the instantaneous Coulomb interaction, i.e., up to $\langle E_1{}^n\rangle$ and $\langle E_2{}^M\rangle$.

Before we can sum the part containing the interactions taking place after the instantaneous Coulomb interaction, we have to simplify the notation for this part also. But we cannot use the same definitions as before since now the momentum of the particles has been changed not only by the interaction with the light quanta, but by their Coulomb interaction as well. The change in momentum of the first particle is equal and opposite to the change in momentum of the second one, and is given from (6) by

$$\begin{aligned}\Delta\mathbf{p}=\mathbf{p}_1{}^{n+1}-\mathbf{p}_1{}^n&=\mathbf{p}_1{}^F-[\mathbf{p}_1{}^0+\mathbf{a}+\mathbf{b}+\cdots+\mathbf{z}]\\&=-\mathbf{p}_2{}^F+[\mathbf{p}_2{}^0+\mathbf{A}+\mathbf{B}+\cdots+\mathbf{Z}].\end{aligned}\tag{9}$$

We define now

$$\langle E_1{}^p\rangle=-p-(p+1)-\cdots-z+E_1{}^F-E_1{}^p\tag{10a}$$

with

$$E_1{}^p=(|\mathbf{p}_1{}^F-\mathbf{p}-\cdots-\mathbf{z}|^2+\mu^2)^{\frac{1}{2}},$$

$$\langle E_2{}^Q\rangle=-Q-(Q+1)-\cdots-Z+E_2{}^F-E_2{}^Q\tag{10b}$$

with

$$E_2{}^Q=(|\mathbf{p}_2{}^F-\mathbf{Q}-\cdots-\mathbf{Z}|^2+\mu^2)^{\frac{1}{2}},$$

where now $\langle E_1{}^p\rangle$ is the energy difference for an intermediate state in which the first particle has interacted with the first $p-1$ quanta and the second particle has interacted with all Z quanta and similarly for $\langle E_2{}^Q\rangle$. The energy difference for an intermediate state in which e_1 has interacted with $p-1$ quanta and e_2 with $Q-1$ quanta, is given by

$$a+b\cdots+(p-1)+A+B\cdots+(Q-1)+E_1{}^0+E_2{}^0-E_1{}^p-E_2{}^Q$$

which from (5) is equal to

$$-p-\cdots-z-Q-\cdots-Z+E_1{}^F+E_2{}^F-E_1{}^p-E_2{}^Q=\langle E_1{}^p\rangle+\langle E_2{}^Q\rangle.\tag{10c}$$

Now we can apply our theorem again if we let $\langle E_1{}^z\rangle$, $\langle E_1{}^{z+1}\rangle\cdots\langle E_1{}^{n+1}\rangle$ correspond to $A_1\cdots A_n$ and $\langle E_2{}^Z\rangle$, $\langle E_2{}^{Z-1}\rangle\cdots\langle E_2{}^{M+1}\rangle$ to $B_1\cdots B_m$.

Hence we obtain finally for the sum of the terms containing the instantaneous Coulomb interaction, omitting for the moment the factors common to the entire series given in (7),

$$\frac{2(u_1{}^{*n}u_1{}^{n+1})(u_2{}^{*M}u_2{}^{M+1})}{|\Delta\mathbf{p}|^2}\times\frac{1}{\langle E_1{}^a\rangle\cdots\langle E_1{}^z\rangle\langle E_2{}^A\rangle\cdots\langle E_2{}^Z\rangle}.\tag{11}$$

We now proceed to the summation of the part of the series containing the retardation. We noted before that whereas the terms containing the instantaneous Coulomb interaction could be separated into the product of a series containing terms up to n and M only and another series containing the terms from $n+1$ and $M+1$ up, no such separation is possible for the terms containing the retardation. Our former definitions for the $\langle E\rangle$'s were consistent as we always had to deal with $\langle E\rangle$'s from the same part of the series. Now, however, we may have an intermediate state where the first particle has already emitted the $(n+1)$th light quantum while the other particle has not yet absorbed the K' quantum (describing the retardation), i.e., we have terms of the form

$$a+\cdots+n+\cdots+(p-1)+A+\cdots+L+E_1{}^0+E_2{}^0-E_1{}^p-E_2{}^L-K'$$

and if we define now

$$\Delta E = +E_1^F - [E_1^0 + a + b + \cdots + z]$$
$$= -E_2^F + [E_2^0 + A + B + \cdots + Z], \quad (12)$$

we can write this term

$$\langle E_1^p\rangle + \langle E_2^L\rangle + \Delta E - K'. \quad (13a)$$

For the case where K' had been emitted by the second particle, we obtain a general term of the form

$$A + \cdots + (Q-1) + a + \cdots$$
$$+ g + E_1^0 + E_2^0 - E_1^g - E_2^Q - K'$$

which can be written

$$\langle E_1^g\rangle + \langle E_2^Q\rangle - \Delta E - K'. \quad (13b)$$

Let us sum first the part corresponding to the emission of K' by the first particle. Calling $\Delta E - K' = X$, we have then a series where each term is the product of expressions defined as before, except that at the step $\langle E_1^n\rangle \to \langle E_1^{n+1}\rangle$ we do not take $\langle E_1^{n+1}\rangle$, but $\langle E_1^{n+1}\rangle + X$, then $\langle E_1^{n+2}\rangle + X$ and so on. However, the step $\langle E_2^M\rangle \to \langle E_2^{M+1}\rangle$ subtracts the X again. The sum can therefore only include those terms where $\langle E_1^{n+1}\rangle$ is reached before $\langle E_2^{M+1}\rangle$.

We split the sum into partial sums according to the point at which the K' is absorbed, i.e., into sums with the same r occurring in the transition $((\langle E_1^r\rangle + \langle E_2^M\rangle + X) \to (\langle E_1^r\rangle + \langle E_2^{M+1}\rangle)$ where r can have any value between $n+1$ and $z+1$ (with $\langle E_1^{z+1}\rangle = 0$). Then each sum can be factored into two sums containing the terms before and after this transition, respectively. Then we can apply our theorem without difficulty up to this transition, with $\langle E_1^a\rangle, \cdots \langle E_1^n\rangle, \langle E_1^{n+1}\rangle + X$, $\langle E_1^{n+2}\rangle + X, \cdots \langle E_1^r\rangle + X$ corresponding to $A_1, \cdots A_n$ and $\langle E_2^A\rangle, \cdots \langle E_2^M\rangle$ corresponding to $B_1, \cdots B_m$. We can furthermore apply it to the terms after this transition, with $\langle E_1^z\rangle, \cdots \langle E_1^r\rangle$ corresponding to $A_1, \cdots A_n$ and $\langle E_2^Z\rangle, \cdots \langle E_2^{M+1}\rangle$ to $B_1, \cdots B_m$. (For the single term obtained for $p = z+1$ we have simply $\langle E_2^Z\rangle, \cdots \langle E_2^{M+1}\rangle$.)

Adding the partial sums again we obtain

$$\frac{1}{\langle E_1^a\rangle \cdots \langle E_1^n\rangle\langle E_2^A\rangle \cdots \langle E_2^Z\rangle}$$
$$\times\left\{\frac{1}{((\langle E_1^{n+1}\rangle + X)\langle E_1^{n+1}\rangle\langle E_1^{n+2}\rangle \cdots \langle E_1^z\rangle}\right.$$
$$+\frac{1}{((\langle E_1^{n+1}\rangle + X)((\langle E_1^{n+2}\rangle + X)\langle E_1^{n+2}\rangle \cdots \langle E_1^z\rangle} + \cdots$$
$$\left.+\frac{1}{((\langle E_1^{n+1}\rangle + X)((\langle E_1^{n+2}\rangle + X) \cdots ((\langle E_1^z\rangle + X)X}\right\}.$$

But the expression in brackets is just equal to the sum $\sum 1/(A, B)$ formed from the sets $0, X$ and $0, \langle E_1^z\rangle \cdots \langle E_1^{n+1}\rangle$, and therefore the entire sum is equal to

$$1/(\langle E_1^a\rangle \cdots \langle E_1^z\rangle\langle E_2^A\rangle \cdots \langle E_2^Z\rangle X).$$

We can sum the second part, corresponding to emission of K' by the second particle, in exactly the same way, only having $-\Delta E - K'$ instead of $\Delta E - K'$. Adding these two sums we obtain

$$\frac{(u_1^{*n}\alpha_{K'}u_1^{n+1})(u_2^{*M}\alpha_{K'}u_2^{M+1})}{K'}$$
$$\times\frac{2K'}{\langle E_1^a\rangle \cdots \langle E_1^z\rangle\langle E_2^A\rangle \cdots \langle E_2^Z\rangle(\Delta E^2 - K'^2)}. \quad (14)$$

Now we have to sum over the directions of polarization of K' and add the expressions (11) and (14). As the independent directions of polarization are mutually perpendicular and perpendicular to the direction of propagation of K', we have

$$\sum (u_1^{*n}\alpha_{K'}u_1^{n+1})(u_2^{*M}\alpha_{K'}u_2^{M+1})$$
$$= (u_1^{*n}\boldsymbol{\alpha}u_1^{n+1}) \cdot (u_2^{*M}\boldsymbol{\alpha}u_2^{M+1})$$
$$- (u_1^{*n}\boldsymbol{\alpha}\mathbf{K}'u_1^{n+1})(u_2^{*M}\boldsymbol{\alpha}\mathbf{K}'u_2^{M+1})/K'^2.$$

Remembering that $\mathbf{K}' = \Delta\mathbf{p}$, we obtain finally for that part of the matrix element (4) which corresponds to the interaction of z quanta with e_1 and Z quanta with e_2 with fixed order of interaction for each particle and which corresponds to Coulomb interaction after n and M as discussed before

$$(-e)^{z+Z+2}[(2\pi\hbar^2c^2)^{z+Z+2}/(a\cdots zA\cdots Z)]^{\frac{1}{2}}\frac{\prod_{1,1}^{n,M}\prod_{n+2,\,M+2}^{z,Z}(u_1^{*n-1}\alpha_n u_1^n)(u_2^{*M-1}\alpha_M u_2^M)}{\langle E_1^a\rangle \cdots \langle E_1^z\rangle\langle E_2^A\rangle \cdots \langle E_2^Z\rangle}$$
$$\times\frac{1}{|\Delta\mathbf{p}|^2 - (\Delta E)^2}2\{(u_1^{*n}u_1^{n+1})(u_2^{*M}u_2^{M+1}) - (u_1^{*n}\boldsymbol{\alpha}u_1^{n+1}) \cdot (u_2^{*M}\boldsymbol{\alpha}u_2^{M+1})$$
$$+[-(\Delta E)^2(u_1^{*n}u_1^{n+1})(u_2^{*M}u_2^{M+1}) + (u_1^{*n}\boldsymbol{\alpha}\Delta\mathbf{p}u_1^{n+1})(u_2^{*M}\boldsymbol{\alpha}\Delta\mathbf{p}u_2^{M+1})]/|\Delta\mathbf{p}|^2\}, \quad (15)$$

where $\Delta\mathbf{p}$ and ΔE are given by (9) and (12), $\langle E_1^a\rangle\cdots\langle E_1^n\rangle$ and $\langle E_2^A\rangle\cdots\langle E_2^M\rangle$ by (8) and $\langle E_1^{n+1}\rangle\cdots\langle E_1^z\rangle$ and $\langle E_2^{M+1}\rangle\cdots\langle E_2^Z\rangle$ by (10).

Equation (15) includes the summation over all the terms of (4) having the same numerator (7), but the summation over all the different possible numerators has still to be done for any particular process as discussed before.

V. DISCUSSION

Comparison of the expression containing the retarded interaction (15) with the expression containing only the instantaneous Coulomb interaction (11) shows that the sole difference is that the factor

$$\frac{4\pi\hbar^2c^2e_1e_2}{|\Delta\mathbf{p}|^2}(u_1^{*n}u_1^{n+1})(u_2^{*M}u_2^{M+1}) \tag{16}$$

is replaced by

$$\frac{4\pi\hbar^2c^2e_1e_2}{|\Delta\mathbf{p}|^2-(\Delta E)^2}\cdot\left\{(u_1^{*n}\cdot u_1^{n+1})(u_2^{*M}\cdot u_2^{M+1})-(u_1^{*n}\boldsymbol{\alpha}u_1^{n+1})\cdot(u_2^{*M}\boldsymbol{\alpha}u_2^{M+1}) + \frac{-(\Delta E)^2(u_1^{*n}\cdot u_1^{n+1})(u_2^{*M}\cdot u_2^{M+1})+(u_1^{*n}\boldsymbol{\alpha}\Delta\mathbf{p}u_1^{n+1})\cdot(u_2^{*M}\boldsymbol{\alpha}\Delta\mathbf{p}u_2^{M+1})}{|\Delta\mathbf{p}|^2}\right\}, \tag{17}$$

independent of the process under consideration. This is in spite of the fact that in (4) the number of terms needed to describe the retardation did depend on the process. Equation (16) is simply the usual matrix element (3a) for the instantaneous Coulomb interaction applied to the particular transition $n\rightarrow n+1$, $M\rightarrow M+1$ of our calculations. We see that by eliminating the virtual quanta, we have obtained an expression (17) analogous to the matrix element (3a) which can be used as a matrix element for the retarded interaction for any process involving the interaction of radiation and two electrons.[7] Using (17) the retarded interaction can then be treated as a single transition in the succession of intermediate states appearing in the total matrix element (4). Therefore the above noted difficulty of a dependence of the form of the retarded interaction on the particular process under consideration was only apparent.

While we have obtained this correspondence of the expressions (16) and (17) for the instantaneous and retarded interactions from the *summed* expressions (11) and (15), we can nevertheless, if desirable, use our "matrix element of retarded interaction" even before summing without introducing an error. The formalism of perturbation theory assigns a meaning to the summed expression only, and we are therefore justified in using any expansion whatsoever, as long as the summed expressions agree. But we could not have obtained the correspondence without *first* summing our series. If we fix our attention on just one term of the series involving the instantaneous interaction, i.e., a fixed order of interactions of the light quanta with the first and second particle, it is impossible to assign a correspondence to a definite part of the series involving the retardation. This is due to the fact that, as pointed out before, the emission and absorption of the virtual quanta need not take place in two consecutive steps.

In the case that no radiation is present (15) reduces to (17) alone. Equation (17) can then be further simplified. $\Delta\mathbf{p}$ is just equal to $\mathbf{p}_1^F-\mathbf{p}_1^0$, and we obtain from Dirac's equation $\boldsymbol{\alpha}(\mathbf{p}_1^F-\mathbf{p}_1^0)u=(E_1^F-E_1^0)u=\Delta E\cdot u$ and therefore our expression reduces to

$$\frac{4\pi\hbar^2c^2e_1e_2}{|\Delta\mathbf{p}|^2-(\Delta E)^2}\times[(u_1^{*0}u_1^F)(u_2^{*0}u_2^F)-(u_1^{*0}\boldsymbol{\alpha}u_1^F)\cdot(u_2^{*0}\boldsymbol{\alpha}u_2^F)]$$

which is just the expression obtained by Møller.[2]

It can be seen similarly that (15) reduces to the expression obtained by Møller for the interaction of two particles and one light quantum,[8] the only other case for which the matrix element has been calculated directly. In this case we have still to include a summation of the expressions (15) over the four possibilities that the retarded interaction can take place before or after the interaction with the light quantum and that the light quantum can interact with either one of the

[7] It should be remembered that our result has been derived for a process involving two particles. Its validity for processes involving more than two particles remains to be proved. See also footnote 6.

[8] C. Møller, Proc. Roy. Soc. 152, 481 (1935).

two particles. In addition to these cases (15) can be used to calculate the emission of several quanta at the collision of two particles. It can also be applied to the problem of creation of an electron-positron pair due to simple or multiple scattering of a γ-ray, as in the Dirac theory the presence of a positron in the final state can be described by the presence of an electron with negative energy in the initial state.

It should be noted that while we started from an expression which involved both particles in intermediate states simultaneously, each factor in the denominator of expression (15) involves one particle in an intermediate state at a time only. Therefore the method of Casimir[9] for summing over both signs of the energy of the particles in the intermediate states can now be applied directly.

Furthermore we can easily see that our expression (15) reduces to the known formula for the case where one particle can be considered as infinitely heavy. Then the energy is conserved for the interaction of the electron and the quanta, but not the momentum, because of the presence of the nucleus. The heavy particle cannot interact with the light quanta, but only with the electron. Therefore it does not pass through any intermediate states, and we have to put $(u_2{}^{*0}u_2{}^F)=1$ and $(u_2{}^{*0}\alpha u_2{}^F)=0$. With these changes, (15) together with the definitions (8a) and (10a) correctly describes the interaction of one electron with radiation or the creation of a pair of particles by the scattering of γ-rays in the field of a nucleus.

In the course of our calculations we have arbitrarily assumed that none of the energy differences in the denominator of (4) vanishes, and we have now to study the limits of applicability of (15). This expression diverges in two cases. The first case occurs when any of the $\langle E_1{}^x\rangle$'s or $\langle E_2{}^y\rangle$'s vanish (found by putting the expressions (8a, b) and (10a, b) equal to zero). As each of these $\langle E\rangle$'s refers to one particle only, the limitation introduced hereby is the same as for a process involving the interaction of radiation and a single particle. In the familiar cases this excludes processes which involve light quanta with the same energy and momentum in both the initial and final states and which could therefore be described by an approximation of lower order in e than the one applied.

The second case occurs when $|\Delta\mathbf{p}|^2-(\Delta E)^2=0$. This is the case for vanishing interaction of the two particles. Then our process splits up into two independent processes each involving a single particle and a certain number of quanta. Again these could have been calculated in a lower order approximation. This case is similar to the situation for the calculation of the scattering of two charged particles in a non-relativistic first order approximation, where we obtain the Rutherford formula, which diverges for zero deviation, i.e., vanishing interaction of the two particles. There, however, the exact solution is known, and it again reduces to the Rutherford formula for all but vanishing deviation angles, for which it also remains finite.[10] Unfortunately the exact solution of the relativistic two-body problem is not known.

Inspecting formulas (8c), (10c), and (13a, b), we might expect still other cases of divergencies if these expressions vanish, while these cases cannot be seen directly from our final Eq. (15). But these expressions can be indefinitely small without any indication of a maximum in the summed expression, due to cancellations. This can be seen directly from the proof of our theorem. If a number A_i of one set is almost equal to the negative of a number B_k of the other set, we can still apply the theorem. If we call $B_k=-A_i+\epsilon$, the sum depends upon ϵ as $1/(-A_i^2+A_i\epsilon)$, with $\epsilon\gtrless 0$, which is a monotonically varying function of ϵ. Therefore (15) does not show any resonance-like behavior in the neighborhood of the values obtained by putting the expressions mentioned above equal to zero.

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[9] H. Casimir, Helv. Phys. Acta **6**, 287 (1933).

[10] W. Gordon, Zeits. f. Physik **48**, 180 (1928), in particular Eq. (7′), p. 190, and the following discussion.