PHYSICAL REVIEW VOLUME 65, NUMBERS 3 AND 4 FEBRUARY 1 AND 15, 1944

On the Currents Carried by Electrons of Uniform Initial Velocity

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(Received December 8, 1943)

The following problem is treated: Electrons enter the space between two infinite parallel planes with uniform velocity at right angles to the planes. The current is studied for all possible values (positive and negative) of the potential difference between the planes. The complete solution can be obtained if this boundary condition for the current is accepted: The number of electrons entering the discharge space must be *equal to or smaller* than a given number N_0 per cm² per sec., and is for each potential as high as the potential permits. The problem depends on two dimensionless parameters, a reduced plate distance ξ_0 (involving the current) and $\eta_0 = V_0/E_0$, where V_0 is the impressed potential difference and eE_0 the energy of the electrons. For each value of ξ_0 the current is space-charge limited below a critical value of η_0. The limitation, however, is not due to the appearance of a potential minimum low enough to stop the electrons, but is due to the fact that the solution does not exist unless the current is limited. The space-charge limited characteristic is a simple generalization of Child and Langmuir's well-known formula [see Eqs. (28) or (35)]. The deviations from this formula are very considerable if $E_0 \sim V_0$ (see Fig. 2). The behavior of the potential distribution is graphed for some typical cases of positive, zero, and negative potentials V_0. Currents can pass with zero or negative potentials only below critical values of the reduced plate distance.

I. INTRODUCTION

It has been understood, for almost 30 years, that the well-known space charge equation of Child and Langmuir represents a limiting law which is valid only when the initial velocities of the electrons approach zero. The influence of the initial velocities, first recognized by Schottky, has been studied theoretically in a very complete way under the assumption of a Maxwellian distribution.[1] However, there remains a gap in the theory. The case where all electrons entering the discharge space have a uniform velocity has not been treated in a satisfactory way, not even in the case of parallel planes where the initial velocity is at right angles to the planes.

This latter problem has been treated in an approximate way by Langmuir and Compton,[2] but their solution is subject to the same criticism as the original Child-Langmuir solution, inasmuch as it leads to an infinite space-charge density (and velocity zero) at the point where the potential is minimum. The following analysis will show, furthermore, that one essential assumption in the derivation of Langmuir and Compton is too restricted. The problem itself has practical implications, and it seemed desirable to the author to attempt its complete solution.

It may appear strange that the more complex problem of Maxwellian distribution has been solved in an entirely satisfactory way, whereas the simpler problem of uniform velocity remains open to discussion. However, there is an essential difference between the two cases. If all velocities are represented, e.g., in the Maxwellian distribution, each electron may be considered as subject to an impressed field treated as continuous and created by electrons of other velocities. Then there is no ambiguity as to how any one electron is going to behave in a field so impressed. This supposition has been clearly recognized and formulated by Epstein.[3] If, however, all electrons have the same velocity, they have to travel in a field created by electrons of the same velocity, and it remains an open question whether these electrons are able to create a minimum of potential which is low enough to stop them. It will be seen that this occurs exclusively in a limiting (unstable) case. In general, the minimum will not be low enough to stop electrons which have entered the discharge space. In spite of this fact the current will be space-charge limited (for sufficiently low values of the potential) because the solution of the problem does not exist for any given value of the plate distance unless the current is below a critical value. Thus the space-

[1] W. Schottky, Physik. Zeits. **15**, 526, 624, 656, 872 (1914); Ann. d. Physik **44**, 1011 (1914); Verh. d. D. Phys. Ges. **16**, 490 (1914). P. S. Epstein, Verh. d. D. Phys. Ges. **21**, 85 (1919). T. C. Frey, Phys. Rev. **17**, 441 (1921); **22**, 445 (1923). I. Langmuir, Phys. Rev. **21**, 419 (1923). E. L. E. Wheatcroft, J. I. E. E. **86**, 473 (1940).

[2] I. Langmuir and K. T. Compton, Rev. Mod. Phys. **3**, 191 (1931), see p. 239.

[3] P. S. Epstein, reference 1, see p. 86.

charge limitation is caused, in this case, not by the fact that some of the electrons return to the cathode, but by the fact that only a limited number of them can get into the discharge space.

The analysis of the problem is quite straightforward. The gist of it is the determination of the limits of validity of the solution. If these are known, it is easy to determine the proper form of boundary condition for the current which will have to replace the artificial boundary condition [i.e., $(dE/dx)_0=0$] of Child and Langmuir.

II. THE EQUATIONS AND THEIR SOLUTION

We shall treat here exclusively the case where the electrodes are infinite parallel planes and where all electrons are supposed to enter the space between the two planes with the same velocity v_0 at right angles to the planes. Let x be measured in this normal direction and let the emitter plane, at $x=0$, be kept at the potential $E=0$, and the receiver plane, at $x=d$, at the potential $E=V_0$ (not necessarily positive).

The equations of the problem are given by Poisson's equation

$$d^2E/dx^2=4\pi ne, \tag{1}$$

by the equation of continuity, and by the equation of motion for the individual electron. We write the latter two equations in the integrated form. If n and v represent the number per cm^3 and the velocity of the electrons, respectively, we have

$$evn=i \tag{2}$$

and

$$v=v_0(1+E/E_0)^{\frac{1}{2}}. \tag{3}$$

Here i represents the current density[4] and E_0 the potential which is characteristic of the initial velocity

$$eE_0=mv_0^2/2. \tag{4}$$

Substituting from (2) and (3) into (1) and integrating once, we obtain

$$(dE/dx)^2=(16\pi iE_0/v_0)(1+E/E_0)^{\frac{1}{2}}+C_1. \tag{5}$$

Here we introduce dimensionless variables

$$\eta=E/E_0,\quad \xi=(16\pi i/(v_0E_0))^{\frac{1}{2}}x, \tag{6}$$

and obtain

$$d\eta/d\xi=\pm((1+\eta)^{\frac{1}{2}}+\alpha_1)^{\frac{1}{2}}, \tag{7}$$

where α_1 is a dimensionless constant of integration replacing C_1.

Finally, we set

$$u=(1+\eta)^{\frac{1}{2}} \tag{8}$$

and find, after a second integration,

$$(4/3)(u-2\alpha_1)(u+\alpha_1)^{\frac{1}{2}}=\pm\xi+\alpha_2, \tag{9}$$

with α_2 as second constant of integration.

It follows from (7) that $d\eta/d\xi$, or dE/dx, may be positive or negative. Limiting ourselves at first to the case $V_0>0$, we have to distinguish two possibilities. Either $d\eta/d\xi\geq 0$, and η will increase in a monotonic way, or $d\eta/d\xi<0$ at first, and η will present a minimum.

First Case, Monotonic Increase

Let $\xi=\xi_0$ correspond to the distance d and let

$$u_0=(1+\eta_0)^{\frac{1}{2}}=(1+V_0/E_0)^{\frac{1}{2}}, \tag{10}$$

then the conditions imposed on the potential demand $u=1$ for $\xi=0$, and $u=u_0$ for $\xi=\xi_0$. Hence, from (9) (with the plus sign)

$$(4/3)(1-2\alpha_1)(1+\alpha_1)^{\frac{1}{2}}=\alpha_2 \tag{11}$$

and

$$(4/3)(u_0-2\alpha_1)(u_0+\alpha_1)^{\frac{1}{2}}=\xi_0+\alpha_2. \tag{12}$$

These two equations determine the constants α_1 and α_2 provided i is known. The field distribution then follows from (9) (with the plus sign).

Second Case, Negative Potential Minimum

Let δ be the value of ξ for which the minimum occurs. The value of δ is obtained by integrating (7) from $\xi=0$ to $\xi=\delta$ and bearing in mind that $d\eta/d\xi=-(u+\alpha_1)^{\frac{1}{2}}=0$ for $\xi=\delta$. Thus it is found that

$$\delta=(4/3)(1-2\alpha_1)(1+\alpha_1)^{\frac{1}{2}}. \tag{13}$$

The solution now becomes

$$(4/3)(u-2\alpha_1)(u+\alpha_1)^{\frac{1}{2}}=-\xi+\alpha_3,\quad 0\leq\xi\leq\delta, \tag{14}$$

and

$$(4/3)(u-2\alpha_1)(u+\alpha_1)^{\frac{1}{2}}=\xi+\alpha_4,\quad \xi\geq\delta, \tag{15}$$

where α_3 and α_4 are new constants of integration. They are determined from the boundary conditions for E, and from the condition of continuity

[4] We count, in the usual way, i as positive if an electron current is flowing in the direction of the positive x axis.

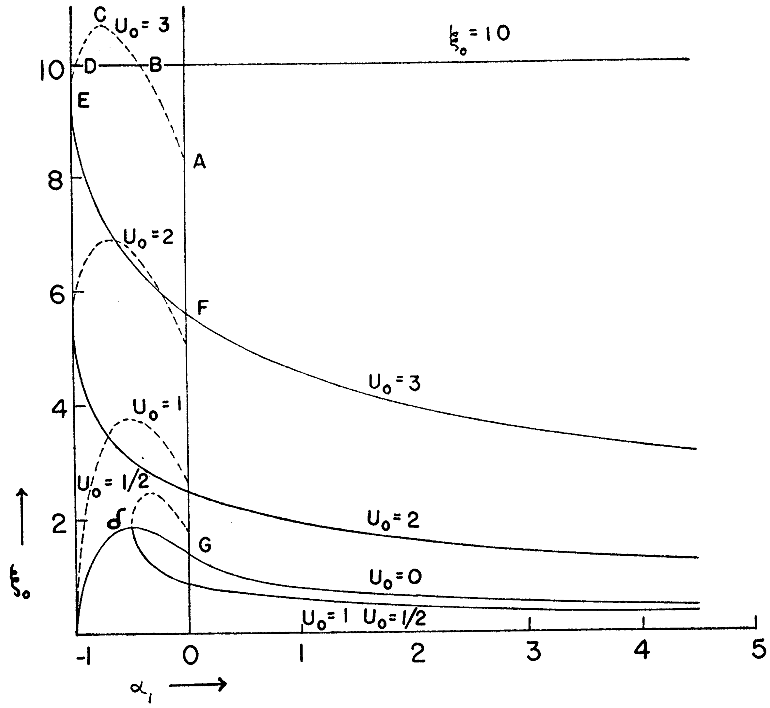

FIG. 1. The curves represent the dependence of the reduced plate distance ξ_0 on the constant α_1 for various values of the external potential $(u_0=(1+\eta_0)^{\frac{1}{2}}=(1+V_0/E_0)^{\frac{1}{2}}=\text{const.})$. Solutions of the problem for prescribed values of ξ_0 and η_0 are given by the intersection of the horizontal $\xi_0=\text{const.}$ with the curve $u_0=\text{const.}$

at $\xi=\delta$. We find

$$\alpha_3=\delta, \quad \alpha_4=-\delta, \tag{16}$$

and, for the determination of α_1,

$$(4/3)(u_0-2\alpha_1)(u_0+\alpha_1)^{\frac{1}{2}}=\xi_0-\delta. \tag{17}$$

When the constants are known, the field distribution follows from (14) and (15).

III. THE BOUNDARY CONDITION FOR THE CURRENT

It will be noticed that the Eqs. (11) and (12) in the first case, and Eqs. (13), (16), and (17) in the second case, determine the constants of integration as functions of η_0 and ξ_0. Thus the solution of the problem depends on these two dimensionless parameters only. However, the solution is not complete since ξ_0 involves the knowledge of the current density i. Now i is a constant of integration itself, originating in the integration of the equation of continuity, and has to be determined by a boundary condition.

We shall assume conditions under which a true saturation current exists. Our problem does not correspond to thermionic emission, but rather to the case where electrons of uniform velocity are shot into the discharge space. Therefore, it is not necessary to include the possibility of a Schottky effect. *Preliminarily* we assume that not only the velocity of the electrons is given, but also their number. Let $N_0=n_0v_0$ signify the number of electrons which enter the discharge space per cm^2 per sec. Then the saturation current will be

$$i=i_0=en_0v_0. \tag{18}$$

With this boundary condition the current will be constant, i.e., independent of the potential V_0, within the limits of validity of our solutions.

It should be noticed that Eq. (9) will give no physical meaning unless the variable u is real. Consequently, from (8), η is subject to the condition $\eta\geq-1$, and the potential minimum cannot become as low as $\eta_{\min}=-1$, and eventually stop the electrons, except in a limiting case. We are going to discuss this limiting case later.

It remains to be seen within what limits our solutions are valid. The practical situation will usually be that η_0 and ξ_0 are given and that the constants α have to be determined. However, the discussion becomes simpler if we assume that, besides η_0, α_1 is given and that, consequently, ξ_0 and the other α's have to be determined. The equations available for the determination of ξ_0 are (11) and (12) in the first case and (13) and (17) in the second case.

It will be seen, from (11), that the constant α_2 in the first case is identical in form with δ in the second case, though it does not have the same physical meaning. Therefore, we can write (12) and (17) in the joint form

$$\xi_0 \pm \delta = (4/3)(u_0 - 2\alpha_1)(u_0 + \alpha_1)^{\frac{1}{2}}, \tag{19}$$

where the plus sign corresponds to the first case. We shall refer to the two cases as "branch I" and "branch II" of the solution.

Because of the last factor in (13), the constant α_1 is subject to the limitation

$$\alpha_1 \geq -1, \text{ for branch I and II.} \tag{20}$$

Furthermore, on branch II, α_1 is limited to negative values as otherwise $d\eta/d\xi$ could not become zero, according to (7). Hence

$$0 \geq \alpha_1 \geq -1, \text{ for branch II.} \tag{21}$$

If α_1 is known, the value of the reduced potential η at the minimum $\eta_{\min}$ is given, from (7), as

$$\eta_{\min} = -(1 - \alpha_1^2). \tag{22}$$

In Fig. 1 the two top curves represent ξ_0 as functions of α_1 for the two values $u_0 = 2$ and $u_0 = 3$ which correspond to the positive values $\eta_0 = 3$ and $\eta_0 = 8$. The curves are calculated from (13) and (19). Branch I is shown by a full line and branch II by a broken line. Branch II has been calculated for the interval (21) only as there is no solution of our problem beyond that interval. A point on branch I represents a possible solution without a potential minimum, a point on branch II a possible solution with a minimum [given by (22)]. A point where the two branches join represents the limiting case where the potential curve starts with a horizontal tangent at the cathode.

It is easy to show that the curves for all values of $u_0 > 1$, i.e., $\eta_0 > 0$, represent the same general character as that exhibited by our two examples: Both branches begin at the value

$$\xi_0 = (4/3)(u_0 + 2)(u_0 - 1)^{\frac{1}{2}} \tag{23}$$

with a vertical tangent. Branch I decreases in a monotonic way and approaches $\xi_0 = 0$ asymptotically. Branch II has a unique maximum at

$$\alpha_1 = -(u_0/(u_0 + 1)), \tag{24}$$

which is given by

$$\xi_{\max} = (4/3)(1 + u_0)^{\frac{3}{2}}, \tag{25}$$

[from substitution of (24) into (19) with the lower sign].

It is the existence of this maximum which is decisive for space-charge limitation. To show this we now return to the usual state of affairs where η_0 and ξ_0 are prescribed. Then the possible solutions will be given by the intersections of the curve $u_0 = \text{const.}$ with the horizontal $\xi_0 = \text{const.}$ Evidently that leads to three possibilities—namely, that there is one solution, that there are two solutions, or that there is no solution at all.

We discuss the last of these possibilities first: it will arise when the given value of ξ_0 is larger than the maximum value (25) corresponding to the given value of u_0. Thus, under our preliminary boundary condition, there will be saturation current for all values of η_0 larger than the critical value η_0^* given by

$$\xi_0 = (4/3)(1 + u_0^*)^{\frac{3}{2}}, \tag{26}$$

and for smaller values there will be no solution. This is a result which cannot be admitted from the physical point of view, and the boundary condition must be modified.

From the definition of the variable ξ [Eq. (6)], it follows that ξ_0 can be lowered by decreasing the value of i involved and that means by decreasing n_0 since v_0 is supposed to be constant. Thus, when the critical value of u_0^* (26) is reached (by decreasing the potential), a solution for lower values of η becomes possible if n_0 is suitably reduced. Therefore, we relax our boundary condition to the following statement: The number of electrons entering the discharge space has to be *equal to or smaller* than the given number $N_0 = n_0 v_0$ per cm² per sec. and is for each potential as high as the potential permits. The last clause has to be added because otherwise the solution

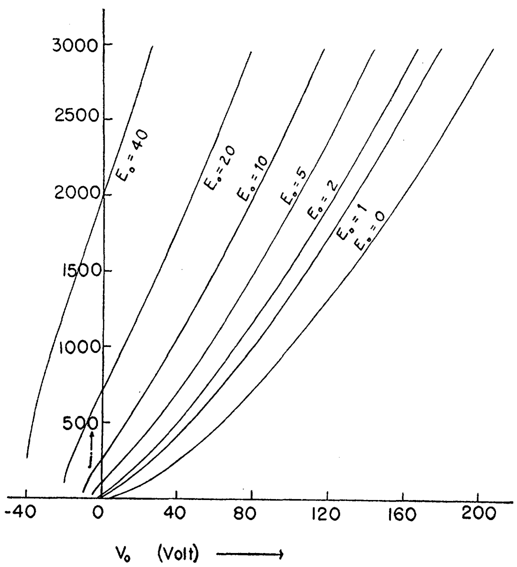

FIG. 2. The curves represent the dependence of the reduced current $j=9\pi i d^2/(2e/m)^{\frac{1}{2}}$ on the external potential V_0 (in volts) for different values of the initial velocity ($E_0=e$ volts).

would not be determined. It would be possible to obtain solutions for all values lower than i_0 before the critical $\eta_0{}^*$ is reached.

With this boundary condition the current will decrease, from the critical value $\eta_0{}^*$ downward in such a way that (26) remains fulfilled since this relation gives the highest value of i which can pass for given values of d and η_0. It should, however, be stressed that the limitation is not due to the appearance of a minimum which makes the electrons (or some of them) return; for the minimum of potential, as given by (22), is less low than $\eta_{\min}=-1$ for all relevant values of α_1 given by (24). Therefore, the space-charge limitation, as stated in the introduction, is due in our case to the fact that, for a given potential, electrons of given velocity can enter the discharge space only in a number limited for each plate distance.

IV. DISCUSSION OF THE SOLUTION

Our Eq. (26) contains the space-charge limited branch of the characteristic. If we bear in mind the definition of u_0 (10) and ξ_0 (6) and introduce the "reduced current density j" by

$$j=9\pi i d^2/(2e/m)^{\frac{1}{2}}, \tag{27}$$

the relation (26) can be written in the form

$$j=V_0^{\frac{3}{2}}[(1+E_0/V_0)^{\frac{1}{2}}+(E_0/V_0)^{\frac{1}{2}}]^3. \tag{28}$$

This is evidently the generalization of the Child-Langmuir law to the case of electrons with uniform initial velocities, and reduces to it in the limit $E_0/V_0\rightarrow 0$. It need hardly be pointed out that our solution does not lead to infinite values of the space-charge density, neither at the cathode nor at the potential minimum. Furthermore, it is automatically limited to the values $i\leq i_0$. As soon as the value $i=i_0$ has been reached by increasing the potential the space-charge limited characteristic, (26) or (28), has to be continued by the horizontal saturation branch.

For small values of E_0/V_0 we obtain

$$j=V_0^{\frac{3}{2}}[1+3(E_0/V_0)^{\frac{1}{2}}+\cdots], \tag{29}$$

and for large values of E_0/V_0

$$j=8E_0^{\frac{3}{2}}[1+(3/4)(V_0/E_0)+\cdots]. \tag{30}$$

Thus, for $V_0=0$ the current reduces to the constant value $j_0=8E_0^{\frac{3}{2}}$. However, the characteristic can be continued into the domain of negative V_0's (see Section VI).

In Fig. 2 we have represented the reduced current density j as a function of V_0 (in volts) for various values of E_0 (also in volts). All curves ultimately (for $V_0\rightarrow\infty$) approach the curve with $E_0=0$, which is the Child-Langmuir law. It will be seen that, for E_0 comparable with V_0, there are very large deviations from the $\frac{3}{2}$-power law, as was known from the study of Maxwellian distribution.

We now revert to the case where the given values of ξ_0 (with $i=i_0$) and η_0 lead to *two* solutions of the problem. As an example, the horizontal $\xi_0=10$ in our Fig. 1 intersects the curve $u_0=3$ in the two points B and D corresponding to $\alpha_1=-0.39$ and $\alpha_1=-0.975$, respectively. The two solutions which are thus possible may be both on branch II, or one on branch I and the other on branch II; the latter case occurs only if $u_0>1.81$.

The two solutions belong to different values of α_1 and, therefore [according to (9), (14), and (15)], have different potential distributions. It is easy to show that the solution with the smaller negative value has the lower minimum (if both solutions are on branch II) and that it leads to

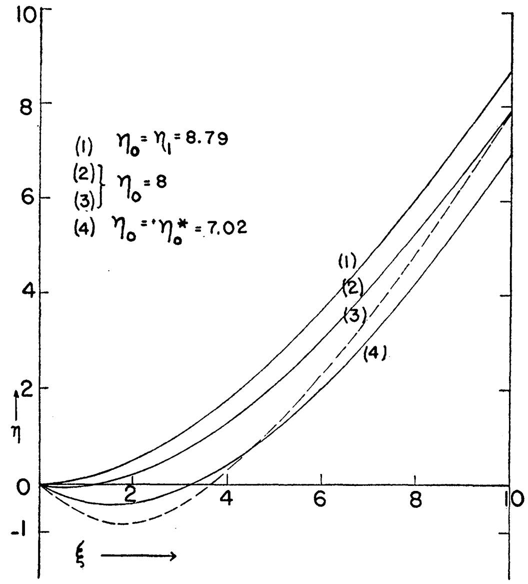

FIG. 3. The curves represent the potential distribution in the case $\xi_0=10$ for typical values of the ratio $\eta_0=V_0/E_0$. At $\eta_0=\eta_1$ the potential minimum begins to appear and at $\eta_0=\eta_0^*$ the current becomes space-charge limited. The broken line corresponds to the unstable form of discharge.

lower values of the potential for any value of ξ except $\xi=0$ and $\xi=\xi_0$. It would require work to change the discharge with the higher potential curve to that with the lower potential curve, and consequently the former discharge is the stable one. Thus we arrive at the result that points to the right of the maximum of branch II, in Fig. 1, represent unstable forms of discharge.

We obtain the following description of the behavior of a discharge for a given value of ξ_0 as the potential is decreased. For very large values of u_0 there will be saturation. The intersection between the curves $\xi_0=$const. and $u_0=$const. will be at large values of α_1, i.e., on branch I, and the potential will be almost linear (see next section). As u_0 is decreased, the intersection remains on branch I until α_1 has reduced to the value -1. This will happen at the value u_1 given by[5]

$$\xi_0=(4/3)(u_1+2)(u_1-1)^{\frac{1}{2}}. \tag{31}$$

At this point the potential curve has a horizontal tangent at $\xi=0$. As the potential is further reduced the intersection will be on branch II, i.e., a maximum begins to make its appearance. The minimum gets lower and δ gets larger (see the curve for δ on Fig. 1) until the point is reached where the horizontal $\xi_0=$const. is tangent to a curve $u_0=$const. This will happen at the critical value of u_0 given by (26). Until this point is reached the current is saturated. If now the potential is further reduced, the representative point will follow downward the line joining the maxima of the various u_0 curves.

We still have to say a few words regarding the limiting case where the potential at the minimum becomes as low as $\eta_{\min}=-1$, i.e., $V_{\min}=-E_0$. This happens exclusively for the value $\alpha_1=0$ [see (22)]. The event is, therefore, characterized by points of branch II located on the ordinate $\alpha_1=0$. We have just seen that such points represent unstable forms of discharge.

Apart from this fact there are strong reasons which speak against the physical admissibility of this limiting case. For any given value of u_0 the case $\alpha_1=0$ (on branch II) occurs for a finite value of ξ_0. Therefore, we must draw the conclusion that the current density i would be finite, as given by this value of ξ_0. Now, the velocity of the electrons naturally becomes zero at a point where $\eta=-1$ [see (3)], and therefore the density n would have to be infinite. We are of the opinion that no solution should be admitted which makes a physically measurable quantity infinite. We, therefore, conclude that the limiting case $\alpha_1=0$ (on branch II) should be excluded, even as a possible form of unstable discharge.

V. THE POTENTIAL CURVE

After the constants α have been determined as indicated above, the potential distribution be-

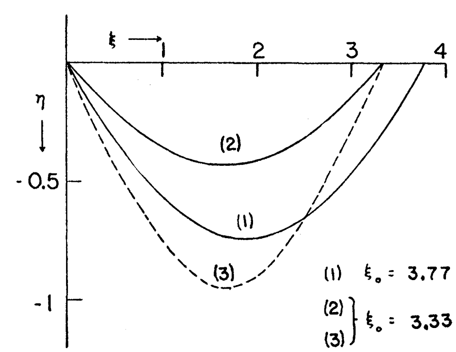

FIG. 4. The curves represent the potential distribution in the case $V_0=0$ for two values of the reduced plate distance. The broken line corresponds to the unstable form of discharge.

[5] Equation (31) represents a cubic for u_1 which is easy to solve. It has one real root only.

tween the plates becomes calculable from (9) or (14), (15) for the two branches. We first discuss the case of very high potentials ($\eta_0 \gg 1$).

We obtain the value of α_1 corresponding to this case by developing (19), with the plus sign, and (13) in powers of u_0/α_1. This yields, to a first approximation,

$$\alpha_1^{\frac{1}{2}} = \eta_0/\xi_0. \qquad (32)$$

Substituting this value into (9) and (11) we find as limiting law

$$\xi = (\xi_0/\eta_0)\eta. \qquad (33)$$

Thus, in the limit $\eta_0 \to \infty$, the potential distribution becomes linear, as has always been assumed and as is evident from the fact that all electrons are swept out of the discharge space with increasing velocity as the potential is increased.

It is not worth while to establish approximate formulae for any other cases as the exact expressions are easy to handle. In Fig. 3 we have graphed some potential curves for the case[6] $\xi_0 = 10$. We have drawn the curves for the value $\eta_0 = \eta_1 = 8.79$ where the minimum begins to appear, for $\eta_0 = 8$, and for the critical value $\eta_0 = \eta_0^* = 7.02$ where the space-charge limitation sets in. For $\eta_0 = 8$ there are two solutions as we have seen above. The unstable form of discharge is given by a broken line.

VI. ZERO POTENTIAL AND NEGATIVE POTENTIALS

In our discussion we have limited ourselves so far to positive values of V_0 though our solutions are valid for all values of $\eta_0 \geq -1$. It is of interest to discuss zero and negative values since the equations developed under the assumption of Maxwellian distribution are not applicable to these cases.[7]

If $V_0 = 0$, or $u_0 = 1$, the branch I of the curve $u_0 = 1$ degenerates into the horizontal axis and the branch II takes the form indicated in Fig. 1. It has its maximum $\xi_{\max} = (4/3)2^{\frac{3}{4}} = 3.771$ at $\alpha_1 = -\frac{1}{2}$. Therefore, currents can pass with $V_0 = 0$ only if $\xi_0 \leq 3.771$. For the interval $2.667 \leq \xi_0 \leq 3.771$ there are two solutions, one of them unstable.

[6] The case $\xi_0 = 10$ does not correspond to any practical case of thermionic emission. Much larger values of ξ_0 would have to be chosen (order of 10^3 to 10^4). However, the minima are much more marked with comparatively low values of η_0 and ξ_0.

[7] I. Langmuir, reference 1, see p. 424.

In Fig. 4 we have drawn the potential distribution for $V_0 = 0$ and for two values of ξ_0, namely, the limiting case $\xi_0 = 3.771$, and the intermediate case $\xi_0 = 3.33$. The unstable distribution is given by a broken line.

If, finally, V_0 becomes negative we have again to distinguish two cases, namely, the case of a potential decreasing in a monotonic way: branch I, and the case of a negative potential minimum: branch II. By calculations which are exactly like

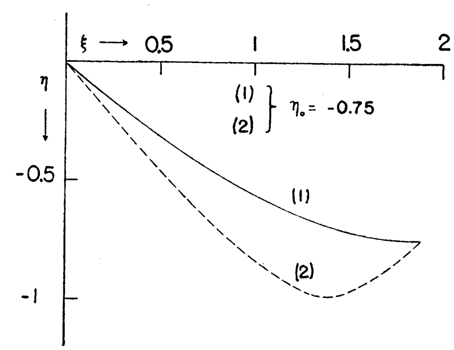

FIG. 5. The curves represent the potential distribution in the case of a negative potential $V_0 = -(3/4)E_0$ for the value $\xi_0 = 1.886$. The broken line corresponds to the unstable form of discharge.

those following Eq. (9), we find for the determination of α_1 on branch I (negative potentials)

$$\delta - \xi_0 = (4/3)(u_0 - 2\alpha_1)(u_0 + \alpha_1)^{\frac{1}{2}}, \qquad (34)$$

where δ retains its definition (13).

For branch II our previous formulae remain valid without any change. Thus α_1 can be determined from (19) with the lower sign. Also the maximum, as defined by (24), (25), retains its significance and the current begins to decrease on the space-charge limited curve as soon as the critical value is reached. Therefore, the characteristic (28) remains valid though it has to be written now in the form

$$j = [(E_0 + V_0)^{\frac{1}{2}} + E_0^{\frac{1}{2}}]^3. \qquad (35)$$

The reduced current decreases according to this equation down to the value $V_0 = -E_0$ giving $j = E_0^{\frac{3}{2}}$. For still lower values of V_0, no solution exists and no current, therefore, can flow. We have continued the characteristics of Fig. 2 down to their limit. The tangents are vertical at the limits.

As for the limits of validity of the solutions, the argument remains similar as for positive potentials. It follows from (34) and (19) (lower sign) that solutions exist only for

$$\alpha_1 \geq -u_0. \tag{36}$$

Both branches of the curves $u_0=\text{const.}$ begin with the value $\alpha_1=-u_0$ at the value $\xi_0=\delta$ and with vertical tangent. They form together a loop which contracts around the point G on Fig. 1 ($\alpha_1=0$, $\xi_0=4/3$) as u_0 decreases from $u_0=1$ to $u_0=0$, and for $u_0=0$ there remains only the branch I. In Fig. 1 the curves for $u_0=\frac{1}{2}$ and $u_0=0$ have been added.

It is now easy to discuss the behavior of the solution. Three intervals have to be distinguished:

(a) $0\leq\xi_0\leq 4/3$. The solution starts, with $V_0=0$, on branch II until, with decreasing potentials, the value $\xi_0=\delta$ is reached. Then the solution changes to branch I (no minimum) and ends on the curve $u_0=0$. The current has the saturation value throughout.

(b) $4/3\leq\xi_0\leq 1.886$. The upper limit of this interval is set by the maximum of δ which occurs for $\alpha_1=-0.5$. The solution begins, with $V_0=0$, on branch II, changes to branch I when the δ curve is intersected for the first time, changes back to branch II when the δ curve is met for the second time, and finally becomes space-charge limited.

(c) $1.886\leq\xi_0\leq 3.771$. In this interval the solution is on branch II throughout. The current is saturated at first and becomes space-charge limited as soon as the critical value (26) is reached. All space-charge limited solutions [in cases (b) and (c)] end at the point G of Fig. 1, i.e., at $\xi_0=4/3$ or $j=E_0^{\frac{3}{2}}$.

In Fig. 5 we have represented the potential distribution in the case of a negative potential V_0. We have taken the value $u_0=\frac{1}{2}$ corresponding to $V_0=-(3/4)E_0$, and have chosen ξ_0 on the δ curve; i.e., $\xi_0=(4/3)2^{\frac{1}{2}}=1.886$. Whenever, in the intervals (a) and (b), the limiting case $\xi_0=\delta$ is realized, the solution of branch I coincides with the (stable) solution of branch II. This case, therefore, corresponds to the case $(d\eta/d\xi)_0=0$ for positive potentials, but the horizontal tangent is now at the receiver plate.

The two solutions on branch II correspond to the values $\alpha_1=-\frac{1}{2}$ and $\alpha_1=-0.025$, respectively. Both potential distributions are represented in Fig. 5, the unstable one by a broken line.

PHYSICAL REVIEW VOLUME 65, NUMBERS 3 AND 4 FEBRUARY 1 AND 15, 1944

The Electrical Oscillations of a Prolate Spheroid. Paper II

Prolate Spheroidal Wave Functions

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(Received October 4, 1943)

The vector wave equation in prolate spheroidal coordinates ξ, η, ϕ is set up, the variables are separated, and the characteristic values (eigenvalues) and characteristic functions (eigenfunctions) of the resulting ordinary differential equations are obtained in series which converge rapidly in the neighborhood of resonance for spheroids of eccentricity near to unity. The coefficients of the two independent primitives in the linear combinations representing diverging and converging waves at infinity are calculated. The zeros of certain of the characteristic values are investigated.

IN a previous paper[1] both the free oscillations of a perfectly conducting prolate spheroid and the oscillations forced by a plane wave with the electric field parallel to the long axis of the spheroid were discussed. The forced oscillations were treated by an approximate method valid only for very eccentric spheroids. For the exact investigation of these oscillations the complete solutions of the vector wave equation in prolate spheroidal coordinates are needed. The object of the present paper is

[1] L. Page and N. I. Adams, Jr., Phys. Rev. **53**, 819 (1938). This paper will be referred to as I and the present paper as II.