Interpretation of the Spinning Electron with Bipolar Coordinates*

MARY HEWLETT PAYNE†
Brown University, Providence, Rhode Island

(Received October 16, 1943)

The bipolar coordinates of a point may be loosely defined as the distances of the point from each of two reference points. Consistent definitions are developed for the bipolar coordinates of a point on a sphere with the reference points at opposite ends of a diameter, and it is shown that the bipolar coordinates so defined transform like Pauli spin functions under rotations. The electron is visualized as a point mass with charge e moving on the sphere with its motion expressed in terms of its bipolar coordinates. It is assumed observable only at two diametrically opposite points of its orbit. This picture allows an exact physical interpretation of the spin variable and the Pauli spin operators. The magnetic moment associated with the motion is $e\hbar/2mc$; the radius of the electron is taken as the radius of the sphere which is $\hbar/2mc$; the wave-length associated with the spinning motion is h/mc. Two related functions must be introduced to take care of reflections and these together with the two original functions transform like Dirac spin functions under rotations, reflections, and Lorentz transformations.

INTRODUCTION

IN this paper an attempt will be made to set up a geometrical picture for the Pauli and Dirac spin theories.[1] The proof of the equivalence of this picture with the analytical theories will be based on the transformation properties of the Pauli and Dirac spin functions. It will be convenient to change the usual notation slightly, and we shall take the Eulerian angles φ, θ, and ψ as in Fig. 1. φ is a rotation about the z axis; θ, a rotation about the new y axis after the φ rotation has been performed; and ψ, a rotation about the final z axis. This differs from the usual conventions where θ is taken as a rotation about the x axis. The other important change is in the time variable. One usually takes the fourth coordinate as ict; we shall take it as ct.

With the notation described in the previous paragraph, one finds that the Pauli spin functions $a_1(\alpha)$ and $a_2(\alpha)$ transform under a rotation through φ, θ, and ψ in that order by a unitary matrix U_p given by

$$U_p=\begin{pmatrix}\cos\frac{1}{2}\theta e^{-\frac{1}{2}i(\varphi+\psi)} & -\sin\frac{1}{2}\theta e^{\frac{1}{2}i(\varphi-\psi)}\\ \sin\frac{1}{2}\theta e^{\frac{1}{2}i(\psi-\varphi)} & \cos\frac{1}{2}\theta e^{\frac{1}{2}i(\varphi+\psi)}\end{pmatrix}. \quad (1)$$

α is the spin variable and is allowed to assume two values α_1 and α_2 corresponding to spin in the positive and the negative z directions, respectively. The spin operators corresponding to the components of spin angular momentum in the x, y, and z directions are, of course, the Pauli matrices multiplied by[2] $\frac{1}{2}\hbar$:

$$s_x=\tfrac{1}{2}\hbar\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad s_y=\tfrac{1}{2}\hbar\begin{pmatrix}0&-i\\i&0\end{pmatrix},$$
$$s_z=\tfrac{1}{2}\hbar\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \quad (2)$$

The significance of the spin variable and the actual form of the spin operators are not given in the Pauli theory.

The Dirac theory makes use of four component wave functions. If we denote the four components by ψ_1, ψ_2, ψ_3, and ψ_4 and investigate the transformation properties, we find that

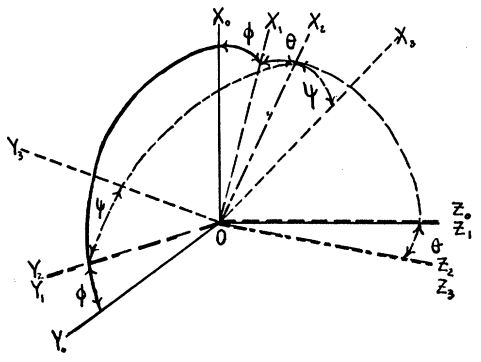

FIG. 1. φ=rotation about Z_0; θ=rotation about Y_1; ψ=rotation about Z_2.

* Extract from a thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in physics at Brown University, Providence, Rhode Island.

† Present address: 816 Picher Avenue, Joplin, Missouri.

[1] W. Pauli, Zeits. f. Physik **43**, 601 (1927); P. A. M. Dirac, Proc. Roy. Soc. **A117**, 610 (1928), **A118**, 351 (1928); E. L. Hill and R. Landshoff, Rev. Mod. Phys. **10**, 87 (1938); B. L. van der Waerden, *Gruppentheoretische Methode* (Julius Springer, Berlin, 1932); L. De Broglie, *L'Electron Magnetique* (Hermann et Cie, Paris, 1934).

[2] $\hbar=h/2\pi$; h=Planck's constant.

under rotations the four components transform by the matrix

$$U_D=\begin{pmatrix} U_p & 0 \\ 0 & U_p \end{pmatrix}. \tag{3}$$

Under the space inversion (reflection in the origin) and the time reversal they transform by the matrices

$$U_I=\pm\begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} \quad \text{and} \quad U_T=\pm\begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}, \tag{4}$$

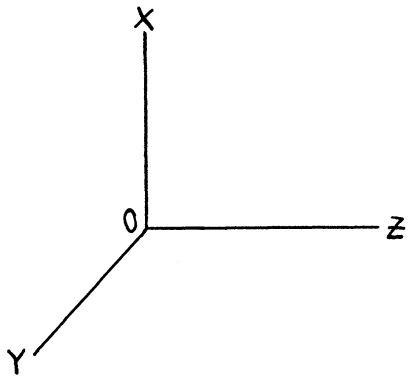

FIG. 2.

respectively, and under a Lorentz transformation they transform by

$$U_L=\begin{pmatrix} e^{-\frac{1}{2}\mu} & 0 & 0 & 0 \\ 0 & e^{\frac{1}{2}\mu} & 0 & 0 \\ 0 & 0 & e^{\frac{1}{2}\mu} & 0 \\ 0 & 0 & 0 & e^{-\frac{1}{2}\mu} \end{pmatrix} \tag{5}$$

where

$$\cosh\mu=1/(1-\beta^2)^{\frac{1}{2}},$$
$$\sinh\mu=-\beta/(1-\beta^2)^{\frac{1}{2}}, \quad \beta=v/c, \tag{6}$$

and v=velocity of the moving system.

PRELIMINARY REMARKS ON BIPOLAR COORDINATES

For the purposes of this discussion we take the Cartesian axes as shown in Fig. 2. Note that this is a right-handed system. Consider the sphere of diameter a with its center at 0, the origin. Points on this sphere have coordinates x, y, and z, such that

$$x^2+y^2+z^2=a^2/4. \tag{7}$$

We define the bipolar coordinates of a point on this sphere as the distances of the point from the points $(0, 0, \frac{1}{2}a)$ and $(0, 0, -\frac{1}{2}a)$ and call them u and v, respectively. The two points from which u and v are measured are called the right-hand reference point and the left-hand reference point, respectively. On occasion we shall call them N and S, respectively. We shall confine ourselves to points on the sphere. This is sufficient for the purpose of developing a model of the electron, and the definitions of the bipolar coordinates become very difficult for points not on the sphere.

Suppose that we have a point P on the sphere. u_P and v_P will depend on φ, θ, and ψ where φ, θ, and ψ are the Eulerian angles defined in the introduction. θ is the angle made by OP and the positive z axis; φ is the angle made by the plane of OP and the z axis with the xz plane; ψ is the angle of rotation about the line OP (see Fig. 3). At first sight one might think that θ and φ are sufficient to determine the position of a point on the sphere uniquely, and it is true that they are. However, if we wish to think of the sphere as a whole and to specify the position of all the points simultaneously, the variable ψ is also necessary. One might think of the ψ dependence as giving the answer to the question: "What rotation would be necessary to get the points from a given initial position to their present position?"

Now it is evident that with our present definition of the bipolar coordinates we have no means of distinguishing among the various points on a circle formed by the intersection of a plane perpendicular to the z axis with the sphere. The next three sections will be devoted

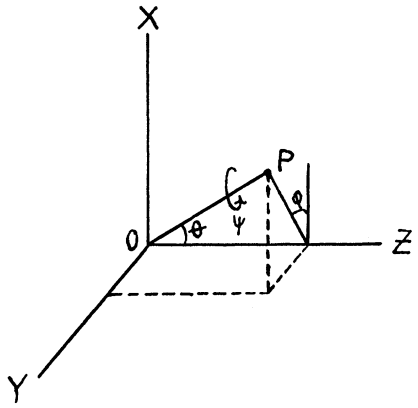

FIG. 3.

to developing a consistent set of conventions for assigning bipolar coordinates to each point on the sphere. To facilitate the discussion we shall introduce the variables

$$x_1=x+iy, \quad x_{-1}=x-iy, \quad x_0=z. \tag{8}$$

THE θ DEPENDENCE OF BIPOLAR COORDINATES[3]

We take a point P on a circle of diameter a with center at the origin in the xz plane. From plane geometry the magnitudes of u and v are given by

$$u=a\sin\tfrac{1}{2}\theta, \quad v=a\cos\tfrac{1}{2}\theta. \tag{9}$$

We shall take (9) as the definition of bipolar coordinates in a plane. Note that we can now distinguish between points placed symmetrically with respect to the z axis (e.g., P and P' in Fig. 4). Note also that each point has two sets of coordinates—that is, the period of u and v is 4π. In other words, the bipolar coordinates of a point in a plane (as here defined) are double valued. We shall observe this characteristic in the φ and ψ dependences as well. In this case we have the relations

$$x_1=x_1^*=x_{-1}=uv/a, \quad x_0=(v^2-u^2)/2a. \tag{10}$$

The matrix for a rotation through $\bar{\theta}$ about the y axis is

$$\Theta=\begin{pmatrix}\cos\frac{1}{2}\bar{\theta} & -\sin\frac{1}{2}\bar{\theta}\\ \sin\frac{1}{2}\bar{\theta} & \cos\frac{1}{2}\bar{\theta}\end{pmatrix}. \tag{11}$$

THE φ DEPENDENCE OF BIPOLAR COORDINATES

We shall assume

$$x_1=uv^*/a, \quad x_1^*=u^*v/a, \quad x_0=(|v|^2-|u|^2)/2a. \tag{12}$$

This is consistent with the equations used in the previous section. For all points on the sphere we have the following equations

$$|x_1|=|x_1^*|=\tfrac{1}{2}a\sin\theta, \quad |u|=a\sin\tfrac{1}{2}\theta, \quad |v|=a\cos\tfrac{1}{2}\theta. \tag{13}$$

Therefore

$$|x_1|=|x_1^*|=|u|\,|v|=|uv|. \tag{14}$$

and Eq. (12) above merely specifies a phase factor.

[3] The development of bipolar coordinates in a plane was first given by W. T. Payne in an unpublished paper.

Now if the point P is such that the plane of OP and the z axis makes an angle φ with the xz plane,

$$x_1=\tfrac{1}{2}a\sin\theta e^{i\varphi}, \quad x_1^*=\tfrac{1}{2}a\sin\theta e^{-i\varphi}, \quad x_0=\tfrac{1}{2}a\cos\theta. \tag{15}$$

We may take

$$u=a\sin\tfrac{1}{2}\theta e^{i\alpha}, \quad v=a\cos\tfrac{1}{2}\theta e^{i\beta}. \tag{16}$$

To proceed further we note that the angle φ will appear positive from the right-hand reference

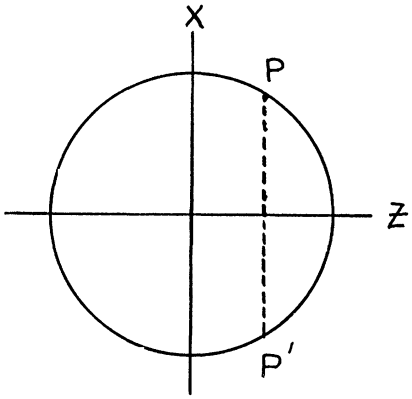

FIG. 4.

point and negative from the left-hand reference point. We take this as sufficient justification to assume that the φ dependence of u and v shall be opposite and set

$$\alpha=-\beta. \tag{17}$$

Making use of this relation, we obtain

$$\alpha-\beta=2\alpha=\varphi: \quad \alpha=-\beta=\tfrac{1}{2}\varphi \tag{18}$$

and

$$u=a\sin\tfrac{1}{2}\theta e^{i\frac{1}{2}\varphi}, \quad v=a\cos\tfrac{1}{2}\theta e^{-i\frac{1}{2}\varphi}. \tag{19}$$

Again we note that half-angles appear. The dependence on φ distinguishes among the points on a circle formed by the intersection of the sphere with a plane perpendicular to the z axis. No two points have the same coordinates, but again each point has two sets of coordinates—that is, the bipolar coordinates are double valued in φ as well as in θ.

The matrix for a rotation through $\bar{\varphi}$ about the z axis is

$$\Phi=\begin{pmatrix}e^{-i\frac{1}{2}\bar{\varphi}} & 0\\ 0 & e^{i\frac{1}{2}\bar{\varphi}}\end{pmatrix}. \tag{20}$$

THE ψ DEPENDENCE OF BIPOLAR COORDINATES

To obtain the ψ dependence we shall consider the bipolar coordinates

$$u = a \sin \tfrac{1}{2}\theta e^{i\frac{1}{2}\varphi}, \quad v = a \cos \tfrac{1}{2}\theta e^{-i\frac{1}{2}\varphi}. \tag{21}$$

First we rotate through φ about the z axis:

$$u' = a \sin \tfrac{1}{2}\theta, \quad v' = a \cos \tfrac{1}{2}\theta \tag{22}$$

and then through θ about the y axis:

$$u'' = 0, \quad v'' = a. \tag{23}$$

Now we apply a rotation through $-\psi$ about the z axis. This is the correct rotation since it is about the line joining the point with the origin. We had to make this line coincide with either the y axis or the z axis since these are the only two axes about which we know how to take rotations so far. The rotation through $-\psi$ gives

$$u''' = 0, \quad v''' = ae^{-i\frac{1}{2}\psi}. \tag{24}$$

Now we get the point back to the position specified by θ and φ by rotating through $-\theta$ about the y axis:

$$u^{iv} = a \sin \tfrac{1}{2}\theta e^{-i\frac{1}{2}\psi}, \quad v^{iv} = a \cos \tfrac{1}{2}\theta e^{-i\frac{1}{2}\psi}; \tag{25}$$

and finally through $-\varphi$ about the z axis:

$$u^{v} = a \sin \tfrac{1}{2}\theta e^{\frac{1}{2}i(\varphi-\psi)}, \quad v^{v} = a \cos \tfrac{1}{2}\theta e^{-\frac{1}{2}i(\varphi+\psi)}. \tag{26}$$

Again we note that half-angles appear, so that the coordinates are double valued in ψ and have period 4π. No two points on the sphere may have the same coordinates, but the ψ dependence makes possible an infinite number of coordinates for each point. We may write the general form for the bipolar coordinates of a point as follows:

$$u = a \sin \tfrac{1}{2}\theta e^{\frac{1}{2}i(\varphi-\psi)}, \quad v = a \cos \tfrac{1}{2}\theta e^{-\frac{1}{2}i(\varphi+\psi)}. \tag{27}$$

GENERAL ROTATION ABOUT THE ORIGIN

Let us perform an arbitrary rotation defined by the Eulerian angles $\bar{\varphi}$, $\bar{\theta}$, and $\bar{\psi}$ in that order. What will be the matrix for the transformation of the bipolar coordinates? Remembering that $\bar{\varphi}$ is a rotation about the z axis, we first operate on u and v with the matrix Φ. $\bar{\theta}$ is a rotation about the y axis, so we operate next with the matrix Θ. $\bar{\psi}$ is a rotation about the final z axis, and so finally we operate with the matrix Ψ, which is

$$\Psi = \begin{pmatrix} e^{-i\frac{1}{2}\bar{\psi}} & 0 \\ 0 & e^{i\frac{1}{2}\bar{\psi}} \end{pmatrix}. \tag{28}$$

Therefore the matrix U for the complete set of rotations is

$$U = \Psi\Theta\Phi \tag{29}$$

and direct substitution shows that this matrix is identical with U_p in Eq. (1).

SPIN FUNCTIONS AND THE SPIN VARIABLE

We shall interpret the spin variable as position on the sphere, and the restriction of the spin variable to two values implies that only two points on the sphere are of importance in discussing electron spin. We shall interpret this as meaning that the electron, which we may visualize as a point mass moving over the surface of the sphere, may be observed only when it occupies one or the other of the two points specified by the spin variable. Now, what is the geometrical relationship of these points? We recall that if the z component of spin is to be observed

$$\begin{aligned} a_1(\alpha_1) &= 1, \quad a_2(\alpha_1) = 0, \\ a_1(\alpha_2) &= 0, \quad a_2(\alpha_2) = 1. \end{aligned} \tag{30}$$

In our bipolar coordinate scheme

$$\begin{aligned} v_N &= a, \quad u_N = 0, \\ v_S &= 0, \quad u_S = a. \end{aligned} \tag{31}$$

If we identify v_N and u_N with $a_1(\alpha_1)$ and $a_2(\alpha_1)$, and v_S and u_S with $a_1(\alpha_2)$ and $a_2(\alpha_2)$, respectively, then α_1 and α_2 refer to the right- and left-hand reference points, respectively, and we may say more generally that α_1 and α_2 refer to points at opposite ends of the diameter parallel to the direction in which spin is being observed.

In the Pauli spin theory α_1 and α_2 are supposed to give information about the direction of spin and refer to spin in the positive and negative z directions, respectively. At the point N in the bipolar coordinate scheme u_N is zero, and v_N is equal to the diameter of the sphere. Now v_N is the coordinate of N relative to S and may be regarded as parallel to the z axis. Similarly u_S may be thought of as antiparallel to the z axis. Thus we may take the bipolar coordinates as associated with the direction of spin. If the axis of the bipolar coordinate picture and the direction of spin do not coincide, then none of the coordinates of the points on the spin axis is zero. In this case we think of the v's as associated with

the parallel and the u's with the antiparallel components of the spin.

There is one further implication of the scheme we have developed which should be mentioned. The electron has been taken as a point charge up to the present. This does not mean, however, that an observation on the diameter of the electron will give the value zero. For in the course of time the electron will occupy two points a distance a apart, and these two points may themselves rotate in space. This means that on the average the electron will appear smeared over the sphere and the observation would record a as the diameter of the electron. We shall compute a in terms of universal constants in a later section.

THE PAULI SPIN OPERATORS

To find the analog of the spin operators[4] in the bipolar coordinate scheme, we must investigate the spin variable somewhat more precisely. So far we have used the spin variable to reduce the configuration space of the electron to two points and to find the geometrical relationship of these two points. That is, we think of the electron as observable at only two points which are at opposite ends of a diameter of the sphere of diameter a. Now we have assumed throughout that only points on the sphere are of any significance, and if we wish to imagine a path for the electron between its observable points, we must think of it as being on the surface of the sphere. Thus we may think of the mechanism for the transfer of the electron from one point in its configuration space to the other as a rotation.

We assume that the electron travels with a velocity whose component in any direction is the velocity of light c. Then the resultant velocity will also be c from the Einstein formula for addition of velocities. The angular velocity will then be given by

$$\omega = c/(a/2) = c/c\Delta t = 1/\Delta t, \tag{32}$$

where Δt is the time required to traverse the radius at velocity c. Now θ, φ, and ψ differ by π at points located at opposite ends of a diameter, so that we may think of the change in α, the angle in question, in going from one point to the other to be of magnitude π. Then

$$\Delta\alpha/2\Delta t = \pi/2\Delta t = \tfrac{1}{2}\pi\omega = \tfrac{1}{2}\pi(\Delta\chi/\Delta t), \tag{33}$$

where χ is the angle implicit in the angular velocity. The term $2\Delta t$ is used in the denominator of (33) since the velocity component along the axis is c and the time required, therefore, to travel from one point to the other is $2\Delta t$.

Now, following the usual quantum mechanical procedure, we take the angular momentum operator to be

$$i\hbar\Delta/\Delta\chi = i(\hbar/2)\pi(\Delta/\Delta\alpha). \tag{34}$$

But since $\Delta\alpha$ is just π, the angular momentum operator reduces to

$$\tfrac{1}{2}i\hbar\Delta \tag{35}$$

or a finite rotation.

There will be three independent rotations about the sphere each giving a path joining the two reference points which the electron might follow. One is a great circle in the xz plane (rotation through π about the y axis); another is a great circle in the yz plane (rotation through π about the x axis); and the third must contain a rotation through π about the z axis. It may be taken as a rotation through $\frac{1}{2}\pi$ about the y axis followed by a rotation through π about the z axis and finally a rotation through $\frac{1}{2}\pi$ about the y axis.

Let us compute these rotations about the x, y, and z axes. The rotation about the x axis is given by a rotation through $\frac{1}{2}\pi$ about the z axis followed by a rotation through π about the new y axis and then by a rotation through $-\frac{1}{2}\pi$ about the final z axis:

$$R_{x,\pi} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \tag{36}$$

where the subscript x means that the rotation is to be taken about the x axis, and the subscript π means that the angle of rotation is π. For the y axis we just rotate through π about the y axis:

$$R_{y,\pi} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{37}$$

For the z axis we carry out the sequence of

[4] The idea of using difference operators was proposed by W. T. Payne.

rotations given above:

$$R_{z,\pi}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}. \quad (38)$$

Now the operators s_x, s_y, s_z of the Pauli theory should be given by

$$s_x=\tfrac{1}{2}i\hbar R_{x,\pi}, \quad s_y=\tfrac{1}{2}i\hbar R_{y,\pi}, \quad s_z=\tfrac{1}{2}i\hbar R_{z,\pi}, \quad (39)$$

as indeed they are.

SOME DEDUCTIONS FROM BIPOLAR COORDINATE THEORY[5]

We may compute the magnetic moment of the electron. If a particle of mass m and charge e moves in an orbit, then its magnetic moment $\mathfrak{y}_0$ is $e/2mc$ times its angular momentum $\mathbf{s}$, if the moment is measured in e.m.u. This is proved by considering the torque on the system due to an external magnetic field. The magnetic forces will act on the electron only when it is moving perpendicular to the field. If the electron is moving in a circle in the plane of the field, then at two diametrically opposite points in its orbit it will move perpendicular to the field and at points halfway between these two points it will move parallel to the field. Elsewhere it will have components of velocity parallel and perpendicular to the field. Thus we may say that the magnetic forces act effectively over only half of the orbit and the factor 2 in the denominator of the gyromagnetic ratio takes account of this fact. Now in the bipolar coordinate model of the electron the orbit is defined only at two diametrically opposite points of its orbit, so that the parts of the orbit over which the magnetic forces would vanish are missing. Hence the factor 2 should be omitted here, and the magnetic moment should be given by

$$\mathfrak{y}_0=(e/mc)\mathbf{s},$$

or for some component

$$\mu=e\hbar/2mc. \quad (40)$$

The magnetic moment in e.s.u. should be the charge of the electron times its velocity times the radius of the orbit. We have taken the velocity to be c and hence

$$c\mu_0=ceh/2mc=eca/2: \quad (a/2)=h/2mc. \quad (41)$$

This is what we have decided to call the radius of the electron. It is (137/2) times the radius deduced from the electrostatic energy of the electron.

Our spin functions are double valued, and hence the wave-length to be associated with the spinning motion should be twice the circumference of the orbit or

$$\lambda=4\pi(a/2)=2\pi h/mc=h/mc, \quad (42)$$

which is the Compton wave-length.

THE RELATIVISTIC PROPERTIES OF BIPOLAR COORDINATES

The relativity transformations include the time as well as the usual space variables. This means that we shall have to examine the introduction of time into the picture rather carefully. In introducing the time in the previous sections we assumed implicitly that the points on the sphere of radius $\frac{1}{2}a$ did not all have the same proper time, for a point is of importance only when we can imagine the electron passing through it, and at a given instant only one point may be occupied by the electron. We shall take the proper time of the right-hand reference point to be zero and that of the left-hand reference point to be 2τ. This means that the time required to travel from the right-hand to the left-hand reference point at velocity c is 2τ where the radius is taken as $\frac{1}{2}a=c\tau$ as before. The proper times of the points elsewhere on the sphere will be between zero and 2τ. Our choice of $t=0$ at N is a little arbitrary; it would do equally well to take $t=0$ at S and $t=2\tau$ at N. This alternative choice will be discussed later.

The bipolar coordinates of a point on a sphere will have exactly the same spatial characteristics as before, and we need not repeat the discussion for rotations. We may, therefore, pass directly to Lorentz transformations. It may be shown immediately that the bipolar coordinates of a point do not transform like Dirac spin functions under a Lorentz transformation. However, a quantity very closely connected with them does.

[5] The considerations in this section are due to W. T. Payne. They were first deduced from a model similar to the bipolar coordinate model of the electron, but much less complete.

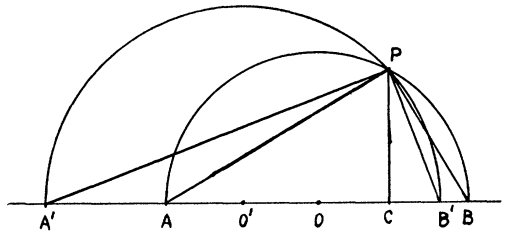

FIG. 5.

We shall consider relative motion in the z direction only since any other motion may be obtained by combining this with the appropriate rotation. We have

$$\begin{aligned} x'&=x \quad z'=z\cosh\mu+ct\sinh\mu,\\ y'&=y \quad ct'=z\sinh\mu+ct\cosh\mu, \end{aligned} \tag{43}$$

where $\cosh\mu$ and $\sinh\mu$ are as defined in the first section. Equation (43) may be written as

$$\begin{aligned} x'&=x \quad (z'+ct')=(z+ct)e^{\mu},\\ y'&=y \quad (-z'+ct')=(-z+ct)e^{-\mu}. \end{aligned} \tag{44}$$

These equations hold for all z and ct whether they are represented on the sphere or not. In particular

$$\begin{aligned} (-z'+c\tau')&=(-z+c\tau)e^{-\mu},\\ (z'+c\tau')&=(z+c\tau)e^{\mu} \end{aligned} \tag{45}$$

holds for all z where $c\tau$ is the radius of the sphere. Of course, the point $(z, c\tau)$ occurs on the sphere only at $z=0$. We have the equations

$$\begin{aligned} z&=(|v|^2-|u|^2)/2a,\\ c\tau&=\tfrac{1}{2}a=(|v|^2+|u|^2)/2a \end{aligned} \tag{46}$$

and hence

$$-z+c\tau=|u|^2/a, \quad z+c\tau=|v|^2/a. \tag{47}$$

Substituting in (44), we obtain

$$\begin{aligned} u'/\sqrt{a'}&=(u/\sqrt{a})e^{-\frac{1}{2}\mu},\\ v'/\sqrt{a'}&=(v/\sqrt{a})e^{\frac{1}{2}\mu}, \end{aligned} \tag{48}$$

where $a'=2c\tau'$. $u/\sqrt{a}$ and $v/\sqrt{a}$ rather than the bipolar coordinates transform like a pair of Dirac spin functions. These quantities are the square roots of the distances between the intersection of the normal from the point on the sphere to the z axis and the reference points.

The transformation of a point on the sphere is shown in Fig. 5. For a positive velocity of the primed system with respect to the unprimed system, $e^{\mu}>1$, so that BC is shortened and AC lengthened in the transformation.

REFLECTIONS AND FOUR COMPONENT SPIN FUNCTION

It is easily verified that under a space inversion

$$u'=v^*, \quad v'=-u^*. \tag{49}$$

Let us change the notation slightly and take

$$u=u_1, \quad v=v_1, \quad u'=u_2, \quad v'=v_2, \tag{50}$$

where now u_1 and v_1 are the bipolar coordinates of some point on the sphere and u_2 and v_2 are the bipolar coordinates of the diametrically opposite point. If now we call u_1, v_1, u_2, and v_2 the four functions to be associated with a given point, we may obtain linear transformations for reflections in space. The matrix for the transformation under an inversion is easily seen to be identical with that given in the first section, Eq. (4). Now it will be recalled that the bipolar coordinates are double valued in all of their variables and, therefore, we may associate two laps with any rotations. The fact that the δ's appear with the same sign in U_I means that we are to think of the two points being in the same lap in the path of the electron.

Let us now consider the time reversal. We have so far assumed that $t_N=0$ and $t_S=2\tau$. A time reversal means that we take $t_N=0$ and $t_S=-2\tau$. Now if $t_S=-2\tau$, we must think of the electron as moving from S to N instead of from N to S as previously. Evidently, then, N and S interchange their roles as in the space inversion. In addition, however, one of the two points at opposite ends of a diameter changes laps because the motion is taking place at essentially 2τ seconds earlier. If originally the electron was moving through P on its way to S from N and now it is moving through P' on its way to N from S, 2τ seconds later it will be at its original position at its original time. But as it goes through N it must change laps. On the other hand, if the electron were originally at P' and the time reversal were performed it would find itself at P 2τ seconds earlier, but it would be on the same lap since it passes only through S and not through N before arriving at the original position. Now the bipolar coordinates of a given point differ only by sign in the two laps, and hence the transformation is to be given by U_T in Eq. (4).

Evidently u_2 and v_2 transform just like u_1 and

v_1 under rotations, and it remains to investigate how they will transform under a Lorentz transformation. u_2 and v_2 will be associated with a point whose z coordinate is the negative of the point associated with u_1 and v_1, and hence if we change the sign of a in Eq. (47) and replace u and v by u_2 and v_2, respectively, we shall obtain the correct transformation. This gives

$$u_2'/\sqrt{a'} = (u_2/\sqrt{a})e^{\frac{1}{2}\mu},$$
$$v_2'/\sqrt{a'} = (v_2/\sqrt{a})e^{-\frac{1}{2}\mu}. \quad (51)$$

The Lorentz transformation for the set of functions u_1, v_1, u_2, and v_2 is thus U_L in Eq. (5), and we have complete agreement with the Dirac theory as far as transformation properties go.

In conclusion the author would like to express her deep appreciation of the many helpful and encouraging discussions with Professor Leon Brillouin, under whose direction the work was carried out, and with Dr. William T. Payne, to whom she is also indebted for the use of his unpublished work on the subject.

PHYSICAL REVIEW VOLUME 65, NUMBERS 1 AND 2 JANUARY 1 AND 15, 1944

Adsorption of Thorium on Tantalum*

CHARLES J. GALLAGHER
Department of Physics, University of Notre Dame, Notre Dame, Indiana
(Received September 30, 1943)

It is known that thorium adsorbed on the surface of tungsten or molybdenum results in a decrease in the work function and also a decrease in the value of A, the emission constant. The behavior of thorium on tantalum was investigated, and similar effects were noted. Thorium was evaporated from thoriated tungsten onto a well-outgassed tantalum ribbon, and the thermionic emission at 1390°K was studied as a function of the time of evaporation. The work function of the ribbon changed from 4.07 volts to 2.52 volts, while A decreased by a factor of thirty. The ribbon was then held at 1550°K, and the emission from the two sides was studied separately. The emission from the activated side decreased, while that of the clean side increased until both reached the same final level after about twenty hours. This indicates that the adsorbed thorium had migrated around the edges of the ribbon and distributed itself uniformly over the emitting surface.

INTRODUCTION

THE thermionic properties of metals are highly sensitive to the presence of adsorbed surface layers. For example, thorium adsorbed on tungsten or molybdenum is known to produce very large increases in the thermionic emission. Similar results are reported for other electropositive elements deposited on these metals.[1] It is also known that these adsorbed atoms are able to migrate over the underlying surface in the direction of decreasing concentration at a rate dependent on the temperature. Evidence for this migration effect has been obtained by measuring the emission from different parts of the surface[2] and by taking photographs of projection tube patterns.[3–5]

It was the object of the present experiments to determine whether thorium would produce similar effects if adsorbed on a surface of tantalum. A tube similar to that used by Brattain and Becker[2] was designed to permit separate measurement of the thermionic emission from either side of a tantalum ribbon. Thoriated tungsten filaments were mounted so that thorium

* Submitted to the Committee on Graduate Study of the University of Notre Dame in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

[1] A. L. Reimann, *Thermionic Emission* (John Wiley & Sons, 1934), pp. 99, 134.

[2] W. H. Brattain and J. A. Becker, Phys. Rev. **43**, 428 (1933).

[3] E. Brüche and H. Mahl, Zeits. f. tech. Physik, No. 12, 623 (1935); also, No. 8, 262 (1936).

[4] M. Benjamin and R. O. Jenkins, Proc. Roy. Soc. **A176**, 262 (1940); also, **A180**, 225 (1942).

[5] A. J. Ahearn and J. A. Becker, Phys. Rev. **54**, 448 (1938).