approximation. Although there is no *a priori* assurance of the over-all reliability of the high temperature expression in general, one is led to expect a suitable computational result for any given case by means of a plausible interpolation between the low and high temperature curves.

The elastic and total (elastic plus inelastic) scattering cross sections are plotted in Fig. 2. Since $(3\pi^2\hbar^2\tau^2/mk_0\Theta)\ll 1$ here $(\cong .007a^2\tau^2)$, the predicted constancy of the total cross section at low temperatures is clearly exhibited. The constancy is strict up to 150°K; but even up to 400°K, the entire variation (a monotonic decrease) is no more than a percent. At 1000°K, the value has decreased to 90 percent of the free elastic cross section, from the maximum of 97 percent at absolute zero. Such a variation should be readily detected by a reasonably sensitive experiment.

VII. ACKNOWLEDGMENT

The author wishes to express sincere appreciation of the assistance and stimulation to this research given by his fellow members of the Physics Department at Stanford University. He owes his particular thanks to Dr. and Mrs. Morton Hamermesh, for numerous pieces of advice and encouragement; to Professor Paul Kirkpatrick for full cooperation in obtaining assistance with the numerical work; and especially to Professor Felix Bloch for suggesting the problem, for having supplied much of the intellectual background necessary to its solution, and for many interesting and valuable personal discussions during the solution.

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The Evolution of Contracting Stars

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By use of the method of successive homology transformations in conjunction with Bialobjeski-Eddington's formula, it is possible to predict the changes of radii and luminosities of massive stars in different stages of their hydrogen and contractive evolution. It is shown that at a certain stage of contraction the star must reach the maximum of its luminosity, and it is calculated that for the star masses of 5, 10, 20, and 40 suns the maximum luminosities are 1×10^5, 4×10^5, 1×10^6, and 4×10^6 suns, respectively. In the later contractive stages total luminosity of the star remains constant whereas its visual luminosity rapidly decreases because of the shift of energy into the ultraviolet part of the spectrum. It is to be expected that for such high values of the (luminosity)/(mass) ratio, radiation pressure becomes strong enough to eject stellar atmospheres into the surrounding space, and it is shown that the ejection will actually take place if the force of gravity on the stellar surface will be somewhat reduced by the centrifugal force due to axial rotation. It is also shown that under the conditions existing in the ejective atmospheres of Wolf-Rayet stars, radiation pressure is primarily due to the light scattering by free electrons. In discussing the motion of the ejected gases, it may be necessary to assume that their original velocity of about 2000 km/sec. can be considerably reduced by the gravitational action of the star. If this is the case, gaseous envelopes which will form around Wolf-Rayet stars in the course of several centuries will possess properties very similar to those of the planetary nebulae. This would indicate a close evolutionary relationship between these two classes of celestial objects. An alternate evolutionary road of a massive star consists in the formation of an energy-producing shell, which will take place in all cases where the convective currents due to axial rotation are not fast enough to secure homogeneity of stellar matter. It is indicated that the growth of such shells will probably lead to the formation of an extensive atmosphere which offers certain possibilities for the interpretation of the so-called red giant stars as intermediate evolutionary stages between the stars of the main sequence and the Wolf-Rayet stars. In conclusion the problem of stellar collapse, which is expected to take place towards the end of contractive evolution, is discussed in some detail.

1. INTRODUCTION

It is generally accepted at present that during the largest part of their evolutionary life stars receive their energy supply from thermonuclear reactions in which hydrogen is being transformed into helium through the "catalytic" action of carbon and nitrogen nuclei (C – N cycle).[1] In the case of the sun, for which the central temperature, central density, and the hydrogen content are determined by the direct numerical integration of the fundamental equilibrium equations, the calculated rate of energy liberation of C – N cycle is in excellent agreement with the observed value of luminosity.

By assuming the sun as the starting point, and using the method of homology transformations, it is possible to calculate the observable characteristics of stars with various masses and various H contents. The fundamental equations of stellar equilibrium can be written in the well-known form:[2]

$$dp/dr = -(GM_r\rho/r^2), \tag{1a}$$

$$dM_r/dr = 4\pi r^2\rho, \tag{1b}$$

$$p = (R/\mu\beta)\rho T, \tag{1c}$$

$$L/4\pi r^2 = -(c\sigma/3k\rho)(dT^4/dr), \quad (k = k_0\rho^\alpha T^{-s}) \tag{1d}$$

$$L = A\cdot x\cdot r_0^3\rho_c^2 T_c^n. \tag{1e}$$

Here ρ, p, and T represent the density, total pressure, and temperature in stellar interior (index c pertaining to the center); μ, k, and $1-\beta$, the mean molecular weight, coefficient of opacity, and relative radiation pressure, respectively; x the H content, r_0 the radius of the small central zone where the energy is mainly produced, and n the exponent in the formula for the temperature dependence of the thermonuclear reaction. (For C – N cycle at solar central temperature $n = 18$, decreasing as $T_c^{-\frac{1}{3}}$ in the case of hotter stars.) The value of β can be assumed to be approximately constant through the body of any given star, being entirely determined by the mass of the star through the well-known Bialobjesky-Eddington equation:[3]

$$[M/M(\odot)]\mu^2 = 18.0\cdot[(1-\beta)/\beta^4]^{\frac{1}{2}}. \tag{2}$$

Applying the method of homology transformations to the equations [(1a)–(1e)], we can express the relative changes of stellar radius R, luminosity L, and central temperature T_c through the changes of M, $\mu\beta$, k_0, and x. We find:

$$R \sim M^{(n+\alpha-s-1)/(n+3\alpha-s+3)}(\mu\beta)^{(n-s-4)/(n+3\alpha-s+3)} \times k_0^{1/(n+3\alpha-s+3)}x^{1/(n+3\alpha-s+3)}, \tag{3a}$$

$$L \sim M^{[(3+2\alpha)n+3\alpha+s+9]/(n+3\alpha-s+3)} \times (\mu\beta)^{[(4+3\alpha)n+3s+12]/(n+3\alpha-s+3)} \times k_0^{-(n+3)/(n+3\alpha-s+3)}x^{-(s-3\alpha)/(n+3\alpha-s+3)}, \tag{3b}$$

$$T_c \sim M\mu\beta/R. \tag{3c}$$

For temperatures between 1×10^7 °C and 5×10^7 °C, and densities between 1 and 100 g/cm^3 (i.e., within the range of temperature and density variations for the normal stars of the main sequence) the changes in opacity can be fairly well represented by the exponent values $\alpha = 0.5$ and $s = 2.75$ (Bethe's approximation), and our expressions become:

$$R \sim M^{(n-3.25)/(n+1.75)}(\mu\beta)^{(n-6.75)/(n+1.75)} \times k_0^{1/(n+1.75)}x^{1/(n+1.75)}, \tag{3'a}$$

$$L \sim M^{(4n+13.25)/(n+1.75)}(\mu\beta)^{(5.5n+20.25)/(n+1.75)} \times k_0^{-(n+3)/(n+1.75)}x^{-(1.25)/(n+1.75)}, \tag{3'b}$$

$$T_c \sim M\mu\beta/R. \tag{3'c}$$

In order to calculate the values of R and L for stars of different masses in the early stages of their H evolution, we assume that all stars possess originally the same hydrogen content $x = 0.35$.[4] To take into account the variability of the exponent n which depends on the central temperature, we carry our calculations step by step for the star masses $M(\odot)$, $5M(\odot)$, $10M(\odot)$, $20M(\odot)$, and $40M(\odot)$, using new values of n in each successive step. The values of β for the masses used are given by Eq. (2) (with $\mu = 1$) and are, respectively: 1.00, 0.95, 0.85, 0.70, and 0.55. As the results of such calculations, we obtain for $\log[R/R(\odot)]$: +0.52, +0.70, +0.86, and +1.02, and for $\log[L/L(\odot)]$: +3.0, +3.9, +4.6 and +5.3.

[1] C. v. Weizsäcker, Physik. Zeits. **39**, 633 (1938); H. Bethe, Phys. Rev. **55**, 434 (1939).

[2] Compare, for example, G. Gamow, Phys. Rev. **53**, 595 (1938); **55**, 718 (1939).

[3] This equation was primarily developed for the so-called standard model. However, it also gives very good results in application to the point-source and contractive models. Compare: S. Chandrasekhar, *Introduction to the Study of Stellar Structure* (Chicago University Press, 1939).

[4] This value corresponds to the sun which, because of its comparatively low luminosity, could not have used but a small fraction of its original H content during a few billion years of its existence.

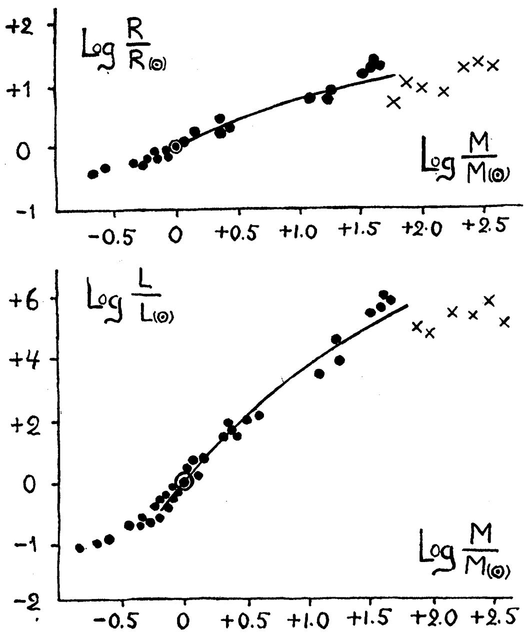

FIG. 1. Dependence of radius and luminosity on the mass for the stars of the main sequence. Continuous curve—theory; solid dots—observation; crosses—the data for Trumpler stars.

The calculated $\log[R/R(\odot)]$ and $\log[L/L(\odot)]$ are plotted in Fig. 1 as the functions of $\log[M/M(\odot)]$ (continuous curves); the separate black dots represent the observed values for individual stars for which reliable data on M, R, and L are at present available.[5] A very good agreement between the observed and calculated values indicates that even for rather massive stars, the method of homology transformations in conjunction with Bialobjesky-Eddington's relation gives rather precise results. We are thus justified in using this method for the calculation of evolutionary changes produced in massive stars by the change of hydrogen content and the subsequent gravitational contraction.

It must be noticed in the conclusion of this section that there is known a special group of very luminous stars which do not seem to fit on the regular $R-M$ and $L-M$ curves.[6] Whereas the radii and luminosities of these so-called "Trumpler stars" seem to indicate normal masses of the order of $50M(\odot)$, mass estimates carried out by Trumpler[7] on the basis of the red-shift analysis lead to the values of several hundred sun masses. We shall also see in the next section that Trumpler's mass values are inconsistent with the alternative hypothesis that these stars represent the later stages of contractive evolution. It seems, therefore, that the high mass values claimed for these stars do not correspond to reality and are due to the erroneous interpretation of the observed shift of spectral lines as a purely gravitational red shift. It may well be that part of the shift is due to the ordinary Doppler effect taking place in the gas masses ejected from the surfaces of these stars, and, in fact, such ejection can be very well expected for the stars of such extremely high luminosity.

2. LUMINOSITY CHANGES DURING THE CONTRACTION

Since the stars of the main sequence produce their energy by transforming hydrogen into helium, their H content must gradually decrease with the time. The change of hydrogen content x with the time is evidently given by:

$$\Delta x = -Lm_{\mathrm{H}}/\epsilon M \cdot \Delta t, \tag{4}$$

where m_{H} is the mass of a hydrogen atom, and ϵ the energy liberation per proton. For the C$-$N cycle $\epsilon = 1.0\times 10^{-5}$ erg. Applying (4) to the sun, we find that during the two or three billion years of its existence it could not have consumed much more than about one percent of its original H content. Since, however, the ratio L/M increases rapidly with the mass of the star $[(L/M)\sim M^{3.3}]$, heavier stars must consume their hydrogen considerably faster. Thus, for example, Sirius A would consume all its hydrogen in just about 3×10^9 years, whereas for the star with the mass $10M(\odot)$ it would require only 4×10^7 years. This fact leads to the well-known difficulty concerning the necessity of the assumption that many stars have been formed at comparatively recent epochs. On the other hand, we are also forced to the conclusion that there must exist a considerable number of massive stars which have by now completely exhausted their H supply and are at present in the state of ultimate contraction.

[5] G. P. Kuiper, Astrophys. J. **88**, 472 (1938).
[6] In Fig. 1 the positions of Trumpler stars are shown by crosses.
[7] R. J. Trumpler, Pub. Astronom. Soc. Pac. **37**, 249 (1935).

We will now try to describe the changes in the luminosities and diameters of the massive stars evolving from their original positions on the main sequence towards the later contractive stages, and we shall use for that purpose the set of homology transformations derived in the previous section. When 35 percent of stellar matter is transformed from H into He (the remaining 65 percent being the so-called Russell mixture of heavier elements) the mean molecular weight μ increases from 1.0 to 1.7. The value of k_0, on the other hand, will remain practically unchanged since both H and He contribute but very little to the total opacity. Taking $n=15$, as an average,[8] we find from (3′) that the radius luminosity and central temperature of any given star will be subject to the relations

$$R \sim (\mu\beta)^{0.52} x^{0.06}, \tag{5a}$$

$$L \sim (\mu\beta)^{6.10} x^{-0.07}, \tag{5b}$$

$$T_c \sim (\mu\beta)^{0.48} x^{-0.06}. \tag{5c}$$

We see that unless x becomes very small as compared with its original value, the changes of R and L will be caused entirely by the variation of μ; the representative point of the star in the $R-L$ diagram will move along the line: $L \sim R^{6.10/0.52} \sim R^{12}$. When most of H is transformed into He, μ rises from 1.0 to 1.7 and, according to (2), the values of β (for the same mass) must increase correspondingly. For the star masses used above, we find the new β-values to be: 1.00, 0.78, 0.62, 0.47, and 0.35. The formulas (5a, b) give us now for $\Delta \log [R/R(\odot)]$: $+0.07$, $+0.05$, $+0.03$, and $+0.02$, whereas for

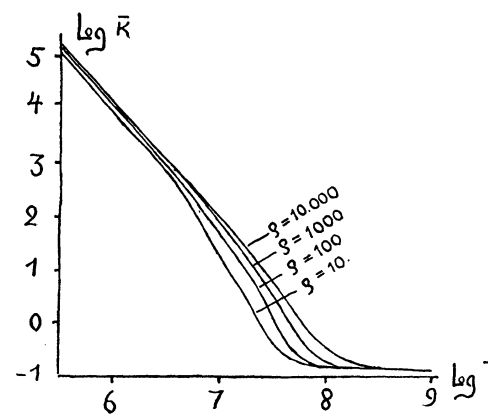

FIG. 3. Morse's curves for the coefficient of opacity at high densities and temperatures.

$\Delta \log [L/L(\odot)]$ we get: $+0.85$, $+0.55$, $+0.3$, and $+0.2$. The central temperatures rise only by a few percent remaining generally in the region of applicability of Bethe's approximation of the opacity law. In Fig. 2 we have plotted the main-sequence positions for different star masses in the frame of the $R-L$ diagram; the parts of evolutionary tracks corresponding to the H consumption are marked as I, and the positions obtained for $\mu=1.7$ are indicated by empty dots.

When the largest part of H is transformed into He, μ becomes practically constant, and further changes of R and L are given by the x power in the expressions (5). The radius begins to decrease whereas the luminosity increases inversely with the radius. Corresponding parts of evolutionary tracks are marked as II in Fig. 2. When x becomes very small, thermonuclear energy production gives place to purely gravitational energy liberation, and the star will slowly readjust itself from the point source to contractive model. This change of the model results in a slight increase

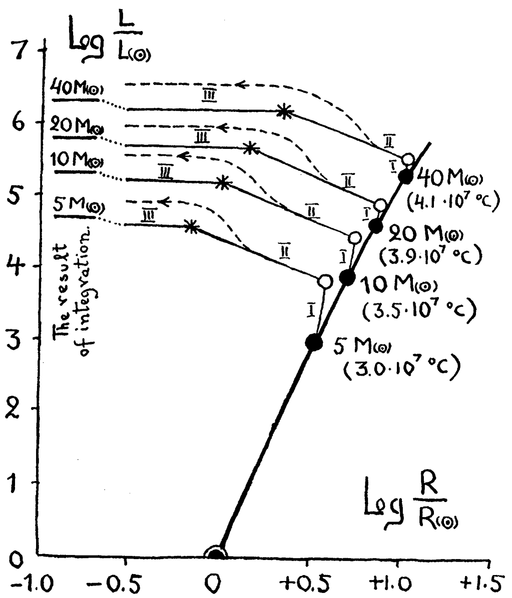

FIG. 2. Construction of evolutionary tracks for massive stars in the luminosity radius diagram.

[8] For the temperature range involved n changes only from 16 to 14.

of the luminosity, which will be taken into account somewhat later.

Because of the rapid decrease of the radius during this stage of evolution, the value of T_c rises very appreciably ($T_c \sim R^{-1}$), and we are soon brought into the region where the ordinary opacity formula is no more applicable. We see, in fact, from Fig. 3, representing the results of Morse's[9] opacity calculations for high densities and temperatures, that near $T=1\times10^8$ °C the value of k suddenly becomes constant. This is due to the fact that for higher temperatures absorption is mainly due to the Compton scattering by free electrons. Although this change of the opacity law begins near the stellar center and then gradually spreads out as the contraction proceeds, we can assume for the purpose of our calculation that this change takes place in the entire body of the star when about one-half of its mass is heated above 1×10^8 °C. Since, further, the temperature in the layer corresponding to $M_r=0.5M$ is about one-half of the central temperature, a reasonable approximation will be achieved by changing the model at the point where $T_c=2\times10^8$ °C. Using the fundamental equilibrium equations (1) with $k=$const., we find easily that under such conditions the luminosity of the star becomes independent of its radius. Estimating the points for which $T_c=2\times10^8$ °C, we find for the changes along these parts of the tracks II: $\Delta \log [R/R(\odot)]=-0.76$, -0.73, -0.68, and -0.67, and $\Delta \log [L/L(\odot)]=+0.76$, $+0.73$, $+0.7$, and $+0.7$. In Fig. 2 these points are indicated by asterisks; beyond them evolutionary tracks run parallel to the R axis. Thus we come to the conclusion that *by the end of the contractive evolution every massive star comes to the state in which its total luminosity becomes constant.*[10] These maximum luminosities for the star masses $5M(\odot)$, $10M(\odot)$, $20M(\odot)$, and $40M(\odot)$ are: $\log L/L(\odot)=+4.6$, $+5.2$, $+5.6$, and $+6.2$.

In order to get some idea of the accumulated errors in our method which uses three successive homology transformations, we can compare the obtained luminosity values with the direct homology relation between L and M for $k=$const. models. Using again the first four equations (1) with $k=$const. we get:

$$L \sim M^3\beta^4. \tag{7}$$

With the above values of β (for $\mu=1.7$) the relation (7) gives:

$$L_{10}/L_5=3.2; \quad L_{20}/L_{10}=2.6; \quad L_{40}/L_{20}=2.5,$$

while the ratios of calculated luminosities are 3.7, 2.7, and 3.7. Taking into account the approximate character of the method used, we can consider the agreement quite satisfactory.

We can also compare our results with the exact numerical integrations of $k=$const. model which were carried out by Keenan[11] (for a complete point-source model) and by Henrich[12] (for a point-source model with a convective core). For $\mu=1.7$ and the star masses used in the present article, Henrich's calculation leads to $\log [L/L(\odot)]=4.7$, 5.3, 5.8, and 6.3. A very good agreement between these exact values and those obtained by the method of successive homology transformations indicates again that *our method can be used safely for the calculation of evolutionary changes of massive stars.*

It must be noticed here that, using the homology transformations based on the sun, we have been working all the time with the point-source model and that the integrations of Henrich, with which we compared our results, have also been carried out for that particular model. As it was already mentioned above, the use of the point-source model is permissible only up to a certain point on the evolutionary track II where nuclear energy generation gives place to purely gravitational contraction. Beyond that stage the star must be described by a contractive model (polytrope of the index 3), and the transition between two different models cannot be in general followed by the methods of homology. It was, however, shown by Biermann[13] that for equal masses, radii, and opacities, the contractive model leads to luminosities which are about twice as high as in the case of point-source model (this is due to the more uniform distribu-

[9] Ph. M. Morse, Astrophys. J. **92**, 27 (1940).

[10] This holds, of course, only for star masses above Chandrasekhar's critical mass: $5.75\mu^{-1} \cong 2M(\odot)$. For lighter stars the electron gas in the interior becomes degenerate at a certain stage of contraction, the luminosity begins to decrease, and the star evolves into a white dwarf.

[11] Ph. C. Keenan, Astrophys. J. **89**, 499 (1939).

[12] L. R. Henrich, Astrophys. J. (Sept. 1943).

[13] L. Biermann, Zeits. f. Astrophys. **3**, 116 (1931).

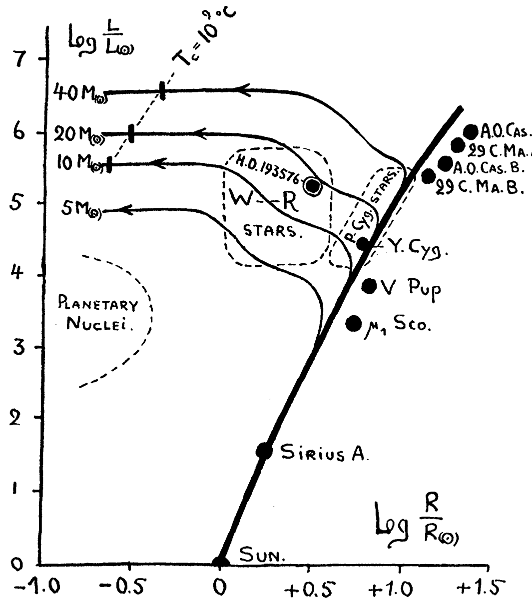

FIG. 4. Comparison of calculated evolutionary curves with the results of observation.

tion of the energy sources in the star). Thus the later parts of evolutionary tracks must run somewhat higher than originally calculated (broken lines in Fig. 2), and the maximum luminosities become approximately: $[\log (L/L(\odot))]_{cor} = 5.0, 5.6, 6.1$, and 6.6. Similar corrections must also be applied to the results of Henrich calculations.

Comparing our results with observational material, we find that there actually exist a number of stellar objects which seem to fit the predicted properties of stars in the late stages of their evolution. While the lower part of the main sequence is represented by a narrow band containing the stars whose M, R, and L values are in a good accord with C−N cycle assumption, the upper part of the sequence is bordered by stars with anomalously small radii, and luminosities which are seemingly too high for their masses. In Fig. 4 we give the comparison between the calculated evolutionary tracks and the observed data for some massive stars. We find that such stars as U Ophiuchi, μ Scorpi, V Puppis, and both components of 29 Canis Majoris and A O Cassiopeiae seem to be normal H-consuming stars. On the other hand in Y Cygni we probably have a case of rather progressed H evolution since its luminosity is definitely too high for its mass. In the same general region of $R-L$ diagram lies the group of stars known as P Cygni stars. Although mass values are not available for any member of that group, we can conclude that their luminosities are abnormally high for their masses from the observations which indicate that all these stars are ejecting their atmospheres into the surrounding space; we shall see indeed in the following sections that such ejection requires rather high (L/M) values. A still more typical group is formed by the so-called Wolf-Rayet stars which are generally similar to P Cygni stars with the difference that the atmospheric ejection process is considerably more vigorous. W-R stars lie considerably farther from the main-sequence line than P Cygni stars, and can be thus assumed to be in a later stage of their contractive evolution. The best available data pertain to one W-R star which happens to be the component of a binary system (H.D.—193576), thus permitting an estimate of its mass. According to Wilson,[14] the mass of the W-R component is $9.7M(\odot)$, its radius $\sim 3R(\odot)$, and its total luminosity about $2\times 10^5 L(\odot)$. Thus the star is about 20 times as luminous as could be expected from the ordinary mass-luminosity relation. Plotting Wilson's data on the diagram of Fig. 4, we find that the representative point of this star is located very close to the calculated evolutionary track for the mass $10M(\odot)$.

3. SOME REMARKS ABOUT SHELL FORMATION

In the considerations of the previous section it was tacitly assumed that chemical changes produced by thermonuclear reactions near the stellar center are propagated through the entire body of the star sufficiently fast to secure the homogeneity of its material in every given stage of the evolution. This is equivalent to the assumption of some convective currents circulating between stellar center and the surface with the period which is shorter than the rate of stellar evolution. It was shown by Eddington[15] that such currents must actually exist within the star, being produced by asymmetry resulting from the axial rotation. He estimated the period of these currents to be of the order of 10^{13} years which

[14] O. C. Wilson, Astrophys. J. **91**, 379 (1940).
[15] A. Eddington, M. N. R. A. S. **90**, 54 (1929).

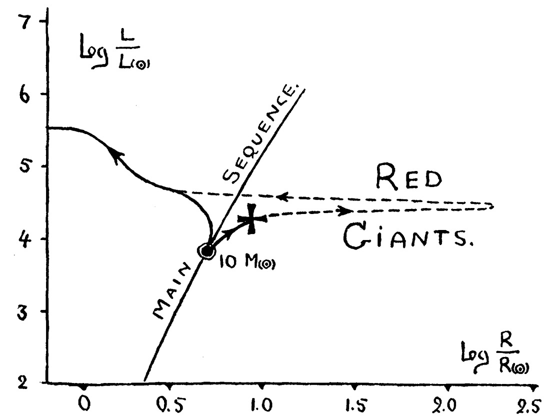

FIG. 5. Possible evolutionary track for a star with a growing energy-producing shell. Vertical cross indicates the place beyond which isothermal and radiative equilibrium solutions cannot be fitted together according to Chandrasekhar and Schoenberg.

is definitely too long to secure any mixing-up of stellar matter. It was, however, indicated by Randers[16] that Eddington's value of circulation period represents most probably an enormous overestimate since it does not take into account the possibility of turbulent motion within the star. If, instead of the ordinary coefficient of gaseous viscosity, used by Eddington, we take the so-called "eddy diffusivity" obtained from the observation of the turbulent gas motion on the solar surface, the value of the circulation period drops from 10^{13} to only 10 years! It is true, of course, that the turbulence inside of the stellar body should be expected to be considerably smaller than on the surface, so that the above value of ten years represents only a lower limit of the circulation period. Thus a more detailed study is necessary before we will be able to decide whether convective currents due to rotation can secure the equipartition of material in the stellar body. Should the period of these currents turn out to be shorter than say 10^6 or 10^7 years, the considerations of the previous sections will hold. In the opposite case the star will develop a "shell source" of energy slowly "growing" in size as more and more hydrogen is being consumed. The stellar model corresponding to this case will consist of: (1) isothermal core with zero hydrogen content ($\mu = 1.7$) deprived of any energy sources; (2) thin energy-producing shell with the temperature suitable for C−N cycle; (3) outer envelope with the original H content ($\mu = 1$). Such a model was studied by Critchfield and the author,[17] and more recently by Chandrasekhar and Henrich,[18] and Chandrasekhar and Schoenberg.[19] The latter investigators discovered a very peculiar fact that the fitting of isothermal-core solution, with the radiative-equilibrium solution of the envelope becomes mathematically impossible as soon as the mass contained in the core exceeds ten percent of the total stellar mass. Since, due to the progressing thermonuclear reactions, the amount of the dehydrogenized material increases steadily, we can ask ourselves what happens when the mass of the growing core trespasses the permitted ten percent. It was suggested by Chandrasekhar and his collaborators that the impossibility of solution for larger core masses will result in a non-equilibrium transition which could be interpreted as the explosion of the star. This point of view is, however, not quite satisfactory since there seem to be no other stable states of lower energy into which such transition can lead. It appears, in fact, that the difficulty with the upper limit of the core mass lies essentially not in the general stability problem, but rather in the boundary conditions usually assumed on the surface of the star. The solutions for the envelope used by Chandrasekhar and his collaborators in their calculations were obtained under the assumption that both density and temperature vanish on the outer boundary of the star, so that the impossibility of obtaining the fit with the isothermal core in the interior means actually that the conditions $\rho_{\mathrm{surf}} = 0$ and $T_{\mathrm{surf}} = 0$ can no longer be fulfilled simultaneously. This result becomes particularly clear if, instead of trying to fit two solutions in the middle of the star, we carry the integration straight forward from the center to the surface. If the mass of the core is sufficiently small, it will always be possible to choose initial conditions (i.e., ρ_c and T_c) in such a way that both density and temperature vanish for certain finite value of stellar radius. If, how-

[16] G. Randers, Astrophys. J. **94**, 109 (1941). The author is grateful to Dr. Randers for the discussion on energy and paraplyen transfer.

[17] Chas. Critchfield and G. Gamow, Astrophys. J. **89**, 244 (1939).

[18] S. Chandrasekhar and L. R. Henrich, Astrophys. J. **94**, 525 (1941).

[19] S. Chandrasekhar and M. Schoenberg, Astrophys. J. **96**, 161 (1942).

ever, the mass of the core becomes too large, the point where the surface conditions can be fulfilled moves into the infinity; this is in accord with Chandrasekhar's result that the rate of radius changes becomes infinite when the mass of the core approaches the critical value of 10 percent. Although this would mean mathematically that the star expands into infinity, physically it is not necessarily so. We know, in fact, that the opacity formula of Kramers' type is valid only for rather high temperatures and that for temperatures below several thousand degrees the opacity of gaseous material drops rather sharply due to the absence of sufficient ionization. It follows that the ordinary solution of stellar interior should be carried out only up to a certain non-vanishing temperature corresponding to the photospheric layer, and that beyond that limit the transparent chromospheric layer should be fitted on.[20] *This will lead to a star model of finite dimensions although its radius will be abnormally large and central condensation of material very high as compared with the stars of the main sequence.* It is at once apparent that such a model can be very useful for the understanding of red giant and supergiant stars, which are located on the right-hand side of the upper part of the main sequence. After the energy-producing shell grows to such an extent that very little material is left in the outer envelope, the escape of radiation from the generating zone will exceed the energy production by nuclear reactions. Nuclear processes will stop for the lack of sufficiently high temperature, and the star, with all its hydrogen practically gone, will go over into the state of gravitational contraction. Finite stages of this contraction will be identical with those expected for the stars retaining their point-source structure. In Fig. 5 we give the possible track of the star evolving according to the shell model, as compared with the normal point-source evolution studied in the previous section.

It is very possible that both modes of evolution are taking place in the stellar world: fast rotating stars with vigorous internal convection may evolve according to the point-source model, whereas slowly rotating stars may develop an energy-producing shell, and pass on their way through the red giant stage. This point of view may gain some support from the observational fact that the upper part of the main sequence is formed almost exclusively by stars possessing abnormally high axial rotations.

[20] Compare: A. Eddington *Internal Constitution of Stars* (Cambridge University Press, 1926), Chap. IV, Section 67.

4. MASS EJECTION BY CONTRACTING STARS

As we have already mentioned in previous sections, the striking property of massive stars in the later stages of their evolution lies in their ability to eject their atmospheres into the surrounding space. The velocity of ejection, estimated from Doppler broadening of emission lines, varies from comparatively low values of about 100 km/sec. for P Cygni stars up to several thousand km/sec. for the typical W-R stars. There can hardly be any doubt that the observed continuous ejection is due to the effect of radiation pressure which overbalances the forces of gravity near the surface of the star. We can form some idea about the material density in the region where the main acceleration of ejected masses takes place by comparing the initial momentum of radiation coming from the stellar interior with the momentum flux of the ejected masses. If L is the total luminosity of the star, and R^* the mean radius of the layer where the primary absorption takes place, the momentum flux of radiation will be

$$L/4\pi R^{*2}\cdot c.$$

If, on the other hand, ρ^* is the mean density in this region and V the observed ejection velocity, the flux of mechanical momentum is

$$\rho^* V\cdot V=\rho^* V^2.$$

According to the law of conservation of momentum we can write:

$$\frac{L}{4\pi R^{*2}c}\cong\rho^* V^2. \tag{8}$$

TABLE I. Acceleration of H and He^+ due to bound-free transitions for $\rho=10^{-12}$.

T	25,000°C	50,000°C	100,000°C	200,000°C
H				
α^{b-f}	4.5×10^{8}	2×10^{10}	2×10^{11}	1×10^{12}
η	2×10^{-8}	3×10^{-10}	2×10^{-11}	3×10^{-12}
$\alpha^{b-f}\cdot\eta$	9	6	4	3
He^+				
α^{bf}	40	1×10^{7}	3×10^{10}	1×10^{12}
η	~1	6×10^{-6}	2×10^{-9}	3×10^{-11}
$\alpha^{bf}\cdot\eta$	40	60	60	30

Taking, as an example, $L=2\times10^5 L(\odot)=8\times10^{38}$ erg/sec., $R^*\cong R\cong 3R(\odot)=2\times10^{11}$ cm.[21] (the data for W-R component of H.D. 193576) and $V\cong 2\times10^8$ cm/sec., we obtain for the density in the accelerative region:

$$\rho^*\cong 1.5\times10^{-12}\ \mathrm{g/cm^3}.$$

This value is in good agreement with Aller's[22] estimate of densities in W-R atmospheres.

Turning to the particular type of the radiation-pressure mechanism which will be of most importance at these low densities and high temperatures (25,000 to 200,000 °C) involved, we must distinguish between three kinds of absorption: (1) absorption due to bound-free transitions (b-f) in highly ionized atoms; (2) absorption due to the so-called free-free transitions (f-f) of electrons in the neighborhood of charged nuclei; and finally (3) absorption due to Compton scattering by free electrons (s).[23]

The absorption cross section for b-f transitions in hydrogen-like ion of atomic number Z is:

$$a_z^{\mathrm{b-f}}=\frac{64\pi^4 m_e e^{10}}{3\sqrt{3}h^6c^4}\cdot\frac{z^4}{\nu^3},\tag{9}$$

where ν is the frequency of incident radiation. Multiplying this expression by Planck's distribution function I_ν, integrating over all frequencies $\geqslant \nu_z^0$ ($\nu_z^0=(2\pi m_e e^4/h^3)z^2$), and dividing by the mass Am_H of the atom in question, we obtain for the radial acceleration of the absorbing atom:

$$\begin{aligned}\alpha_z^{\mathrm{b-f}}&=\int_{\nu_z0}^{\infty}\pi\frac{a_z^{\mathrm{b-f}}}{Am_\mathrm{H}}\cdot\frac{I_\nu}{c}d\nu\\ &=\frac{256\pi^5 m_e e^{10}Z^4}{3\sqrt{3}h^6c^4m_\mathrm{H}A}\\ &\qquad\cdot kT\cdot\ln\left(1-\exp\left(-h\nu_z^0/kT\right)\right)^{-1}\\ &=7.3\cdot10^{22}\cdot\frac{Z^4}{A}\\ &\qquad\cdot kT\ln\left(1-\exp\left(-h\nu_z^0/kT\right)\right)^{-1}.\end{aligned}\tag{10}$$

[21] On the basis of the absorption coefficient in W-R atmospheres one arrives at the conclusion that the thickness of the accelerative layer must be comparable with the stellar radius.

[22] L. H. Aller, Astrophys. J. **97**, 135 (1943).

[23] Compare, for example, S. Chandrasekhar, *Introduction into the Study of Stellar Structure*, Chap. VII.

On the other hand the fraction η of all atoms of the given element which retain their K electron is given by the thermal ionization formula:

$$\begin{aligned}\eta&=\frac{2}{1+2\dfrac{(2\pi m_e)^{\frac{3}{2}}}{N_eh^3}(kT)^{\frac{3}{2}}\exp\left(-h\nu_z^0/kT\right)}\\ &=\frac{2}{1+1\times10^{28}(kT)^{\frac{3}{2}}\exp\left(-h\nu_z^0/kT\right)},\end{aligned}\tag{11}$$

where N_e is the electron density. Taking $N_e\cong10^{12}$ electrons/cm^3 (which corresponds to $\rho^*\cong10^{-12}$ g/cm^3) we obtain the following values of $\alpha_z^{\mathrm{b-f}}$, and η for H and He$^+$, respectively (see Table I). The third and sixth lines of the table represent the actual acceleration (in cm/sec.^{2}) expected in the stellar atmosphere built entirely from the given gas. We notice that the effect on He$^+$ is somewhat larger than that on H, but the calculations show that beyond helium radiative acceleration again decreases rather rapidly. It may be mentioned here that in his work on radiation pressure in hot stars, Gerasimovič[24] comes to much higher values of expected accelerations. This is, however, due to the fact that he assumes the densities in accelerative regions to be much higher (by a factor of about 10,000) than is assumed in the present article; such high densities would be, in fact, entirely inconsistent with the momentum conservation law as expressed by our Eq. (8).

Turning now to f-f transition, we have the following formula for the effective cross section:

$$a_z^{\mathrm{f-f}}=\frac{4\sqrt{2}\cdot\pi^{\frac{1}{2}}e^6N_eZ^4}{3\sqrt{3}m_e^{\frac{3}{2}}hc(kT)^{\frac{1}{2}}}\cdot\frac{1}{\nu^3}.\tag{12}$$

The total acceleration of stellar matter comes out to be:

$$\begin{aligned}\alpha_{\mathrm{total}}^{\mathrm{f-f}}&=\frac{8\sqrt{2}\cdot\pi^{\frac{3}{2}}e^6N_e}{3\sqrt{3}m_e^{\frac{3}{2}}hc^4m_\mathrm{H}}(kT)^{\frac{1}{2}}\sum\frac{c_zZ^2}{A}\\ &=1.8\cdot10^5(kT)^{\frac{1}{2}}\sum\frac{c_zZ^2}{A},\end{aligned}\tag{13}$$

where c_z is the relative abundance of element Z. For an atmosphere consisting of 65 percent of Russell mixture of heavier elements the sum in

[24] B. P. Gerasimovič, M. N. R. A. S. **94**, 737 (1934).

(13) is equal to 4. Taking again $N_e \cong 10^{12}$, we find from (13) that for the temperatures from 25,000 °C to 200,000 °C, total acceleration due to free-free transitions will be only between 0.1 and 0.6 cm/sec.².

In case of scattering by free electrons the cross section is given by the well-known formula:

$$a_s = \frac{8\pi e^4}{3 m_e c^4} = 6.6 \times 10^{-25} \text{ cm}^2 \qquad (14)$$

and the corresponding acceleration:

$$\alpha_s = \frac{a_s}{\mu_e m_{\mathrm{H}}} \cdot \frac{\sigma T^4}{2c} = 3.8 \times 10^{-16} \frac{T^4}{\mu_e}, \qquad (15)$$

μ_e being the mass per free electron. Because of the strong ionization we can assume μ_e to be not much larger than 3 or 4. The formula (15) gives then that with T varying from 25,000 to 200,000 °C, acceleration α_s will increase from 40 to 160,000 cm/sec.².

Thus we come to the conclusion that *under the conditions existing in the accelerative layers of massive contracting stars radiation pressure due to scattering by free electrons must play the most important role in the process of ejection of stellar atmospheres.*

For the ratio of radiation pressure to the force of gravity on the stellar surface we have:

$$\frac{\alpha_s}{\alpha_{\mathrm{grav}}} = \left(\frac{a_s L}{\mu_e m_{\mathrm{H}} 8\pi R^2 c}\right) : \left(\frac{GM}{R^2}\right)$$
$$= 4 \times 10^{-6} \frac{(L/L(\odot))}{(M/M(\odot))}. \qquad (16)$$

Thus the condition that radiation pressure overbalances gravity may be written in the form:

$$(L/L(\odot))/(M/M(\odot)) > 2.5 \times 10^5. \qquad (17)$$

Applying this condition to the normal stars of the main sequence, we find that even for most luminous stars $[L \cong 2 \times 10^5 L(\odot);\ M \cong 40 M(\odot)]$ the left side of (17) is only 5×10^3. Since, however, luminosities of contracting stars reach much higher values for the same mass, conditions for the ejection become more favorable. Using maximum luminosities calculated above for stellar masses $10M(\odot)$, $20M(\odot)$, and $40M(\odot)$, we find for the ratio of radiation pressure to the force of gravity 0.1, 0.2, 0.3, respectively. These values are still smaller than unity, and, since the mass ejection by W-R stars is a well-established fact, the conclusion seems to be necessary that in our estimate of the ratio $\alpha_{\mathrm{rad}}/\alpha_{\mathrm{grav}}$ we have either underestimated the value of radiation pressure or overestimated the acting force of gravity. The latter possibility seems quite reasonable since the gravity on the stellar surface can be largely reduced by the centrifugal force resulting from the axial rotation of the star; and it is known, in fact, that most stars belonging to the upper part of the main sequence rotate very rapidly with equatorial velocities easily reaching the values as high as 200 km/sec. Such high rotation velocities, increased still further by the steady contraction of stellar body, will help the radiation pressure in the process of atmospheric ejection, and will make this process possible already for smaller values of star masses. Further observational studies in this direction are evidently very desirable.

It must be noticed in the conclusion of this section that all the treatment of the dynamical state of stellar atmosphere under the action of escaping radiation has been carried out only in a very schematic form. In a more exact treatment it would be necessary to take into account the effect of re-emission of the absorbed energy, leading to the backflow of radiation through the stellar atmosphere, and exercising "breaking action" on the outgoing gas masses. Some interesting contributions to this problem can be found in the works of Gerasimovič.

5. LATE STAGES OF CONTRACTIVE EVOLUTION

Since in the later stages of the evolution the energy liberation in the star is due exclusively to gravitational forces, we can work out an evolutionary time scale on the basis of the self-evident equation:[25]

$$1.3 \cdot MG^2\left(\frac{1}{R} - \frac{1}{R_0}\right) = L \cdot \Delta t. \qquad (18)$$

The resulting time changes of stellar radius for star masses $10M(\odot)$, $20M(\odot)$, and $40M(\odot)$

[25] The numerical coefficient 1.3 in the expression of potential energy corresponds to the mass distribution in the contractive model.

are shown in Fig. 6a. We see that the contraction of the star becomes quite noticeable a few thousand years after the luminosity has reached its maximum value. Since the total luminosity does not change any more, the surface temperature of the star must increase inversely with the square root of the radius (Fig. 6b). With the increasing surface temperature, the maximum of the energy curve will move further and further into the ultraviolet, and the visible part of the spectrum will be more and more weakened. The visual luminosity of the star will be proportional to the surface temperature and to the square of the stellar radius; since $T_{\text{surf}} \sim R^{-\frac{1}{2}}$ we have $L_{\text{vis}} \sim R^{+\frac{3}{2}}$. These luminosity changes, expressed in terms of stellar magnitudes, are shown graphically in Fig. 6c, and we see that a contracting star must become visually fainter by two or three magnitudes in the period of a few thousand years.

We can now ask ourselves what becomes of the gases which are being continuously ejected from the surface of a contracting star. Let us suppose first that they continue to move outwards with the velocity of several thousand km/sec. observed near the surface of the star. According to the continuity law

$$\rho r^2 V = \text{const.}, \tag{19}$$

the density in the expanding envelope will vary inversely with the square of the distance from the star. Since the density of gases in the accelerative layer near the surface of the star is about 10^{-12} g/cm^3, the density of the envelope will drop to 10^{-20} g/cm^3 at the distance of 10^4 stellar radii (or 200 astr. units), and to 10^{-22} g/cm^3 at the distance of 10^5 radii (2000 astr. units).[26] The later value represents the average size of the celestial objects known as planetary nebulae, and consisting of a very hot central star (planetary nucleus) surrounded by a luminous gaseous envelope. The study of planetary envelopes carried out by Menzel and Aller[27] leads to the densities about one hundred times lower than the above-given estimate of the expected densities for the W-R ejection. Since the luminosity of the gaseous envelopes is due entirely

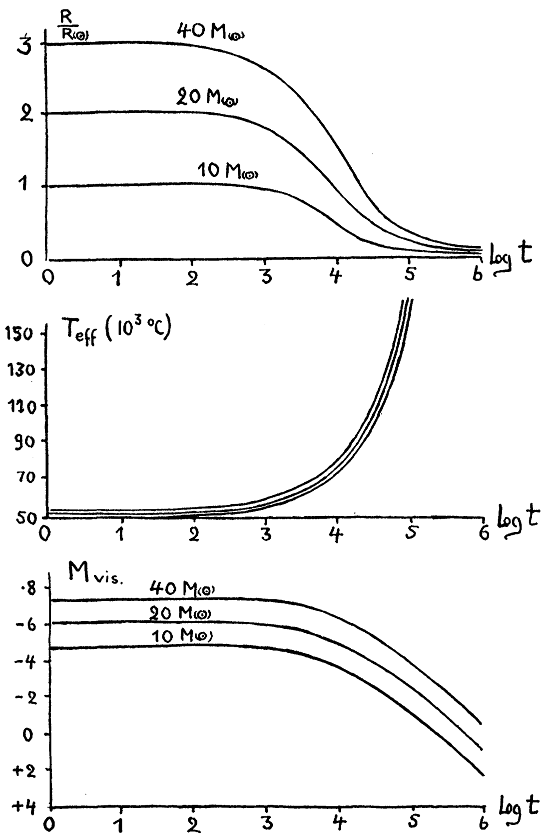

FIG. 6. Time changes of the properties of a contracting star which has reached the maximum of its total luminosity. (a) (*top*) Radius, (b) (*middle*) surface temperature, (c) (*bottom*) visual magnitude.

to the re-emission of light coming from the central star and must be roughly proportional to the amount of matter forming the envelope, we should expect W-R envelopes to be relatively about one hundred times fainter than those of the planetary nebulae. The relative intensity can be still further reduced because of the fact that W-R stars seem to have somewhat lower surface temperatures than the nuclei of planetary nebulae, and correspondingly possess less intensity in the active ultraviolet part of their spectrum. It is difficult to give the exact estimate of the expected luminosity of W-R envelopes without more detailed calculations, but it seems that one should be able to observe them at least as a faint glow around the star.

On the other hand, observational studies of the known W-R stars do not give even a slight indication of the existence of such envelopes. If this

[26] Because of the high velocities of ejected gases, it would take only 5 years before the envelope will expand to that size.

[27] D. H. Menzel and L. H. Aller, Astrophys. J. 93, 195 (1941).

conclusion will be confirmed by more careful observations and more detailed calculations of the expected intensity of W-R envelopes, the conclusion will be inevitable that the ejected gases do not continue to move with the original velocity, but are probably slowed down by the gravitational forces of the star. Such an assumption seems quite reasonable since the escape velocities for W-R stars, as calculated from their masses and radii, are of the same order of magnitude as the observed original ejection-velocities. Thus, for example, for $M=10M(\odot)$ and $R=3R(\odot)$ we have $V_{\text{escape}}=1200$ km/sec.

If we assume that the gases ejected from the typical W-R stars are decelerated by gravity to such an extent that their original velocity is reduced by a factor of a hundred, the equation of continuity (18) will lead to the densities of W-R envelopes comparable with those of planetary nebulae for equal distances. Furthermore, this would reduce the expansion velocity to only 20 km/sec., thus making it of the same order of magnitude as the actually observed expansion velocities of planetary envelopes. On the other hand, the time period necessary to reach the size of a planetary nebula will be lengthened to several hundred or a thousand years. With the further progress of the contraction, visual luminosity of the central star will decrease, and the gaseous envelope will become more luminous because of the relative strengthening of the ultraviolet part of the spectrum. All in all, *the aged W-R star will look rather similar to a typical planetary nebula, and the suggestion may be put forward that the planetary nebulae observed at present actually represent the results of the evolution of the W-R stars of the past.*

6. ULTIMATE FATE OF THE CONTRACTING STAR

In considering the ultimate fate of a contracting star, we must first of all take into consideration the dissipation of mass due to the atmospheric ejection. Using the data given in the previous section, we find that the outgoing flow of gases (both in case of W-R stars and planetary nebulae) carries away about 10^{28} g/year or 10^{-5} sun mass per year. Thus far a star of $10M(\odot)$ substantial mass changes cannot be expected to take place in less than say a hundred thousand years. On the other hand, the central temperature of the contractive star increases inversely with its radius, and as it can be seen from Fig. 6a, will increase by a factor of 5 in about 10^4 or 10^5 years after the star has reached the maximum of its luminosity. Assuming, as in previous sections, that the maximum luminosity is reached for the central temperature of about 2×10^8 °C, this would correspond to the interior of the star heated up to 1×10^9 °C.

Now, it has been shown by the present author and Schoenberg[28] that the temperature of this order represents a threshhold for the so-called urca-processes by which a large number of neutrinos can be produced in the stellar interior. Escaping through the body of the star these neutrinos carry away the heat energy of the central regions at a rate which can be as high as 10^{10} or even 10^{15} erg/g sec. Thus the formation and the escape of neutrinos act as a kind of thermostatic arrangement preventing the central temperature of the star from rising much above 10^9 °C. Under such conditions, "adiabatic" dependence of gas pressure on gas density in the stellar interior will be reduced to simple proportionality (instead of the usual relation $p\sim\rho^{5/3}$), and the star will become radially unstable. This can be simply seen from the fact that the increase of gas pressure ($p_{\text{gas}}\sim\rho$) produced by a small contraction of stellar body will be unable to counteract the increase of interior pressure due to gravity ($p_{\text{grav}}\sim\rho^{4/3}$). Thus the contraction will continue, and go over into a major collapse. In Fig. 4 we have indicated by vertical bars the points of contractive-evolution tracks where such a collapse is expected to take place.

The treatment of the dynamical problem of a collapsing star is, of course, immensely difficult, and we can give here only some preliminary qualitative considerations concerning this process.

A comparatively simple solution could be obtained under the assumption of the spherical symmetry of the collapse motion. In this case we would get a rapidly but uniformly shrinking model with the radiation streaming outwards along the radii in accordance with the ordinary opacity formula. Since the collapse is expected towards the end of the evolution, when the opacity of stellar matter had reached its mini-

[28] G. Gamow and M. Schoenberg, Phys. Rev. **59**, 539 (1941); G. Gamow, Phys. Rev. **59**, 617 (1941).

mum value (scattering opacity), and since the temperature in the center cannot rise above 10^9 °C, the flux of the radiant energy through the body of the spherically-symmetrically collapsing star cannot be substantially larger than that existing before the collapse started. Thus almost the entire gravitational energy liberated in the collapse will have to be removed by neutrinos generated in the urca-region, and no sudden flare up characterizing stellar explosion can be expected. The star would simply shrink very rapidly while keeping its original luminosity.

It must be clear, however, that the spherical-symmetrical solution cannot possibly correspond to the reality, in the case of a collapse taking place within only a few hours. It is, in fact, much more likely that the lack of supporting pressure in the stellar interior will cause a very irregular motion of gas masses, some of them precipitating towards the center and the other being "squeezed out" and thrown away in the form of vigorous prominences. The rotation of the star will exercise some kind of directing influence on this phenomenon since the presence of centrifugal force will hinder the inward motions of matter in equatorial regions. We may expect that the polar regions of a collapsing star will, so to speak, cave in, ejecting masses of central and equatorial regions. Radiation energy, which is saturating the material of stellar interior, will be thus bodily carried out and set free in the outer regions of the star. A rough estimate of the amount of radiative energy which can be liberated in such a process and of the velocities of ejected gas masses seems to be in satisfactory agreement with the observational data on supernovae explosions. In particular it is evident that mass ejection by supernovae must be of a purely mechanical character since the total mechanical momentum of the ejected envelope considerably exceeds the total momentum of the emitted radiation. In fact, according to Baade[29] and Minkowski[30] the velocity of gases ejected by the Chinese supernova of the year A.D. 1054 is about 1000 km/sec., whereas the total mass of the gas shell may be as large as 15 sun masses. These numbers lead to the value of 3×10^{42} erg·sec./cm for the total mechanical momentum of the shell. On the other hand the total amount of light energy emitted by a typical supernova is of the order of magnitude of 10^{48} erg corresponding to the momentum of only 10^{38} erg·sec./cm. One can argue, of course, that the above-given number does not take into account the invisible part of supernova spectrum, but it is extremely unlikely that the actual luminosity of a supernova could exceed by a factor of ten thousand its visible luminosity. Thus we can conclude with a large degree of certainty that in contrast to the mass ejection by a W-R star, nuclei of planetary nebulae, and possibly in contrast to the ordinary novae, the explosion of supernovae is of more mechanical character and is directly connected with the dynamical peculiarities of a collapse process.

In conclusion the author considers it to be his pleasant duty to express his gratitude to his astronomical friends Dr. J. A. Hynek and Dr. Ph. C. Keenan for many helpful discussions on the subject.

[29] W. Baade, Astrophys. J. **96**, 188 (1942).
[30] R. Minkowski, Astrophys. J. **96**, 199 (1942).